MICAS921: Multi-user communications

“Practical schemes and Resource Allocation”

Philippe Ciblat

Telecom Paris, Institut Polytechnique de Paris, Palaiseau, France

Outline

1. Introduction and motivation

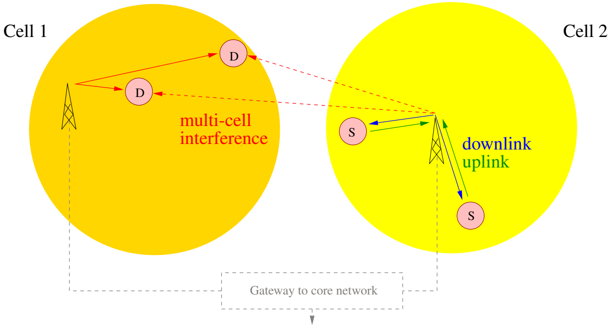
2. Practical multiple access schemes
   - TDMA, FDMA, OFDMA, CDMA
   - Orthogonality loss
   - Receivers
   - Link with Information Theory (capacity region)
   - Extension to NOMA

3. Resource allocation algorithms
   - Which parameters: power, bandwidth, scheduling
   - Which problems: sum-capacity, power minimization, energy efficiency, ...
   - Technical optimization issues: often non-convex optimization
   - Technical solutions: geometric programming, fractional programming, signomial programming, difference of convex/successive convex approximation, alternate strategies/block-coordinate descent, relaxation, monotonic programming, biconvex programming, ...

Section 1 : Introduction and Motivation

Interference issues: where do they come from?

Cellular network (3G/4G/5G)



- Several information flows to manage
- Different kinds of interference: multi-cell (red), uplink (green), downlink (blue)

$\Rightarrow$ **Multi-User Interference (MUI)**

Example 1

Consider two users

$$\begin{cases} x_1(t) &= s_1\phi_1(t) \\ x_2(t) &= s_2\phi_2(t) \end{cases}$$

with ϕ_1 and ϕ_2 two orthonormal functions
We receive (uplink: users $\rightarrow$ BTS)

$$\begin{aligned} y(t) &= (h_1 \star x_1)(t) + (h_2 \star x_2)(t) + w(t) \\ &= s_1\psi_1(t) + s_2\psi_2(t) + w(t) \end{aligned}$$

with $\psi_n(t) = (h_n \star \phi_n)(t)$.
We get

$$\begin{aligned} < \psi_1 | \psi_2 > &= \int (h_1 \star \phi_1)(t)\overline{(h_2 \star \phi_2)(t)}dt \\ &= \iiint h_1(\tau_1)\overline{h_2(\tau_2)}\phi_1(t-\tau_1)\overline{\phi_2(t-\tau_2)}dtd\tau_1 d\tau_2 \\ &= \int H_1(f)\overline{H_2(f)}\Phi_1(f)\overline{\Phi_2(f)}df \end{aligned}$$

Usually channel leads to orthogonality loss

Example 2

- Consider MISO downlink with two users.
- Apply beamforming $\mathbf{v}_1$ and $\mathbf{v}_2$

We get

$$\begin{cases} y_1 = \mathbf{h}_1^{\mathrm{T}}\mathbf{v}_1 s_1 + \mathbf{h}_1^{\mathrm{T}}\mathbf{v}_2 s_2 \\ y_2 = \mathbf{h}_2^{\mathrm{T}}\mathbf{v}_1 s_1 + \mathbf{h}_2^{\mathrm{T}}\mathbf{v}_2 s_2 \end{cases}$$

Maximizing SINR is ideally equivalent to get

- $\mathbf{v}_1 \in \mathrm{span}(\mathbf{h}_1)$, $\mathbf{v}_2 \in \mathrm{span}(\mathbf{h}_2)$,
- and $\mathbf{v}_1 \in \ker(\mathbf{h}_2)$, $\mathbf{v}_2 \in \ker(\mathbf{h}_1)$

It happens iff $\mathbf{h}_1 \perp \mathbf{h}_2$. If not, by keeping the signal power maximization, we get

$$\begin{cases} y_1 = \|\mathbf{h}_1\|^2 s_1 + \gamma s_2 \\ y_2 = \gamma s_1 + \|\mathbf{h}_2\|^2 s_2 \end{cases}$$

with

$$\gamma = \mathbf{h}_1^{\mathrm{T}}\mathbf{h}_2 = <\mathbf{h}_1|\mathbf{h}_2>$$

MU-MIMO with beamforming leads to non-orthogonality

Naive solution 1: do nothing

- One user of interest but $N-1$ interferers (with same power):

$$y = x + \sum_{k=1}^{N-1} \gamma x_k + w$$

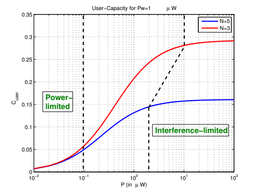
- **Assumption :** interference seen as an extra (Gaussian) noise:

$$C_{\text{user}} = \log_2\left(1 + \frac{P}{(N-1)\gamma^2 P + P_w}\right)$$

with user power P and noise power P_w

Result

- $C_{\text{user}} \to \log_2\left(1 + \frac{1}{(N-1)\gamma^2}\right)$ when $P \to \infty$
- C_{target} achievable iff $N \leq 1 + \frac{1}{(2^{C_{\text{target}}}-1)\gamma^2}$



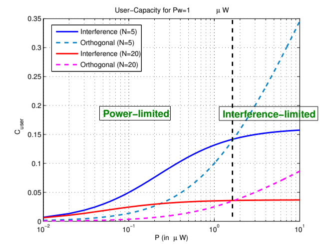
Naive solution 2: take margin

- TDMA with time margin of $\gamma^2\%$
- FDMA with frequency margin of $\gamma^2\%$

Result

$$C_{\text{user}}^{\perp} = \frac{1}{N(1+\gamma^2)} \log_2\left(1 + \frac{P}{P_w}\right)$$

- $C_{\text{user}}^{\perp} \to \infty$ when $P \to \infty$, no upper bound
- For low and medium P (depending on N) $C_{\text{user}} > C_{\text{user}}^{\perp}$.



Comments

Two regimes:

- Interference-limited: if SNR large enough
- Power-limited: if SNR low enough

⇒ **Orthogonality can not be used for any flow (N too large)**
  - in practice in downlink and uplink only, ...

⇒ **Even if orthogonality used, partially broken at the receiver**
  - in practice multi-path, Doppler effect, ...

Degrees of freedom:

- Multiple access techniques
- Receivers
- Resource allocation (scheduling, power)

Section 2 : Practical multiple access schemes and related receivers

Section Outline

- TDMA, FDMA, OFDMA, CDMA, MC-CDMA
- Orthogonality loss
- Receivers
- Practical performances
- Link with Information Theory (capacity region)
- Extension to NOMA (power-domain and code-domain)

- **Exercise session:** code-domain NOMA performance analysis

Orthogonal schemes

Easily to translate the **orthogonality principle** in time and frequency

- TDMA (2G); if random access, same idea in CSMA/CA (Wifi)
- FDMA (2G); if coupled with OFDM, then **OFDMA** (4G/5G)
  - in one subcarrier, only one user is assigned
  - each user may have several subcarriers

Rate per user:

- B: total bandwidth
- N: users
- T: channel use duration ($= 1/B$)

TDMA

$$\text{user rate} = \frac{1}{NT} = \frac{B}{N}$$

FDMA

$$\text{user rate} = \frac{B}{N}$$

same spectral efficiency

Orthogonal schemes : another way to build them

Idea: translating the **orthogonality principle** into a signal structure

- CDMA (IS95, 3G, HSPA); direct sequence implementation (DS)
- For user n, instead of sending s_n, we send N consecutive samples stacked into

$$\mathbf{x}_n = [c_n^{(1)} s_n, \cdots, c_n^{(N)} s_n] = \underbrace{[c_n^{(1)}, \cdots, c_n^{(N)}]}_{\mathbf{c}_n} s_n$$

- Orthogonality property:

$$\mathbf{x}_n \perp \mathbf{x}_{n'} \Leftrightarrow \mathbf{c}_n \perp \mathbf{c}_{n'}$$

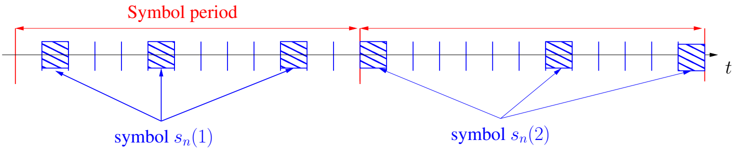
Rate per user:

$$\text{user rate} = \frac{1}{NT} = \frac{B}{N}$$

Actually, any orthogonal scheme offers the same spectral efficiency
but $\neq$ in robustness to orthogonality loss, diversity gain, complexity, ...

Almost-orthogonal schemes: CDMA-like

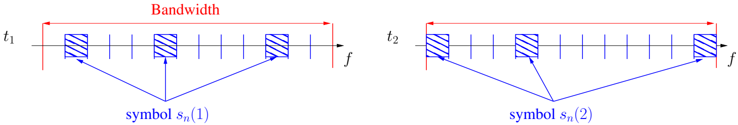
- Time Hopping (TH) (UWB - IEEE 802.15.4a)



Assigned "slots" depend on user code

Collision occurs but is mitigated due to the user code

- Frequency Hopping (FH) (4G or military application)



Assigned "subcarriers" depend on user code

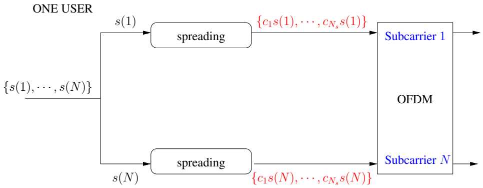
Small interference allowed (collision and missynchronization mitigation)

Coupling between OFDM and CDMA-like

- MC-DS-CDMA: $\perp$ loss <u>but</u> time diversity
- MC-CDMA: $\perp$ loss <u>but</u> frequency diversity
- OFDMA: no $\perp$ loss <u>but</u> no inherent diversity

MC-DS-CDMA transmitter

- One DS-CDMA per subcarrier with spreading factor N_s
- Time diversity over $N_s NT$



Consider subcarrier 1, we have $\mathbf{y} = [y_1(1), \cdots, y_{N_s}(1)]^{\mathrm{T}}$.

$$\mathbf{y} = \mathbf{H}\mathbf{c}_1 s(1) + \text{other users} + \mathbf{w}$$

where

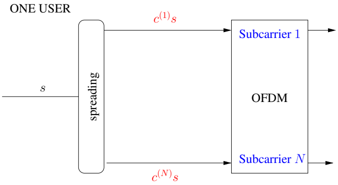
- $\mathbf{H} = \mathrm{diag}(H_1(1), \cdots, H_{N_s}(1))$
- $\mathbf{c}_1 = [c_1, \cdots, c_{N_s}]^{\mathrm{T}}$

If $H_k(1)$ independent of k (no time diversity),

- channel matrix is proportional to identity
- do correlation with signature: then no MUI

MC-CDMA transmitter

- DS-CDMA for each user spread over all the subcarriers
- Frequency diversity over N subcarriers



Let $\mathbf{y} = [y(1), \cdots, y(N)]^{\mathrm{T}}$.

$$\mathbf{y} = \mathbf{H}\mathbf{c}_k s + \text{other users} + \mathbf{w}$$

where

- $\mathbf{H} = \mathrm{diag}(H(1), \cdots, H(N))$ a diagonal matrix
- $\mathbf{c}_k = [c_k^{(1)}, \cdots, c_k^{(N)}]^{\mathrm{T}}$
- $\mathbf{C} = [\mathbf{c}_1, \cdots, \mathbf{c}_N]$ a unitary matrix
- $\mathbf{s} = [s_1, \cdots, s_N]^{\mathrm{T}}$ and so $\mathbf{y} = \mathbf{HCs} + \mathbf{w}$

Matrix HC not diagonal and even non-unitary matrix $\Rightarrow \perp$ loss

How introducing diversity in OFDMA?

Problem: channel fading for some subcarriers

- FH-OFDMA
- Coded-OFDMA
  - coding
  - time-frequency interleaving
- LP-OFDMA (Linear Precoding). Spreading symbols over the assigned subcarriers

$$\mathbf{y} = [y(n_1), \cdots, y(n_K)]^{\mathrm{T}} = \mathbf{HWs} + \mathbf{w}$$

  with
  - $\mathbf{H} = \mathrm{diag}(H(n_1), \cdots, H(n_K))$ diagonal matrix of the channel
  - $\mathbf{W}$ precoding matrix providing diversity

  but **HW** neither diagonal nor unitary matrix $\Rightarrow$ intra-ISI
  MC-CDMA is just a specific LP (in addition to user separator)
- MIMO-OFDM (but additional antennas to exhibit space diversity)

LP-OFDM: why does it work?

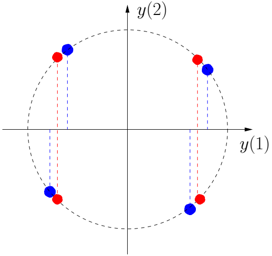
$$\begin{cases} y(1) = h(1)x(1) + w(1) \\ y(2) = h(2)x(2) + w(2) \end{cases}$$

with independent BPSK $x(1)$ and $x(2)$. Then

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$

with $\mathbf{y} = [y(1), y(2)]^{\mathrm{T}}$, $\mathbf{H} = \mathrm{diag}(h(1), h(2))$.

- Diversity 1
- Instead of sending $\mathbf{x}$, we send $\mathbf{W}\mathbf{x}$ with a rotation matrix $\mathbf{W}$
- Diversity 2 but coding rate 1



Receivers: refresher

General framework:

$$\mathbf{y} = \underbrace{\mathbf{HC}}_{\underline{\mathbf{H}}}\mathbf{s} + \mathbf{w}$$

with **H** channel matrix, **C** multiple access matrix.
If we get a pseudo-unitary matrix (cf. *MICAS903*)

$$\underline{\mathbf{H}}^{\mathrm{H}}.\underline{\mathbf{H}} \propto \mathbf{Id}$$

then ZF is optimal.

Conditions for pseudo-unitary property:

- no channel ($\mathbf{H} = \mathbf{Id}$)
- **C** pseudo-unitary

Sequence design

- If $\mathbf{C}$ is forced to be pseudo-unitary: Walsh-Hadamard sequence.
  Let $N = 2^P$ and $\mathbf{C}_0 = [1]$

$$\mathbf{C}_p = \begin{bmatrix} \mathbf{C}_{p-1} & \mathbf{C}_{p-1} \\ \mathbf{C}_{p-1} & -\mathbf{C}_{p-1} \end{bmatrix}$$

  $\Rightarrow$ $\mathbf{C}_P/\sqrt{N}$ unitary matrix
- If $\mathbf{C}$ is not forced to be pseudo-unitary:
  - why ? often $\mathbf{HC}$ closer to pseudo-unitary if $\mathbf{C}$ not
  - how? Gold or Kasami Pseudo-Noise (PN) sequence.

  Let $\mathbf{c}^{(n)}(\tau) = [c_\tau^{(n)}, \cdots, c_{(N-1+\tau) \bmod N}^{(n)}]$ be shifted sequence by τ.
  For $n \neq n'$ or $\tau \neq \tau'$, we want

$$< \mathbf{c}^{(n)}(\tau) | \mathbf{c}^{(n')}(\tau') > \approx \frac{1}{\sqrt{N}}$$

  rather than

$$< \mathbf{c}^{(n)}(0) | \mathbf{c}^{(n)}(0) >= 1 \text{ and } < \mathbf{c}^{(n)}(0) | \mathbf{c}^{(n')}(0) >= 0$$

Receivers design

- If we get a pseudo-unitary matrix

$$\underline{\mathbf{H}}^{\mathrm{H}}.\underline{\mathbf{H}} \propto \mathbf{Id}$$

then ZF (apply $\underline{\mathbf{H}}^{\#}$ and then thresholding) is optimal.

- If we do not get a pseudo-unitary matrix
  - Single-user detection (SUD)
    $\Rightarrow$ multi-user interference seen as a noise
  - Multi-user detection (MUD)
    $\Rightarrow$ multi-user interference structure used (codes required)

Single user detection

$$\mathbf{y} = \mathbf{H}_k \mathbf{C}_k \mathbf{s}_k + \text{other users} + \text{noise}$$

The oldest receiver: Rake receiver [1958]

- Apply $(\mathbf{H}_k \mathbf{C}_k)^{\mathrm{H}}$ (matched filter) and then thresholding
- it works well if flat fading channel (as then MUI-free)
- it does not work well if non-flat fading
  - Floor effect
  - Near-far effect: power control required (IS95, 3G)

Let $\mathbf{z}_{k'} = (\mathbf{H}_{k'} \mathbf{C}_{k'})^{\mathrm{H}} \mathbf{y}$, then

$$\mathbf{z}_{k'} = (\mathbf{H}_{k'} \mathbf{C}_{k'})^{\mathrm{H}} \mathbf{H}_k \mathbf{C}_k \mathbf{s}_k + \text{other users} + \text{noise}$$

$\mathbf{z}_k$ non-exhaustive statistics for user k [1983]

Example: SUD for MC-CDMA

Subcarrier n, we have

$$y(n) = H_1(n)c_1^{(n)}s_1 + \sum_{k=2}^{K} H_k(n)c_k^{(n)}s_k + \text{noise}$$

(if downlink $H_k(n)$ does not depend on k at user 1)

Idea: linear recombination between subcarriers

$$z_1 = \sum_{n=1}^{N} c_1^{(n)} w(n) y(n)$$

with

- Maximum Ratio Combiner (matched filter/Rake) : $w(n) = \overline{H_1(n)}$
- Equal Gain Combiner : $w(n) = \overline{H_1(n)}/|H_1(n)|$
- ZF : $w(n) = 1/H_1(n)$
- MMSE : $w(n) = \overline{H_1(n)}/\left(|H_1(n)|^2 + \sigma_w^2\right)$

Multi-user detection: optimal detector (ML)

$$\mathbf{y} = \mathbf{HCs} + \mathbf{w}$$

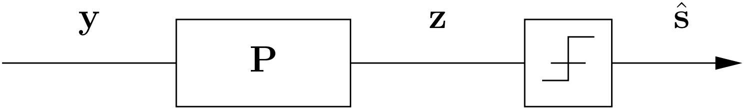
with

- $\mathbf{y}$ received samples (during a frame of length M)
- $\mathbf{s}$ transmit symbols for all users (length KM)
- $\mathbf{w}$ zero-mean white Gaussian noise

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg\min_{\mathbf{s}} \|\mathbf{y} - \mathbf{HCs}\|^2$$

- Exhaustive search: $\mathcal{O}(\mathrm{card}(\mathbb{S})^{KM})$ with the constellation $\mathbb{S}$
- Viterbi algorithm: when applying it ? if ISI of length L, then $\mathcal{O}(KM(\mathrm{card}(\mathbb{S}))^{KL}) \Rightarrow$ still huge

Zero-Forcing (ZF)



$$\mathbf{P}_{\mathrm{ZF}} = (\mathbf{HC})^{\#}$$

with $(\bullet)^{\#} := (\bullet^{\mathrm{H}}.\bullet)^{-1}\bullet^{\mathrm{H}}$ the left-pseudo-inverse of $\bullet$

Then

$$\mathbf{z} = \mathbf{s} + (\mathbf{HC})^{\#}\mathbf{w}$$

- MUI completely vanishes
- but noise enhancement issue

Minimum Mean Square Error (MMSE)

$$\mathbf{P}_{\mathrm{MMSE}} = E_s\mathbf{C}^{\mathrm{H}}\mathbf{H}^{\mathrm{H}}(E_s\mathbf{H}\mathbf{C}\mathbf{C}^{\mathrm{H}}\mathbf{H}^{\mathrm{H}} + \sigma_w^2\mathbf{Id})^{-1}$$

- If low SNR, close to Rake (MUI almost not treated)
- If high SNR, close to ZF

Remark 1: if **C** is unitary,

- Equalize the channel
- then correlate with the user signature

Remark 2: Expensive large matrix inversion (size $KM \times KM$)

Example: MMSE for MC-CDMA

Previous slide applies directly

$$\mathbf{z} = \mathbf{R}_s \mathbf{C}^{\mathrm{H}} \underbrace{\mathbf{H}^{\mathrm{H}}(\mathbf{H}\mathbf{C}\mathbf{R}_s\mathbf{C}^{\mathrm{H}}\mathbf{H}^{\mathrm{H}} + \sigma_w^2 \mathbf{Id})^{-1}}_{\mathbf{W}}$$

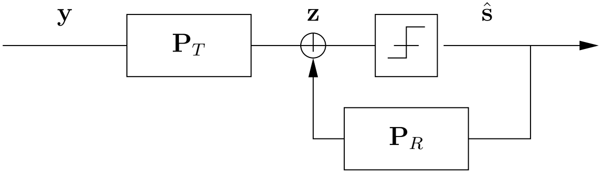
with

- $\mathbf{R}_s = \mathrm{diag}(P_1, \cdots, P_K)$ the power allocation of the users
- **H** now diagonal (due to OFDM)

Remark: **W** is non-diagonal except if

- Same power for any user
- Fully loaded system ($K = N$)

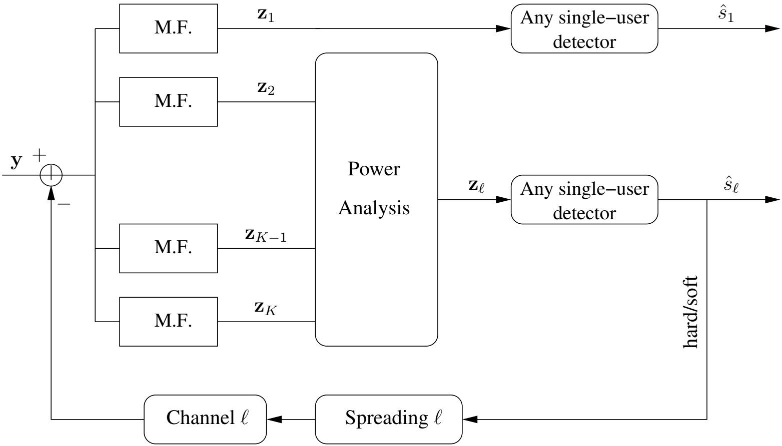
Decision feedback equalizer principle (DFE)



How $\hat{\mathbf{s}}$ available for feeding back the information

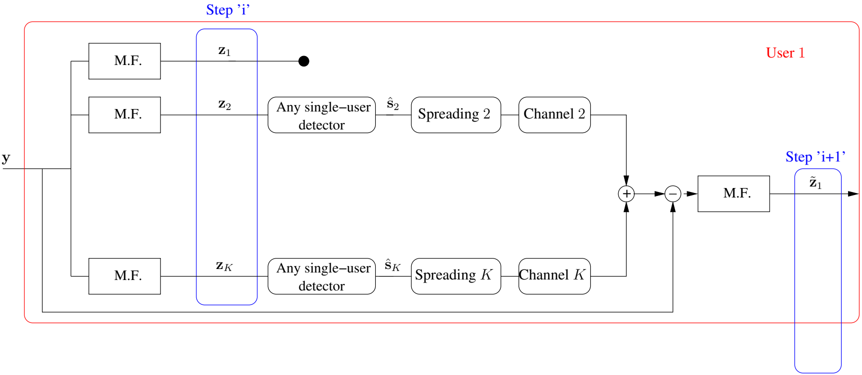
- Time causality: straightforward
- Multi-user causality:
  - the first one
  - the strongest one

Example: Successive Interference Canceller (SIC)



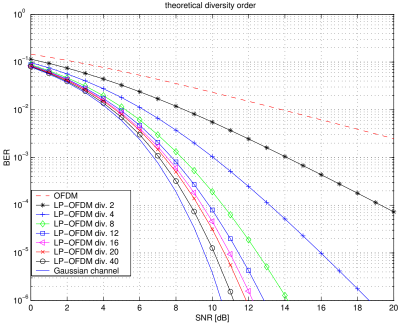
- A lot of combinations via any single-user detector

Example: Parallel Interference Canceller (PIC)

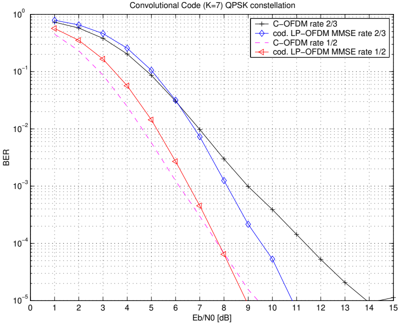


- High complexity
- Parallel processing possible

Performances: diversity point-of-view



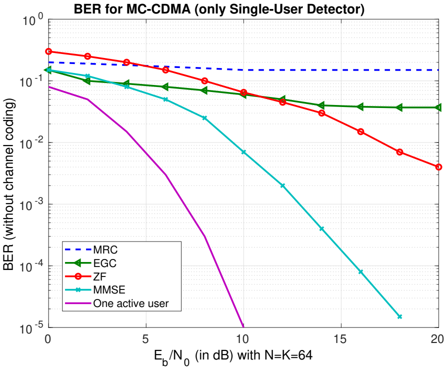
(uncoded) LP-OFDM



C-OFDM vs coded LP-OFDM

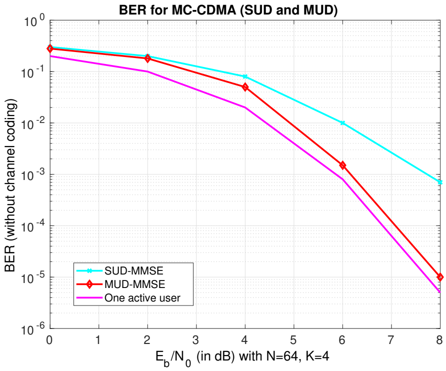
Source: Prof. Debbah (CentraleSupélec)

MC-CDMA with Single-User Detector



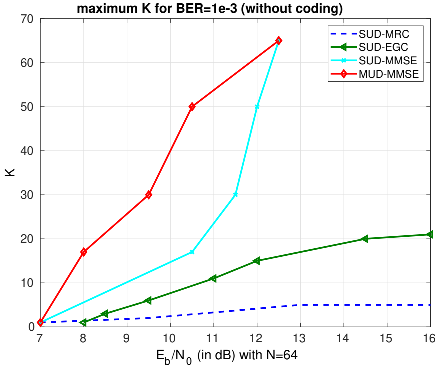
- Large loss in performance with SUD

MC-CDMA with Multi-User Detector



- MUD significantly improves the performance

MC-CDMA: load system



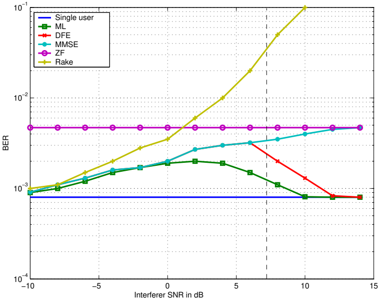
- At mid-SNR, MUD enables us to support more users
- Increasing BTS complexity decreases the BTS number
- MUD-MMSE diagonal matrix if fully-loaded $=$ SUD-MMSE

Performances: system-level

- $N = 7$, $K = 4$
- AWGN channels ($\mathbf{H} = \mathbf{Id}$)
- Non-orthogonal codes

$$\text{code correlation} = \frac{1}{7}\begin{bmatrix} 7 & 3 & -1 & -1 \\ 3 & 7 & -1 & 3 \\ -1 & -1 & 7 & -1 \\ -1 & 3 & -1 & 7 \end{bmatrix}$$

- User of interest with fixed SNR of 7dB
- Interferers with variable SNRs



Link with IT: General scheme

Rates depend on power, receiver algorithm, multiple access, ...

Question

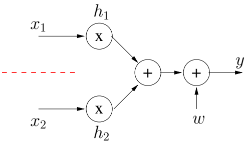
achievable rates regardless of the technique ?

$\Rightarrow$ **Multi-user Information Theory**

Example: Multi Access Channel (MAC/uplink)

$$y = h_1 x_1 + h_2 x_2 + w$$

$\rightsquigarrow$ Decode x_1 and x_2 from y



Capacity region [1974]

$$R_1 \leq \log_2\left(1 + \frac{|h_1|^2 P_1}{P_w}\right), R_2 \leq \log_2\left(1 + \frac{|h_2|^2 P_2}{P_w}\right),$$

$$R_1 + R_2 \leq \log_2\left(1 + \frac{|h_1|^2 P_1 + |h_2|^2 P_2}{P_w}\right)$$

Capacity for T/F/C-DMA and Time-Sharing

- T/F/C-DMA (example with 2 users for TDMA)
  - U1: α% of time with average power P_1. (power $\frac{P_1}{\alpha}$ when active)
  - U2: $(1-\alpha)$% of time with average power P_2. (power $\frac{P_2}{1-\alpha}$ when active)

Result

$$R_1 = \alpha \log_2 \left(1 + \frac{|h_1|^2 P_1}{\alpha P_w}\right) \text{ and } R_2 = (1-\alpha) \log_2 \left(1 + \frac{|h_2|^2 P_2}{(1-\alpha) P_w}\right)$$

- Time-Sharing (**warning:** $\neq$ T/F/C-DMA)
  - U1: α% of time with power P_1 when active
  - U2: $(1-\alpha)$% of time with power P_2 when active

Result

$$R_1 = \alpha \log_2 \left(1 + \frac{|h_1|^2 P_1}{P_w}\right) \text{ and } R_2 = (1-\alpha) \log_2 \left(1 + \frac{|h_2|^2 P_2}{P_w}\right)$$

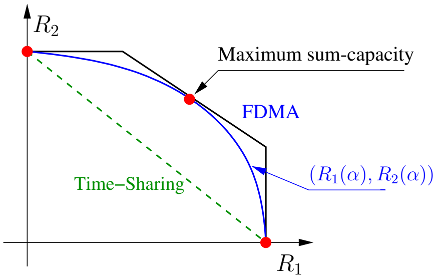
Any orthogonal scheme (in flat-fading channel) offers same capacity region

Sum-capacity

$$C^{\perp} = \alpha \log\left(1 + \frac{|h_1|^2 P_1}{\alpha P_w}\right) + (1-\alpha)\log\left(1 + \frac{|h_2|^2 P_2}{(1-\alpha)P_w}\right)$$

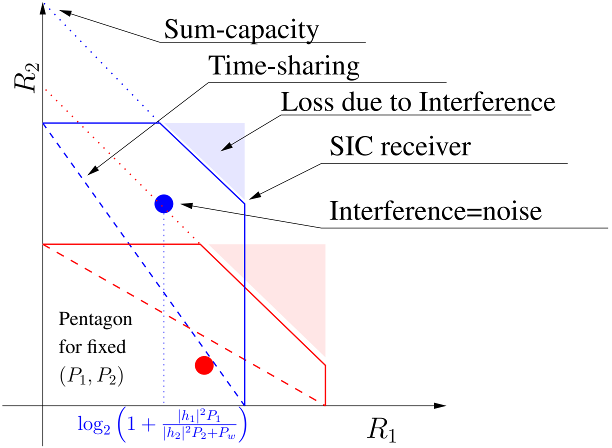
$$C^{\text{Time}-\text{Sharing}} = \alpha \log\left(1 + \frac{|h_1|^2 P_1}{P_w}\right) + (1-\alpha)\log\left(1 + \frac{|h_2|^2 P_2}{P_w}\right)$$

with $\alpha \in [0, 1]$



$C^{\perp}$ reachs the sum-capacity for $\alpha^{\star} = \frac{|h_1|^2 P_1}{|h_1|^2 P_1 + |h_2|^2 P_2}$

Capacity region



⇒ **Loss due to interference is the triangle (weak or strong)**

⇒ **Large loss if nothing done (the points)**

NOMA

NOMA $=$ Non-Orthogonal Multiple Access

Remark: Typically with SIC, interference can be managed

Consequence:

- Interference can be tolerated
- Multiple access can accept collision in advance
  - with appropriate coding scheme
  - with appropriate receiver

Power-domain NOMA

Downlink context (here with 2 users)

- x_1 be the symbol (normalized, i.e., $\mathbb{E}[|x_1|^2] = 1$) for user 1.
- x_2 be the symbol (normalized, i.e., $\mathbb{E}[|x_2|^2] = 1$) for user 2.

The basestation sends the following signal

$$x = \sqrt{P_1}x_1 + \sqrt{P_2}x_2$$

and the user $u \in \{1, 2\}$ receives

$$y_u = h_u x + w_u$$

with w_u a zero-mean unit-variance Gaussian noise ($\mathbb{E}[|w_u|^2] = 1$).

Decoder

- user 1:
  - decode user 2 by considering the signal of user 1 as noise,
  - remove the decoded user 2 from the received signal,
  - decode finally user 1.
- user 2: interference from user 1 viewed as noise.

Power-domain NOMA: result

Assuming $|h_1| > |h_2|$

$$\begin{aligned} R_1 &= \log_2(1 + |h_1|^2 P_1) \\ R_2 &= \log_2\left(1 + \frac{|h_2|^2 P_2}{1+|h_2|^2 P_1}\right) \end{aligned}$$

If we swap the decoders of users 1 and 2,

$$\begin{aligned} R_1 &= \log_2\left(1 + \frac{|h_2|^2 P_1}{1+|h_2|^2 P_2}\right) \\ R_2 &= \log_2(1 + |h_2|^2 P_2) \end{aligned}$$

Numerical applications: $P_1 = P_2 = 1$, $h_1 = 2$, $h_2 = 1$.
$R_{1,\perp} = (1/2)\log_2(1 + 2|h_1|^2 P_1)$ and $R_{2,\perp} = (1/2)\log_2(1 + 2|h_2|^2 P_2)$

- NOMA(1): $R_1 = 2.32$ and $R_2 = 0.58$
- NOMA(2): $R_1 = 0.58$ and $R_2 = 1$
- OMA: $R_1 = 1.58$ and $R_2 = 0.79$

Sketch of proof

- Let R_2' be the rate for error-free in its decoding algorithm

$$R_2' = \log_2(1 + \frac{|h_2|^2 P_2}{1 + |h_2|^2 P_1})$$

- Let R_2'' be the rate for error-free user 2 in SIC decoder of user 1

$$R_2'' = \log_2(1 + \frac{|h_1|^2 P_2}{1 + |h_1|^2 P_1})$$

- The rate for user 2 is $R_2 = \min(R_2', R_2'')$. As $h_1 \geq h_2$,

$$R_2 = R_2' = \log_2(1 + \frac{|h_2|^2 P_2}{1 + |h_2|^2 P_1})$$

- Then

$$R_1 = \log_2(1 + |h_1|^2 P_1)$$

Code-domain NOMA

Code-domain = based on signature (actually, CDMA is back!)

Plenty of solutions for next G:

- *Sparse Code Multiple Access (SCMA):* sparse spreading sequence (to avoid large collisions)
- *Non-orthogonal Code Multiple Access (NCMA):* CDMA with non-orthogonal codes
- *Resource Spread Multiple Access (RSMA):* codeword with low data rates and spread in time and frequency
- ...

Section 3 : Resource allocation algorithms

Section Outline

- Reminder on convex optimization
- Extension to non-convex optimization
- Some criteria for resource allocation (fairness dilemma)
- Problem 1: Waterfilling
- Problem 2: SINR-target based problem
- Problem 3: Sum-rate maximization with interference
- Problem 4: Joint power and scheduling optimization
- Problem 5: Energy efficiency optimization
- Problem 6: Power minimization with nonlinear interference

- **Exercise session:** Sum-throughput maximization
- **Lab:** power minimization in multi-cell context

Reminder on convex optimization

Optimization problem

$$\min_{\mathbf{x}} f(\mathbf{x})$$

s.t.

$$\forall \ell,\ g_\ell(\mathbf{x}) \leq 0$$

$$\forall \ell',\ h_{\ell'}(\mathbf{x}) = 0$$

with f and g_ℓ ($\forall \ell$) convex, and $h_{\ell'}$ ($\forall \ell'$) affine

Resolution tools:

- Mathematically : KKT conditions (seldom feasible)
- Numerically : algorithms such as gradient-descent, newton, interior-point method, etc

Reminder on non-convex optimization

Typically,

- we keep the convex constraints set
- but f is not convex anymore

Special case (seen in MICAS901)

If

$$f(\mathbf{x}) = f(\mathbf{x}_1, \cdots, \mathbf{x}_N)$$

with $\bullet \mapsto f(\cdots, \mathbf{x}_{k-1}, \bullet, \mathbf{x}_{k+1}, \cdots)$ strongly convex, then

- Use Block-Coordinate Descent (BCD) approach
- Convergence to a stationary point

but other strong assumption: constraint set is convex **separable**!

Counter-example: downlink (power constraint: $\sum_{k=1}^{N} P_k \leq P$)

Reminder (cont'd)

When no structure on the constraint set (no decoupling)

Successive Convex Approximation (SCA) (seen in MICAS901)

At each iteration i, solve

$$\mathbf{x}_{i+1}^* = \arg\min_{\mathbf{x}\in\mathcal{D}} \overline{f}_i(\mathbf{x}, \mathbf{x}_i^*)$$

with $\overline{f}_i$ an upper-bound approximating convex function of f

- $\overline{f}_i(\mathbf{x}_i^*, \mathbf{x}_i^*) = f(\mathbf{x}_i^*)$, $\nabla_{\mathbf{x}}\overline{f}_i(\mathbf{x}, \mathbf{x}_i^*)_{|\mathbf{x}=\mathbf{x}_i^*} = \nabla_{\mathbf{x}} f(\mathbf{x})_{|\mathbf{x}=\mathbf{x}_i^*}$,
- $\forall \mathbf{x} \in \mathcal{D}$, $f(\mathbf{x}) \leq \overline{f}_i(\mathbf{x}, \mathbf{x}_i^*)$.

Then SCA converges to a stationary point of f

Problem: how finding $\overline{f}_i$?

Special case: Difference of Convex (DoC) $\Rightarrow$ easy to exhibit $\overline{f}_i$

- $f(\mathbf{x}) = f_1(\mathbf{x}) - f_2(\mathbf{x})$
- $\overline{f}_i(\mathbf{x}, \mathbf{x}_i^*) = f_1(\mathbf{x}) - f_2(\mathbf{x}_i^*) - \nabla_{\mathbf{x}} f_2(\mathbf{x})_{|\mathbf{x}=\mathbf{x}_i^*}(\mathbf{x} - \mathbf{x}_i^*)$

Extension to other non-convex optimization

Nevertheless, there are some other special cases for non-convex optimization

Geometric Programming (GP): f and g_ℓ are posynomial, and $x_n \geq 0, \ \forall n$.

$$f(\mathbf{x}) = \sum_m \beta_m \prod_{n=1}^{N} (x_n)^{\alpha_{m,n}}$$

with $\alpha_{m,n} \in \mathbb{R}$ and $\beta_m \in \mathbb{R}_+$

- $g_\ell(\mathbf{x}) \leq 1$
- Change of variables $y_n = \log(x_n)$
- Work on $\log(f)$ and $\log(g_\ell)$
- New problem is convex

Example

$$f(\mathbf{x}) = x_1 x_2$$

- Not jointly convex:

$$\text{Hessian:} \quad \nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  It is not a positive matrix! ($[1, -1].(\nabla^2 f).[1, -1]^{\mathrm{T}} = -2$)

- but $j : \mathbf{y} \mapsto \log(f(e^{\mathbf{y}}))$ is **convex** since

$$\begin{aligned} j(\mathbf{y}) &= \log(e^{y_1} e^{y_2}) \\ &= y_1 + y_2 \end{aligned}$$

Extension to other non-convex optimization (cont'd)

- Fractional Programming (FP):

$$f(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

  with p a convex function and q a concave function
  - Dinkelbach algorithm
  - Converges to a stationary point
- Complementary Geometric Programming (CGP): f and g_ℓ ratio of posynomials
  - SCA and converges to a stationary point
- Signomial Programming (SP): as for CGP but $\beta_m \in \mathbb{R}$
  - SCA and converges to a stationary point
- Monotonic Programming (MP): $\mathbf{x}_1 \geq \mathbf{x}_2$ elementwise, then

$$f(\mathbf{x}_1) \geq f(\mathbf{x}_2)$$

  - Branch-Reduce Bound (BRB) algorithm
  - Converges to the optimal point
- ...

Principle for resource allocation

Play with

- power
- subcarrier assignement
- modulation and coding scheme
- ...

to improve the rate, the energy consumption, the latency, ...

In context of multi-user/multi-cell/... interference-disturbed communications, try to mitigate the interference degradation

Some functions

- R_k rate for user k
- SINR_k the SINR for user k
- P_k power for user k

Functions to be optimized

- (weighted) Sum rate: $\sum_k w_k R_k$
- Proportional fairness: $\sum_k \log(R_k)$
- Maxmin fairness: $\max \min_k R_k$
- Sum Energy Efficiency: $\sum_k \frac{R_k}{P_k + P_{\mathrm{circuitry}}}$
- Power minimization: $\sum_k P_k$

with typically

$$R_k = \log_2(1 + \mathrm{SINR}_k).$$

and

$$\mathrm{SINR}_k = \frac{P_k}{\sum_{m, m \neq k} \gamma_m P_m + P_w}$$

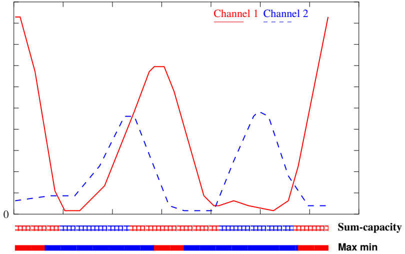
Example about fairness

- $H_k(n)$ channel response for user k on subcarrier n
- $P_k(n)$ power for user k on subcarrier n
- $a_k(n)$ assignement policy:
  - $a_k(n) = 1$ iff subcarrier n assigned to user k,
  - $a_k(n) = 0$ otherwise

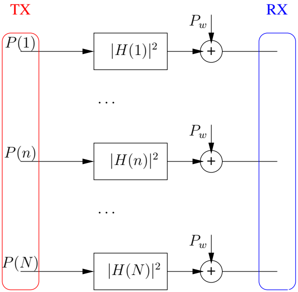
$$R_k = \sum_n a_k(n) \log_2(1 + |H_k(n)|^2 P_k(n))$$

$$\max_{\{a_k(n)\}} \sum_k R_k$$

$$\max_{\{a_k(n)\}} \min_k R_k$$



Problem 1: Waterfilling [1948]



- Sum rate maximization
- Power constraint:

$$\sum_{n=1}^{N} P(n) = P_{\max}$$

  with maximum power $P_{\max}$.
- Perfect CSIT
- $P_w = 1$

Problem: maximum capacity?

$$[P(1)^*, \cdots, P(N)^*] = \arg \max_{P(1),\cdots,P(N)} \sum_{n=1}^{N} \log_2(1 + |H(n)|^2 P(n))$$

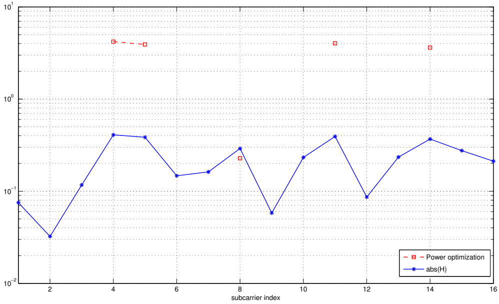
s.t. $P(n) \geq 0$, and $\sum_{n=1}^{N} P(n) \leq P_{\max}$.
Convex optimization problem ($\Rightarrow$ KKT conditions)

Problem 1: Result

$$P(n)^* = \left(\nu - \frac{1}{|H(n)|^2}\right)^+$$

with

- ν chosen s.t. $\sum_{n=1}^{N} P(n)^* = P_{\max}$.
- $(\bullet)^+ = \max(0, \bullet)$.



Problem 1: *sketch of proof*

Lagrangian function

$$\mathcal{L}(\mathbf{P}, \lambda, \mu_n) = -\sum_n \log_2(1+|H(n)|^2 P(n)) + \lambda(\sum_n P(n) - P_{\max}) - \sum_n \mu_n P(n)$$

KKT conditions

$$\begin{cases} -\frac{|H(n)|^2}{1+|H(n)|^2 P(n)} + \lambda - \mu_n = 0 \Leftrightarrow P(n) = \frac{1}{\lambda - \mu_n} - \frac{1}{|H(n)|^2}, \forall n \\ \lambda(\sum_n P(n) - P_{\max}) = 0 \\ \mu_n P(n) = 0, \ \forall n \end{cases}$$

- If $\mu_n \neq 0$, then $P(n) = 0$
- If $\mu_n = 0$, then $P(n) = \frac{1}{\lambda} - \frac{1}{|H(n)|^2}$ if this term is positive.

So

$$P(n) = \left(\frac{1}{\lambda} - \frac{1}{|H(n)|^2}\right)^+$$

Problem 2: Power allocation with target SINR [1992]

K TX and K RX with interference

$$y_k = h_{k,k}x_k + \sum_{n\neq k} h_{k,n}x_n + w_k$$

with $P_k = \mathbb{E}[|x_k|^2]$

Problem statment: find $\{P_n\}$ satisfying target SINR β_k for user k, $\forall k$

$$\frac{|h_{k,k}|^2 P_k}{\sum_{n\neq k} |h_{k,n}|^2 P_n + P_w} \geq \beta_k, \ \forall k$$

$$\Updownarrow$$

Problem statment: find $\mathbf{p} = [P_1, \cdots, P_K]^{\mathrm{T}}$ s.t.

$$(\mathbf{Id}_K - \mathbf{F})\mathbf{p} = \mathbf{b}$$

with $\mathbf{F} = \{\bar{\delta}_{k,n}|h_{k,n}|^2\beta_k/|h_{k,k}|^2\}_{k,n}$, $\mathbf{b} = P_w[\beta_1/|h_{1,1}|^2, \cdots, \beta_K/|h_{K,K}|^2]^{\mathrm{T}}$

Problem 2: cont'd

- $\mathbf{F}$ non negative matrix (NNM)
- $\mathbf{b}$ and $\mathbf{p}$ non-negative vectors

Result

If $\mathbf{F}$ primitive ($\exists m$ s.t. $\mathbf{F}^m > 0$), then we get

i) if the eigenvalue with the largest absolute value of $\mathbf{F}$ lies in $(0, 1)$

ii) then $(\mathbf{Id}_K - \mathbf{F})^{-1}$ exists and is strictly non-negative

iii) so $\mathbf{p}^\star = (\mathbf{Id}_K - \mathbf{F})^{-1}\mathbf{b}$ is the strictly non-negative solution of problem statment

- Problem is sometimes not feasible
- Can be overcome by orthogonalization or receiver improvement

Problem 2: *sketch of proof*

i) $\rightarrow$ ii) As the largest eigenvalue in absolute value is less than 1, $\ker\{\mathbf{Id}_K - \mathbf{F}\} = \emptyset$ and we get

$$(\mathbf{Id}_K - \mathbf{F})^{-1} = \sum_{\ell} F^{\ell} < +\infty$$

Consequently, $(\mathbf{Id}_K - \mathbf{F})^{-1}$ is strictly non-negative

ii) $\rightarrow$ iii) straightforward

Problem 2: example

We have

$$\begin{cases} \frac{|h_1|^2 P_1}{\gamma |h_2|^2 P_2 + P_w} = \beta \\ \frac{|h_2|^2 P_2}{\gamma |h_1|^2 P_1 + P_w} = \beta \end{cases}$$

which is equivalent to

$$\begin{bmatrix} |h_1|^2 & -\beta\gamma |h_2|^2 \\ -\beta\gamma |h_1|^2 & |h_2|^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \beta P_w \\ \beta P_w \end{bmatrix}$$

or

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{|h_1 h_2|^2 (1 - \beta^2 \gamma^2)} \begin{bmatrix} |h_2|^2 & \beta\gamma |h_2|^2 \\ \beta\gamma |h_1|^2 & |h_1|^2 \end{bmatrix} \begin{bmatrix} \beta P_w \\ \beta P_w \end{bmatrix}$$

Non-negative solution exists iff $\beta < 1/\gamma$

Numerical illustrations: $\beta = 1$, $h_1 = h_2 = 1$ and $P_w = 1\mu$W

- if $\gamma = 1/6$, then $P_1 = P_2 = 1.2\mu$W
- if $\gamma = 1/2$, then $P_1 = P_2 = 2\mu$W

Problem 3: Rate optim. with interference [2007]

Several problems with

$$R_k = \log_2(1 + \mathrm{SINR}_k)$$

where

$$\mathrm{SINR}_k = \frac{G_{k,k}P_k}{\sum_{m\neq k} G_{k,m}P_m + P_w}$$

Sum-rate	**Power**	**Maxmin**
$\max \sum_k R_k$	$\min \sum_k P_k$	$\max \min_k R_k$
s.t.	s.t.	s.t.
$\sum_k P_k \leq P_{\max}$	$R_k \geq R_k^t, \forall k$	$\sum_k P_k \leq P_{\max}$

Problem 3: Extension of Geometric Programming

Complementary Geometric Programming (CGP)

$$\min_{\mathbf{P}} \frac{p_0(\mathbf{P})}{q_0(\mathbf{P})} \quad \text{s.t.} \quad \frac{p_i(\mathbf{P})}{q_i(\mathbf{P})} \leq 1 \quad \forall i = 1, \cdots, K$$

where p_i and q_i are posynomial functions $\forall i = 0, \cdots, K$.

- CGP are nonconvex and become GP when q_i are monomials.
- SCA by replacing posynomial denominator with approximate monomial

Signomial Programming (SP)

$$\min_{\mathbf{P}} \frac{a_0(\mathbf{P}) - b_0(\mathbf{P})}{c_0(\mathbf{P}) - d_0(\mathbf{P})} \quad \text{s.t.} \quad \frac{a_i(\mathbf{P}) - b_i(\mathbf{P})}{c_i(\mathbf{P}) - d_i(\mathbf{P})} \leq 1 \quad \forall i = 1, \cdots, K$$

where a_i, b_i, c_i and d_i are posynomial functions $\forall i = 0, \cdots, K$.

- SP problems are nonconvex and can be converted into CGP

$$\min_{\mathbf{P},t} t \quad \text{s.t.} \quad \frac{a_i(\mathbf{x}) - b_i(\mathbf{x})}{c_i(\mathbf{x}) - d_i(\mathbf{x})} \leq t^{\delta_{0,i}} \to \frac{a_i(\mathbf{x}) + t^{\delta_{0,i}} d_i(\mathbf{x})}{b_i(\mathbf{x}) + t^{\delta_{0,i}} c_i(\mathbf{x})} \leq 1.$$

Problem 3: power minimization

$$\min_{\mathbf{P}} \sum_k P_k \text{ s.t. } R_k \geq R_k^t, \forall k$$

Non convex optimization due to the constraint

Rewrite the constraint

$$R_k \geq R_k^t \Leftrightarrow \log_2\left(1 + \frac{G_k P_k}{\sum_{m\neq k} G_m P_m + P_w}\right) \geq R_k^t$$

$$\Leftrightarrow \sum_{m\neq k} G_m G_k^{-1} P_m P_k^{-1} + G_k^{-1} P_w P_k^{-1} \leq \frac{1}{2^{R_k^t} - 1}$$

Geometric programming

Remark: SINR_k^{-1} is a posynomial

Problem 3: maxmin

$$\max_{\mathbf{P}} \min_k R_k \text{ s.t. } \sum_k P_k \leq P_{\max}$$

Non convex optimization due to the cost function

$$\max_{\mathbf{P}} \min_k R_k \Leftrightarrow \max_{\mathbf{P}} \min_k \mathrm{SINR}_k$$

Change of variables

$$\max_{\mathbf{P},t} t$$

s.t.

$$\mathrm{SINR}_k \geq t, \forall k$$

$$\sum_k P_k \leq P_{\max}$$

Non convex optimization due to the first constraints

Solution:

- use the same trick as in previous slide (and do $\min_{\mathbf{P},t} t^{-1}$)
- then **Geometric Programming**

Problem 3: rate maximization (high SINR case)

$$\max_{\mathbf{P}} \sum_k R_k \text{ s.t. } \sum_k P_k \leq P_{\max}$$

$$\Updownarrow$$

$$\min_{\mathbf{P}} \prod_k (1+\mathrm{SINR}_k)^{-1} \text{ s.t. } \sum_k P_k \leq P_{\max}$$

Main remarks:

- $(1+\mathrm{SINR}_k)^{-1}$ is not a posynomial!
- but SINR_k^{-1} is a posynomial!

At high SINR:

- $(1+\mathrm{SINR}_k)^{-1} \approx \mathrm{SINR}_k^{-1}$
- then **Geometric Programming**

Problem 3: general case

Go back to the original problem

$$\max_{\mathbf{P}} \sum_k \log_2 \left(1 + \mathrm{SINR}_k\right) \text{ s.t. } \sum_k P_k \leq P_{\max}$$

- no specific structure (especially since coupling constraint)
- resorting to SCA
  - basic SCA
  - difference of convex
  - CGP

Problem 3: general case (basic SCA)

- Typically we can approximate by lower-bounding the function in such a way

$$\log_2(1+\mathrm{SINR}_k(\mathbf{P})) \geq \alpha_{i,k} + \beta_{i,k}\log_2(\mathrm{SINR}_k(\mathbf{P}))$$

with $\alpha_{i,k}$ and $\beta_{i,k}$ well chosen to satisfy SCA properties at the current iteration i around the powers $\mathbf{P}^{(i)}$

- then works as in high SNR case since

$$\max \sum_k \beta_{i,k}\log_2(\mathrm{SINR}_k(\mathbf{P})) \Leftrightarrow \max \prod_k \mathrm{SINR}_k(\mathbf{P})^{\beta_{i,k}}.$$

So

$$\max \prod_k t_k \quad \text{s.t.} \quad \mathrm{SINR}_k(\mathbf{P})^{\beta_{i,k}} \geq t_k$$

then

$$\min \prod_k t_k^{-1} \quad \text{s.t.} \quad t_k^{\frac{1}{\beta_{i,k}}}\mathrm{SINR}_k(\mathbf{P})^{-1} \leq 1$$

Problem 3: general case (DoC)

$$\begin{aligned}
\sum_k R_k &= \sum_k \log_2\left(1 + \frac{G_k P_k}{\sum_{m\neq k} G_m P_m + P_w}\right) \\
&= \sum_k \log_2\left(\frac{\sum_m G_m P_m + P_w}{\sum_{m\neq k} G_m P_m + P_w}\right) \\
&= \underbrace{\sum_k \log_2(\sum_m G_m P_m + P_w)}_{\text{concave}} - \underbrace{\sum_k \log_2(\sum_{m\neq k} G_m P_m + P_w)}_{\text{concave}}
\end{aligned}$$

SCA applied with DoC
but poor approximation

Problem 3: general case (CGP) - 1

Another systematic way to apply SCA (then with GP): we remind

$$\max_{\mathbf{P}} \prod_k \left(1 + \frac{G_k P_k}{\sum_{m \neq k} G_m P_m + P_w}\right) \text{ s.t. } \sum_k P_k \leq P_{\max}$$

$$\Updownarrow$$

$$\min_{\mathbf{P}} \prod_k \left(\frac{\sum_{m \neq k} G_m P_m + P_w}{\sum_m G_m P_m + P_w}\right) \text{ s.t. } \sum_k P_k \leq P_{\max}$$

$$\Updownarrow$$

$$\min_{\mathbf{P}} \frac{\prod_k (\sum_{m \neq k} G_m P_m + P_w)}{\prod_k (\sum_m G_m P_m + P_w)} \text{ s.t. } \sum_k P_k \leq P_{\max}$$

Main remarks:

- Ratio of posynomial (CGP)
- Easy to solve if denominator was a monomial!

Problem 3: general case - 2

How transforming a sum of monomials into a monomial: that's the question!
Let $Q_m(\mathbf{P}) = \beta_m \prod_{n=1}^{N} P_n^{\alpha_{m,n}}$ be a monomial

Result

$$Q(\mathbf{P}) := \sum_m Q_m(\mathbf{P}) \geq \tilde{Q}(\mathbf{P}) := \prod_m \left(\frac{Q_m(\mathbf{P})}{\delta_m}\right)^{\delta_m}$$

In addition, if $\delta_m = Q_m(\mathbf{P}_0)/Q(\mathbf{P}_0)$, then

- $Q(\mathbf{P}_0) = \tilde{Q}(\mathbf{P}_0)$,
- and

$$\frac{\partial Q}{\partial \mathbf{P}}_{|\mathbf{P}=\mathbf{P}_0} = \frac{\partial \tilde{Q}}{\partial \mathbf{P}}_{|\mathbf{P}=\mathbf{P}_0}$$

Problem 3: *sketch of proof*

- Comparison between arithmetic mean and geometric mean:

$$\sum_m \delta_m x_m \geq \prod_m x_m^{\delta_m}$$

with $\delta_m \geq 0$ and $\sum_m \delta_m = 1$.

- Consider $x_m = Q_m(\mathbf{P})/\delta_m$ and $\delta_m = Q_m(\mathbf{P}_0)/Q(\mathbf{P}_0)$

- $\tilde{Q}(\mathbf{P}_0) = \prod_m (Q(\mathbf{P}_0))^{Q_m(\mathbf{P}_0)/Q(\mathbf{P}_0)} = Q(\mathbf{P}_0)^{\sum_m Q_m(\mathbf{P}_0)/Q(\mathbf{P}_0)} = Q(\mathbf{P}_0)$
- $\frac{\partial \log Q}{\partial \mathbf{P}}_{|\mathbf{P}=\mathbf{P}_0} = \frac{\sum_m \partial Q_m/\partial \mathbf{P}_{|\mathbf{P}=\mathbf{P}_0}}{\sum_m Q_m(\mathbf{P}_0)}$
- $\log \tilde{Q}(\mathbf{P}) = \sum_m \delta_m \log_2(Q_m(\mathbf{P})/\delta_m)$
  $\frac{\partial \log \tilde{Q}}{\partial \mathbf{P}}_{|\mathbf{P}=\mathbf{P}_0} = \sum_m \delta_m \frac{\partial Q_m/\partial \mathbf{P}_{|\mathbf{P}=\mathbf{P}_0}}{Q_m(\mathbf{P}_0)} = \sum_m \frac{\partial Q_m/\partial \mathbf{P}_{|\mathbf{P}=\mathbf{P}_0}}{\sum_m Q_m(\mathbf{P}_0)}$

Problem 3: general case - 3

Original problem around $\mathbf{P}_0$ can be replaced with

$$\min_{\mathbf{P}} \underbrace{\frac{\prod_k (\sum_{m \neq k} G_m P_m + P_w)}{\prod_k (\prod_m (G_m P_m)^{\delta_m} P_w^{\delta_0})}}_{\text{posynomial}} \text{ s.t. } \sum_k P_k \leq P_{\max}$$

with

- $\delta_m = G_m P_m(0)/(\sum_m G_m P_m(0) + P_w), \forall m > 0$, and
- $\delta_0 = P_w/(\sum_m G_m P_m(0) + P_w)$.

Function to be optimized

- upper-bounded original function
- same value in $\mathbf{P}_0$
- same derivative function in $\mathbf{P}_0$

SCA properties are satisfied!
One GP per iteration!

Problem 4: Power and scheduling optim. [2013]

- Two cells
- OFDMA in each cell with N subcarriers and K users ($K \leq N$)
- Uplink context

At base station 1,

$$R_1 = \sum_{k=1}^{K} \sum_{n=1}^{N} \log_2 \left(1 + \frac{a_k^1(n) G_k^1(n) P_k^1(n)}{\sum_{k=1}^{K} a_k^2(n) G_k^{2\rightarrow 1}(n) P_k^2(n) + P_w} \right)$$

where

- $a_k^j(n) \in \{0, 1\}$: assignement for user k in subcarrier n for cell j
- $P_k^j(n)$: power used by user k in subcarrier n for cell j
- $G_k^j(n)$: channel gain between from user k in cell j to BTS j at subcarrier n
- $G_k^{i\rightarrow j}(n)$: channel gain between from user k in cell i to BTS j at subcarrier n

Problem 4: Optimization problem

$$\max_{\{a_k^1(n),a_k^2(n),P_k^1(n),P_k^2(n)\}_{k,n}} R_1 + R_2$$

s.t.

$$\begin{aligned}
R_1 &\geq R_1^t \\
R_2 &\geq R_2^t \\
a_k^j(n)P_k^j(n) &\leq P_{\max}, \ \forall j \\
\sum_{k=1}^{K} a_k^j(n) &= 1, \ \forall j, n \\
P_k^j(n) &\geq 0, \ \forall j, k, n \\
a_k^j(n) &\in \{0, 1\}, \ \forall j, k, n
\end{aligned}$$

Mixed Integer Non Convex Programming (MINCP)

Problem 4: Trick 1

Iterate between basestation allocations:

- fix $\{a_k^2(n), P_k^2(n)\}$,
- optimize $\{a_k^1(n), P_k^1(n)\}$, and so on

$$\max_{\{a_k^1(n), P_k^1(n)\}_{k,n}} R_1 + R_2$$

s.t.

$$\begin{aligned} R_1 &\geq R_1^t \\ R_2 &\geq R_2^t \\ a_k^1(n) P_k^1(n) &\leq P_{\max} \\ \sum_{k=1}^{K} a_k^1(n) &= 1,\ \forall n \\ P_k^1(n) &\geq 0,\ \forall k, n \\ a_k^1(n) &\in \{0, 1\},\ \forall k, n \end{aligned}$$

Still too complex (except at high SINR by removing $1+$: DoC)

Problem 4: Trick 2

Remove denominator (related to interference from cell 1 to 2) but add maximum interference level (across the entire spectrum) on BTS

$$\max_{\{a_k^1(n),P_k^1(n)\}_{k,n}} R_1 \Leftrightarrow \max_{\{a_k^1(n),P_k^1(n)\}_{k,n}} \sum_{k=1}^{K}\sum_{n=1}^{N} \log_2\left(1+\frac{a_k^1(n)G_k^1(n)P_k^1(n)}{P_c}\right)$$

with $P_c = \sum_{k=1}^{K} a_k^2(n)G_k^{2\to 1}(n)P_k^2(n) + P_w$, s.t.

$$\begin{aligned}
\sum_{k=1}^{K}\sum_{n=1}^{N} a_k^1(n)G_k^{1\to 2}(n)P_k^1(n) &\leq I_{1\to 2}\\
a_k^1(n)P_k^1(n) &\leq P_{\max}\\
\sum_{k=1}^{K} a_k^1(n) &= 1,\ \forall n\\
P_k^1(n) &\geq 0,\ \forall k,n\\
a_k^1(n) &\in \{0,1\},\ \forall k,n
\end{aligned}$$

Still non convex (as *a* binary and product *aP*)

Problem 4: Tricks 3 & 4

- Perspective function: let f be a concave function, then

$$(x, y) \mapsto x.f\left(\frac{y}{x}\right)$$

is still a joint concave function.
Application: if $a \in \{0, 1\}$, then

$$\log_2(1 + aP) = a\log_2(1 + P)$$

Do the change of variable $(a, P) \to (a, Q := aP)$, then

$$\log_2(1 + aP) = a\log_2(1 + P) = a\log_2\left(1 + \frac{Q}{a}\right)$$

Consequently

$$(a, Q) \mapsto a\log_2\left(1 + \frac{Q}{a}\right)$$

is jointly concave !

- Relaxation: $a_k^1(n) \in \{0, 1\} \Rightarrow a_k^1(n) \in [0, 1]$

Problem 4: Final optimization problem

$$\max_{\{a_k^1(n), Q_k^1(n)\}_{k,n}} \sum_{k=1}^{K} \sum_{n=1}^{N} a_k^1(n) \log_2 \left(1 + \frac{G_k^1(n) Q_k^1(n)}{a_k^1(n) P_c}\right)$$

s.t.

$$\begin{aligned}
\sum_{k=1}^{K} \sum_{n=1}^{N} G_k^{1\to 2}(n) Q_k^1(n) &\leq I_{1\to 2} \\
Q_k^1(n) &\leq P_{\max} \\
\sum_{k=1}^{K} a_k^1(n) &= 1, \ \forall n \\
Q_k^1(n) &\geq 0, \ \forall k, n \\
a_k^1(n) &\in [0, 1] \ \forall k, n
\end{aligned}$$

Convex optimization problem

Problem 5: Energy efficiency [2015]

- K users
- Downlink communications
- FDMA (with equal bandwidth for each user)

$$\max_{\{P_k\}} \frac{\sum_{k=1}^{K} \overbrace{\log_2(1+G_k P_k)}^{R_k}}{\sum_{k=1}^{K} P_k + P_{\text{circuitry}}}$$

s.t.

$$\begin{aligned} \sum_{k=1}^{K} P_k &\leq P_{\max} \\ P_k &\geq 0, \ \forall k \end{aligned}$$

Remarks:

- Warning: the power constraint is not necessary saturated.
- Ratio between concave function and convex function
- Linear constraints

$\Rightarrow$ Resorting to **Fractional programming (FP)**

Problem 5: Review on Fractional Programming - 1

$$\max_{\mathbf{x}\in\mathcal{C}} \frac{f(\mathbf{x})}{g(\mathbf{x})}$$

with concave function f, and positive convex function g.

- Let q^* be the maximum value (assumed non-negative).
- Let $\mathbf{x}^*$ be the argmax value.

Lemma 1

$\mathbf{x}^*$ is achieved iff

$$\max_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}) - q^*.g(\mathbf{x}) = 0$$

Consequence:

- If q^* is known in advance, just solve

$$\max_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}) - q^*.g(\mathbf{x})$$

- As f concave and g convex, $f - q^*.g$ is concave
  Convex optimization

Problem 5: *Sketch of proof*

- Let $\mathbf{x}^*$ be the optimal solution of RHS of Lemma 1. It means

$$f(\mathbf{x}^*) - q^* g(\mathbf{x}^*) = 0 \Rightarrow \frac{f(\mathbf{x}^*)}{g(\mathbf{x}^*)} = q^*$$

  Moreover $\forall \mathbf{x} \neq \mathbf{x}^*$,

$$f(\mathbf{x}) - q^* g(\mathbf{x}) \leq 0 \Rightarrow \frac{f(\mathbf{x})}{g(\mathbf{x})} \leq q^*$$

  which proves that $\mathbf{x}^*$ is the optimal solution of FP.
- Let $\mathbf{x}^*$ be the optimal solution of FP. As $q^* = f(\mathbf{x}^*)/g(\mathbf{x}^*)$, we get

$$\frac{f(\mathbf{x})}{g(\mathbf{x})} < q^* \Rightarrow f(\mathbf{x}) - q^* g(\mathbf{x}) \leq 0$$

  for any $\mathbf{x} \neq \mathbf{x}^*$.

Problem 5: Review on Fractional Programming - 2

Lemma 2

Let

$$F(q) := \max_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}) - q.g(\mathbf{x})$$

is a strictly decreasing function in $q \in \mathbb{R}_+$

Consequence:

- $q \mapsto F(q)$ is continuous
- $\lim_{q\to 0} F(q) = \max_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}) > 0$ (as q^* is non-negative)
- $\lim_{q\to\infty} F(q) = -\infty$ (as g is non-negative)
- q^* is the unique root of F
- Any root-finding algorithm works!
- but each computation of F requires a convex optimization

Problem 5: *Sketch of proof*

- Assume $q_1 > q_2$
- $\mathbf{x}_1^*$ the argmax with q_1, and $\mathbf{x}_2^*$ the argmax with q_2

$$
\begin{aligned}
F(q_1) = f(\mathbf{x}_1^*) - q_1 g(\mathbf{x}_1^*) &\overset{(a)}{<} f(\mathbf{x}_1^*) - q_2 g(\mathbf{x}_1^*) \\
&\overset{(b)}{<} f(\mathbf{x}_2^*) - q_2 g(\mathbf{x}_2^*) = F(q_2)
\end{aligned}
$$

(a) $q_1 > q_2$ and g is a positive function. Strict inequality.

(b) $\mathbf{x}_2^*$ is the argmax for q_2

Problem 5: Practical algorithm

Dinkelbach algorithm [1967]

Start with $q_0 = 0$, select an arbitrary small ε.
Iterate over n

1. Given q_n, find
$$\mathbf{x}_n^* = \arg\max_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}) - q_n.g(\mathbf{x})$$
2. Then
$$q_{n+1} = \frac{f(\mathbf{x}_n^*)}{g(\mathbf{x}_n^*)}$$
3. Stop when $F(q_{n+1}) < \varepsilon$

Result: this algorithm converges to $(q^*, \mathbf{x}^*)$ up to ε.

Problem 5: *Sketch of proof*

Step 1: sequence $\{q_n\}_n$ is strictly increasing.

- Assuming $F(q_n) = f(\mathbf{x}_n^*) - q_n g(\mathbf{x}_n^*) > 0$ (True for $F(q_0)$)
- $f(\mathbf{x}_n^*) - q_{n+1} g(\mathbf{x}_n^*) = 0$

$$q_{n+1} - q_n = \frac{F(q_n)}{g(\mathbf{x}_n^*)} > \frac{F(q_n)}{g_{\max}} > 0 \qquad (1)$$

with $g_{\max} = \max_{\mathbf{x} \in \mathcal{C}} g(\mathbf{x})$ (it exists if $\mathcal{C}$ compact)

Step 2: convergence to q^*

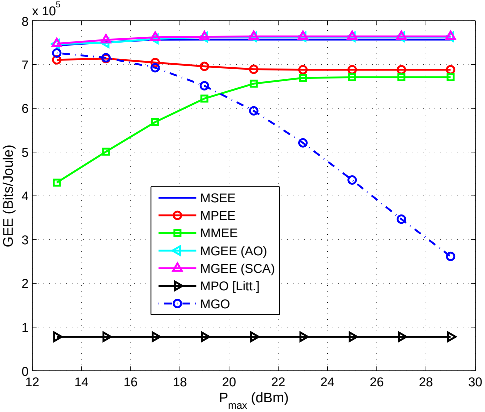
- Due to stopping criterion, bounded increasing sequence, and so $\lim_{n\to\infty} q_n = \overline{q}$
- Assuming that $\lim_{n\to\infty} F(q_n) = F(\overline{q}) > \varepsilon$, i.e, $\overline{q} < q^* - \delta$ with $F(q^* - \delta) = \varepsilon$.
- but as q_n converges, $(q_{n+1} - q_n)$ converges to 0, and Eq. (1) implies

$$F(q_n) \to 0 \Rightarrow q_n \to q^*$$

which leads to a contradiction.

Problem 5: Numerical illustrations

- Here R_k: throughput with HARQ and practical modulation and coding scheme [2018]
- Global Energy Efficiency (GEE) versus $P_{\max}$



Problem 6: nonlinear interference [2021]

- OFDMA-based return link between terrestrial distributed antennas and satellite (gain G_k for user k)
- Then gateway between satellite and basestation
- Assumption: nonlinear amplifier on satellite board.

$$y_c(t) = \gamma_1 x_c(t) + \gamma_3 x_c(t) x_c(t) \overline{x_c(t)} + w_c(t)$$

For sample n of user/band k, we get

$$z_k(n) = z_k^{\mathrm{L}}(n) + z_k^{\mathrm{NL}}(n) + w_k(n)$$

Let

- $\mathcal{P}_{\mathrm{L}}(k) = \mathbb{E}[|z_k^{\mathrm{L}}|^2]$ be the auto-correlation of the linear part,
- $\mathcal{P}_{\mathrm{NL}}(k) = \mathbb{E}[|z_k^{\mathrm{NL}}|^2]$ be the auto-correlation of the nonlinear part,
- $\mathcal{P}_{\mathrm{LNL}}(k) = \mathbb{E}[z_k^{\mathrm{L}} \overline{z_k^{\mathrm{NL}}}]$ be the cross-correlation between the linear and nonlinear parts.

Problem 6: capacity expressions - 1

Assuming optimal decoder and Gaussian codebooks

$$C(k) = \log_2\left(1 + Q(k)\right)$$

with

$$Q(k) = \frac{\mathcal{P}_{\mathrm{L}}^2(k) + 2\mathcal{P}_{\mathrm{L}}(k)\Re\{\mathcal{P}_{\mathrm{LNL}}(k)\} + |\mathcal{P}_{\mathrm{LNL}}(k)|^2}{\mathcal{P}_{\mathrm{L}}(k)\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{L}}(k)\mathcal{P}_{\mathrm{W}} - |\mathcal{P}_{\mathrm{LNL}}(k)|^2}$$

Assuming nonlinear part as noise and Gaussian codebooks

$$\underline{C}(k) = \log_2\left(1 + \underline{Q}(k)\right)$$

with

$$\underline{Q}(k) = \frac{\mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}}$$

Remark: if $z_k^{\mathrm{NL}}(n) = 0$, then

$$Q(k) = \underline{Q}(k) = \frac{\mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{W}}}$$

Problem 6: capacity expressions - 2

$$\begin{aligned}
\mathcal{P}_{\mathrm{L}}(k) &= \gamma_1^2 G_k P_k, \\
\mathcal{P}_{\mathrm{NL}}(k) &= 4\gamma_3^2 \alpha^{(1)} G_k P_k \sum_{k',k''} G_{k'} G_{k''} P_{k'} P_{k''} \\
&+ 2\gamma_3^2 \alpha^{(2)} \sum_{k_1,k_2,k_3 | k=k_1+k_2-k_3} G_{k_1} G_{k_2} G_{k_3} P_{k_1} P_{k_2} P_{k_3} \\
&+ 4\gamma_3^2 \beta^{(1)} (\delta^c_{k,1} G_{k-1} P_{k-1} + \delta^c_{k,K} G_{k+1} P_{k+1}) \sum_{k',k''=1}^{K} G_{k'} G_{k''} P_{k'} P_{k''} \\
&+ 2\gamma_3^2 \beta^{(2)} \sum_{k_1,k_2,k_3 | k=k_1+k_2-k_3 \pm 1} G_{k_1} G_{k_2} G_{k_3} P_{k_1} P_{k_2} P_{k_3} \\
\mathcal{P}_{\mathrm{LNL}}(k) &= 2\gamma_1 \gamma_3 \lambda G_k P_k \sum_{k'} G_{k'} P_{k'}
\end{aligned}$$

- All $\alpha^{(1)}$, $\alpha^{(2)}$, $\beta^{(1)}$, $\beta^{(2)}$, λ are positive.
- All terms are posynomials

Problem 6: power minimization (with $\underline{C}$)

$$\min_{\mathbf{P}} \sum_{k=1}^{K} P_k \quad \text{s.t.} \quad \log_2\left(1+\frac{\mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{NL}}(k)+\mathcal{P}_{\mathrm{W}}}\right) \geq R_k \quad \forall k$$

which is equivalent to

$$\min_{\mathbf{P}} \sum_{k=1}^{K} P_k$$

s.t.

$$\mathcal{P}_{\mathrm{L}}(k)^{-1}\left(\mathcal{P}_{\mathrm{NL}}(k)+\mathcal{P}_{\mathrm{W}}\right) \leq \frac{1}{2^{R_k}-1} \quad \forall k=1,\ldots,K$$

Last problem is GP

Problem 6: maxmin data rate (with $\underline{C}$)

$$\max_{\mathbf{P}} \min_{k} \frac{\mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}}$$

which is equivalent to

$$\max_{\mathbf{P},t} t$$

s.t.

$$\frac{\mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}} \geq t \quad \forall k$$

$$\min_{\mathbf{P},t} t^{-1}$$

s.t.

$$t\mathcal{P}_{\mathrm{L}}(k)^{-1}\left(\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}\right) \leq 1 \quad \forall k$$

Last problem is GP

Problem 6: sum-rate (with $\underline{C}$)

$$\max_{\mathbf{P}} \sum_{k=1}^{K} \log_2\left(1 + \frac{\mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}}\right) = \max_{\mathbf{P}} \prod_{k=1}^{K} \frac{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}} + \mathcal{P}_{\mathrm{L}}(k)}{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}}$$

$$= \min_{\mathbf{P}} \prod_{k=1}^{K} \frac{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}}}{\mathcal{P}_{\mathrm{NL}}(k) + \mathcal{P}_{\mathrm{W}} + \mathcal{P}_{\mathrm{L}}(k)}$$

$$= \min \frac{\text{posynomial}}{\text{posynomial}}$$

- Apply GP, then convex/convex: not a good shape
- Apply results of general problem related to Problem 3

Problem 6: sum-rate (with C)

Due to sign $-$ in $Q(k)$, we have

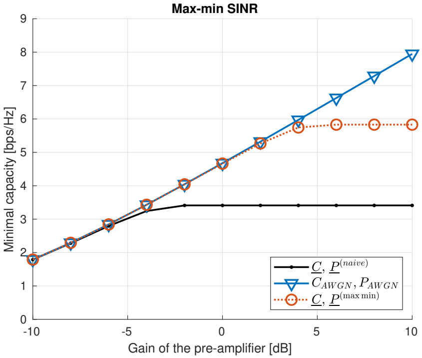
$$\min \frac{\text{signomial}}{\text{signomial}}$$

under ratio of signomials.

- Solution: Signomial Programming

Problem 6: Numerical illustrations

- $K = 6$ users
- Rainy weather (G_k strongly different between users)
- $P_{\max} = 50$W (47dBm)
- $\gamma_3 = 0.05$



Conclusion

- Multi-user communications are a crucial issue
- We omit to discuss about
  - no CSIT available for doing resource allocation
  - numerous other problems : actually one problem per configuration
  - distributed optimization (partial knowledge of functions per node)
  - some mathematical techniques: game theory, deep learning, ...

Another direction: game theory

Exemple : uplink or multi-cell interference γ with $(P_1, P_2) \in [0, P_{\max}]^2$

$$\left\{ \begin{array}{lll} y_1 = h_1 x_1 + \gamma h_2 x_2 + w_1 & \Rightarrow & R_1 = \log_2\left(1 + \frac{|h_1|^2 P_1}{\gamma^2 |h_2|^2 P_2 + P_w}\right) \\ y_2 = \gamma h_1 x_1 + h_2 x_2 + w_2 & \Rightarrow & R_2 = \log_2\left(1 + \frac{|h_2|^2 P_2}{\gamma^2 |h_1|^2 P_1 + P_w}\right) \end{array} \right.$$

"Social" optimization
maximization of $R = R_1 + R_2$
If $|h_2| > |h_1|$, $P_1^* = 0$ et $P_2^* = P_{\max}$

Individual optimization
game theory with (R_1, R_2)
Nash eq. if $P_1^* = P_2^* = P_{\max}$

Numerical evaluations: $\gamma^2 = 0.8$, $P_w = 1$, $P_{\max} = 1$, $h_1 = 1$, $h_2 = 2$

- Centralized: $R_1^* = 0$, $R_2^* = 2.32$, and $R^* = 2.32$
- Game theory: $R_1^* = 0.3$, $R_2^* = 1.68$, and $R^* = 1.98$

References

T. Cover, “Elements of Information Theory”, 1991

S. Verdú, “Multi-user detection”, 1998

S. Kaiser, K. Fazel, “Multicarrier and Spread spectrum systems”, 2003

H. Homa, A. Toskala, “LTE for UMTS, OFDMA and SC-FDMA based radio access”, 2009

D. Tse, “Fundamentals of wireless communications”, 2005

M. Chiang, D. Palomar, et al., “Power control by GP”, 2007

S. Boyd et al., “A tutorial on GP”, 2007

S. Stanczak et al., “Fundamentals fo resource allocation in wireless networks; theory and algorithms”, 2009

A. Abdelnasser, E. Hossain, “Joint subchannel and power allocation in two-tier OFDMA HetNet with clustered femtocells”, 2013

W. Dinkelbach, “On nonlinear fractional programming”, 1967

A. Louchart et al., “Power allocation in Uplink Multiband Satellite System with Nonlinearity-Aware Receiver”, 2021