MICAS904-ROSP/DCOM1

Introduction to Communications Theory

Part on ”Inter-Symbol Interference mitigation”

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Outline

- Section 1: Inter-Symbol interference model
- Section 2: Optimal receiver
  - Refresher on detection theory
  - Viterbi algorithm
- Section 3: Suboptimal receivers
  - ZF (with or without CSIT)
  - MMSE
  - DFE
- Section 4: OFDM

Section 1 : Inter-Symbol interference model

Model on one polarization

At the Transmitter side

$$x(t) = \sum_k s_k g(t - kT_s)$$

with

- s_k symbols from any QAM/PSK/PAM/OOK constellation
- T_s symbol rate
- $g(t)$ square-root Nyquist shaping filter
  - does not lead to Inter-Symbol Interference
  - employs in 3G (with small roll-off)
  - employs in WDM Nyquist based system (with very small roll-off)

On the channel

$$y(t) = c(t) \star x(t) + w(t)$$

with

- $c(t)$ channel impulse response (multipath wireless, CD, PMD)
- $w(t)$ white Gaussian noise
- Nonlinear effect neglected (optical fiber, non-linear amplifier)

Model on one polarization (cont'd)

At the Receiver side

$$y(t) = \sum_k s_k h(t - kT_s) + w(t)$$

with $h(t) = c(t) \star g(t)$

- Implement the matched filter (to either $h(t)$ or $g(t)$)
- Sample at symbol rate T_s

Final model

$$\begin{aligned} z(n) &= \overline{h(-t)} \star y(t)_{|t=nT_s} \\ &= \sum_\ell \tilde{h}_\ell s_{n-\ell} + \tilde{w}(n) \end{aligned}$$

with $\tilde{h}_\ell = \overline{h(-t)} \star h(t)_{|t=\ell T_s}$ and $\tilde{w}(n) = \overline{h(-t)} \star w(t)_{|t=nT_s}$

General model

As $\tilde{h}_\ell \neq \delta_{0,\ell}$,

- Inter-Symbol Interference
- Colored noise $\Rightarrow$ whitening filter $y(n) = f \star z(n)$

Goal: decode the data from $n = 0, \cdots, N$

$y(n) = \sum_{\ell=0}^{L} h_\ell s_{n-\ell} + w(n)$	$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{w}$
with $w(n)$ white Gaussian	with $\mathbf{y} = [y(0), \ldots, y(N)]^{\mathrm{T}}$ and $\mathbf{s} = [\underbrace{s_{-L}, \cdots, s_{-1}}_{known}, s_0, \ldots, s_N]^{\mathrm{T}}$

Matrix model also valid for wireless MIMO, PolMux, multimode, multiple access (<u>but</u> with different matrix structures)

Section 2 : Optimal receiver

Refresher on detection theory

Main result

If data are equilikely, then the optimal receiver (minimizing the error probability) is the Maximum Likelihood (ML)

$$\hat{\mathbf{s}}_{\text{ML}} = \arg\max_{\mathbf{s}} p(\mathbf{y}|\mathbf{s})$$

Application to linear model

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{w}$$

with

- square matrix of size N_H (for sake of simplicity)
- $\mathbf{w}$ white Gaussian noise

ML = Least Square (LS)

$$\hat{\mathbf{s}}_{\text{ML}} = \arg\min_{\mathbf{s}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2$$

Optimal detector

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg\min_{\mathbf{s}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2$$

Simple optimization without constraint on $\mathbf{s}$

Optimization problem consists in finding out the minimum of a (positive) quadratic form under (nonconvex) conditions

- Exhaustive search
  - complexity in $\mathcal{O}(M^{N_H})$ with M-QAM
  - ex.: 4×4 MIMO with 64-QAM leads to 16M $\mathbf{s}$
- Special simple cases: $N_H = 1$ or unitary matrix
- Tree approach
  - suitable for wireless MIMO with small $N_H \Rightarrow$ Sphere decoding
- Using the structure of $\mathbf{H}$
  - suitable for ISI since $\mathbf{H}$ Toeplitz $\Rightarrow$ Viterbi algorithm
- Suboptimal detectors
  - Remove constraint and threshold ($\hat{\mathbf{s}} = \mathrm{threshold}(\mathbf{H}^{-1}\mathbf{y})$) $\Rightarrow$ ZF
  - MMSE, DFE

Optimal detector

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg \min_{\mathbf{s} \in \mathcal{C}^{N_H}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2$$

Hard optimization due to constraint on **s**

Optimization problem consists in finding out the minimum of a (positive) quadratic form under (nonconvex) conditions

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  - complexity in $\mathcal{O}(M^{N_H})$ with M-QAM
  - ex.: 4×4 MIMO with 64-QAM leads to 16M **s**
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  - MMSE, DFE

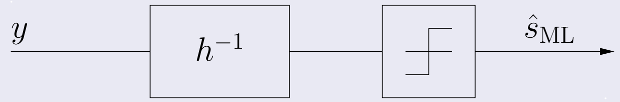
Special case: $N_H = 1$

We consider scalar signal

$$y = hs + w$$

We have

$$\begin{aligned}\hat{s}_{\mathrm{ML}} &= \arg\min_{s\in\mathcal{C}} |y - hs| \\ &= \arg\min_{s\in\mathcal{C}} |h|.|h^{-1}y - s| \\ &= \arg\min_{s\in\mathcal{C}} |h^{-1}y - s| \\ &= \mathrm{threshold}(h^{-1}y)\end{aligned}$$



- ZF=ML
- Easy to implement

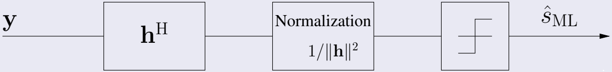
Special case: SIMO

We consider a multivariate signal (typically L antennas)

$$y(\ell) = h_\ell s + w(\ell) \text{ with } \ell = 1, \cdots, L \Longleftrightarrow \mathbf{y} = \mathbf{h}s + \mathbf{w}$$

with $\mathbf{h}$ a $L \times 1$ column-vector
We have

$$\begin{aligned} \hat{s}_{\mathrm{ML}} &= \arg\min_{s \in \mathcal{C}} \|\mathbf{y} - \mathbf{h}s\|^2 \\ &= \arg\min_{s \in \mathcal{C}} \|\mathbf{y}\|^2 + \|\mathbf{h}\|^2 |s|^2 - \mathbf{y}^{\mathrm{H}}\mathbf{h}s - \bar{s}\mathbf{h}^{\mathrm{H}}\mathbf{y} \end{aligned}$$



- ML=MRC=ZF (with pseudo-inverse $\mathbf{h}^{\#} = (\mathbf{h}^{\mathrm{H}}\mathbf{h})^{-1}\mathbf{h}^{\mathrm{H}} = \mathbf{h}^{\mathrm{H}}/\|\mathbf{h}\|^2$)
- Easy to implement

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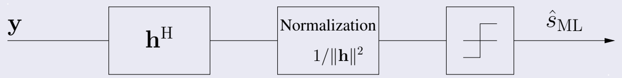
with $\mathbf{h}$ a $L \times 1$ column-vector

We have

$$\begin{aligned}
\hat{s}_{\mathrm{ML}} &= \arg\min_{s \in \mathcal{C}} \|\mathbf{y} - \mathbf{h}s\|^2 \\
&= \arg\min_{s \in \mathcal{C}} \|\mathbf{h}\|^2 \left\| \frac{\mathbf{h}^{\mathrm{H}}\mathbf{y}}{\|\mathbf{h}\|^2} \right\|^2 + \|\mathbf{h}\|^2 \left(|s|^2 - \frac{\mathbf{y}^{\mathrm{H}}\mathbf{h}}{\|\mathbf{h}\|^2} s - \overline{s} \frac{\mathbf{h}^{\mathrm{H}}\mathbf{y}}{\|\mathbf{h}\|^2} \right)
\end{aligned}$$



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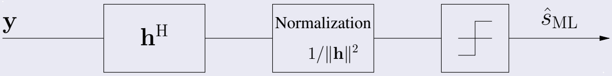
We have

$$\begin{aligned} \hat{s}_{\mathrm{ML}} &= \arg\min_{s \in \mathcal{C}} \|\mathbf{y} - \mathbf{h}s\|^2 \\ &= \arg\min_{s \in \mathcal{C}} \left\| \frac{\mathbf{h}^{\mathrm{H}}\mathbf{y}}{\|\mathbf{h}\|^2} - s \right\|^2 \end{aligned}$$



- ML=MRC=ZF (with pseudo-inverse $\mathbf{h}^{\#} = (\mathbf{h}^{\mathrm{H}}\mathbf{h})^{-1}\mathbf{h}^{\mathrm{H}} = \mathbf{h}^{\mathrm{H}}/\|\mathbf{h}\|^2$)
- Easy to implement

Special case: unitary matrices

We consider $\mathbf{H}\mathbf{H}^{\mathrm{H}} = \mathbf{I}_{N_H}$, or equivalently, $\|\mathbf{H}\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x}$

We have

$$\begin{aligned}\hat{\mathbf{s}}_{\mathrm{ML}} &= \arg\min_{\mathbf{s}\in\mathcal{C}^{N_H}} \|\mathbf{y}-\mathbf{H}\mathbf{s}\| \\ &= \arg\min_{\mathbf{s}\in\mathcal{C}^{N_H}} \|\mathbf{H}\left(\mathbf{H}^{\mathrm{H}}\mathbf{y}-\mathbf{s}\right)\| \\ &= \arg\min_{\mathbf{s}\in\mathcal{C}^{N_H}} \|\mathbf{H}^{-1}\mathbf{y}-\mathbf{s}\|\end{aligned}$$

- ZF=ML
- Easy to implement
- Not true in general since usually $\|\mathbf{A}\mathbf{B}\| \neq \|\mathbf{A}\|.\|\mathbf{B}\|$

Go back to ISI problem

Idea

- Due to filtering, $\mathbf{H}$ is Toeplitz
- Such a structure enables a low-complex optimal detector

Dynamic programming $\Rightarrow$ Viterbi's algorithm `[Viterbi1973]`

We remind that

$$\|\mathbf{y}-\mathbf{H}\mathbf{s}\|^2 = \sum_{n=0}^{N}\left|y(n)-\sum_{\ell=0}^{L}h_\ell s_{n-\ell}\right|^2 \stackrel{def.}{=} J_N(\underbrace{[s_0,\cdots,s_N]}_{\mathbf{s}^{(N)}})$$

Fundamental property

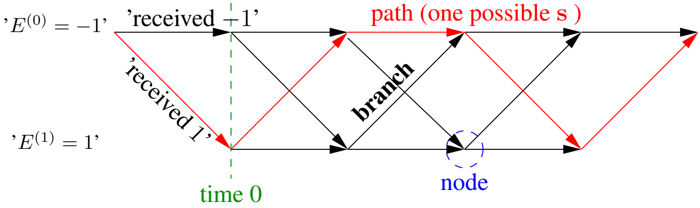
$$J_N(\mathbf{s}^{(N)}) = J_{N-1}(\mathbf{s}^{(N-1)}) + \Delta J(s_N, E_N) = \sum_{n=0}^{N}\Delta J(s_n, E_n)$$

with $\Delta J(s_n, E_n) = |y(n)-\sum_{\ell=0}^{L}h_\ell s_{n-\ell}|^2$ and $E_n = [s_{n-1},\cdots,s_{n-L}]^{\mathrm{T}}$

Definition of Trellis

- enables to view $\mathbf{s}^{(N)}$ through states $\{E_n\}_n$
- corresponds to the possible transitions between states

Example : $M = 2$, $L = 1 \Rightarrow$ two states $E^{(0)} = [-1]$ et $E^{(1)} = [1]$

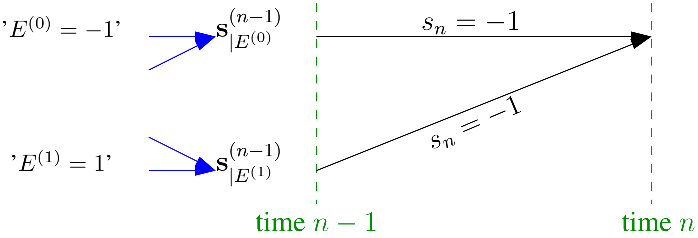


- Branch:
  - only characterized by s_n and E_n (at time n)
  - ΔJ : Branch metric (cost to go through this branch)
- **Goal:** find optimal path (with the lowest sum of branch metrics)

Algorithm principle

For sake of simplification

- two states only $E^{(0)} = [-1]$ et $E^{(1)} = [1]$
- inspecting between time $n-1$ and n



Amongst paths arriving at $E^{(0)}$, only one minimizes $J_{n-1}(.)$

- $\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(0)}}$ s.t. $J_{n-1}(\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(0)}}) \leq J_{n-1}(\mathbf{s}^{(n-1)}_{|E^{(0)}})$
- $\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(1)}}$ s.t. $J_{n-1}(\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(1)}}) \leq J_{n-1}(\mathbf{s}^{(n-1)}_{|E^{(1)}})$ (similar result for $E^{(1)}$)

Algorithm principle (cont'd)

- What's happened at time n in state $E^{(0)}$?
- In other words, what do we know about $\mathbf{s}_{|E^{(0)}}^{(n)}$ and $\mathbf{s}_{\text{opt.}|E^{(0)}}^{(n)}$

$$J_{n-1}(\mathbf{s}_{\text{opt.}|E^{(0)}}^{(n-1)}) \leq J_{n-1}(\mathbf{s}_{|E^{(0)}}^{(n-1)})$$

Algorithm principle (cont'd)

- What's happened at time n in state $E^{(0)}$?
- In other words, what do we know about $\mathbf{s}^{(n)}_{|E^{(0)}}$ and $\mathbf{s}^{(n)}_{\text{opt.}|E^{(0)}}$

$$J_{n-1}(\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(0)}}) + \Delta J(-1, E^{(0)}) \leq J_{n-1}(\mathbf{s}^{(n-1)}_{|E^{(0)}}) + \Delta J(-1, E^{(0)})$$

Algorithm principle (cont'd)

- What's happened at time n in state $E^{(0)}$?
- In other words, what do we know about $\mathbf{s}^{(n)}_{|E^{(0)}}$ and $\mathbf{s}^{(n)}_{\text{opt.}|E^{(0)}}$

$$J_n([\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(0)}}, -1]) \leq J_n([\mathbf{s}^{(n-1)}_{|E^{(0)}}, -1])$$

Any suboptimal path at time $n-1$ remains suboptimal

Algorithm principle (cont'd)

- What's happened at time n in state $E^{(0)}$?
- In other words, what do we know about $\mathbf{s}^{(n)}_{|E^{(0)}}$ and $\mathbf{s}^{(n)}_{\text{opt.}|E^{(0)}}$

$$J_n([\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(0)}}, -1]) \leq J_n([\mathbf{s}^{(n-1)}_{|E^{(0)}}, -1])$$

Any suboptimal path at time $n-1$ remains suboptimal

Finally,

$$\mathbf{s}^{(n)}_{\text{opt.}|E^{(0)}} = [\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(0)}}, -1] \text{ or } [\mathbf{s}^{(n-1)}_{\text{opt.}|E^{(1)}}, -1]$$

according to respective values of J_n

Complexity analysis

- Complexity in $\mathcal{O}((N+1).M^{L+1})$
  - linear in N
  - polynomial in M
  - exponential in L

  So M and L have to be small enough

- Refresher: $\max_k \tau_k = LT_s$, $D = \log_2(M)/T_s$ and $B = 1/T_s$

- Increasing data rate leads to
  - either increasing $M \Rightarrow$ bad news for Viterbi's algorithm
  - or increasing B and thus $L \Rightarrow$ bad news for Viterbi's algorithm

Numerical example

- Filter: $h_0 = 1$ and $h_1 = 0.25$ ($L = 1$)
- Noiseless case

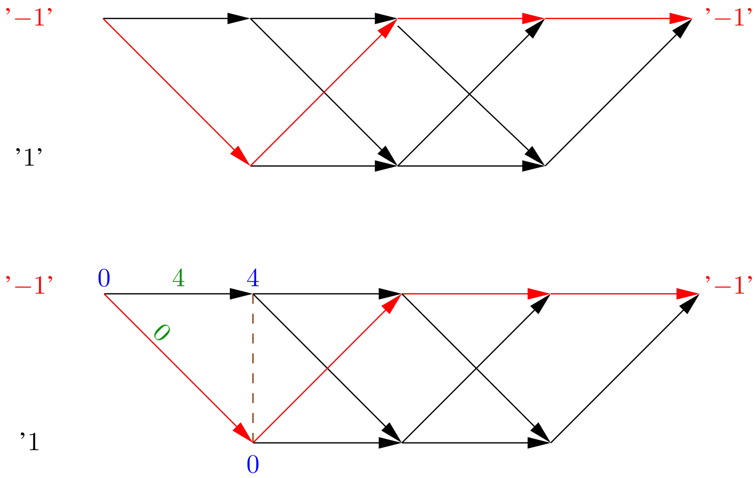
- Opening: '-1'
- Data: '$1, -1, -1$'
- Closing: '-1'

- Sought vector: $[-1, s_0?, s_1?, s_2?, -1]$
- Received vector: $[y(-1)?, 0.75, -0.75, -1.25, -1.25, y(4)?]$

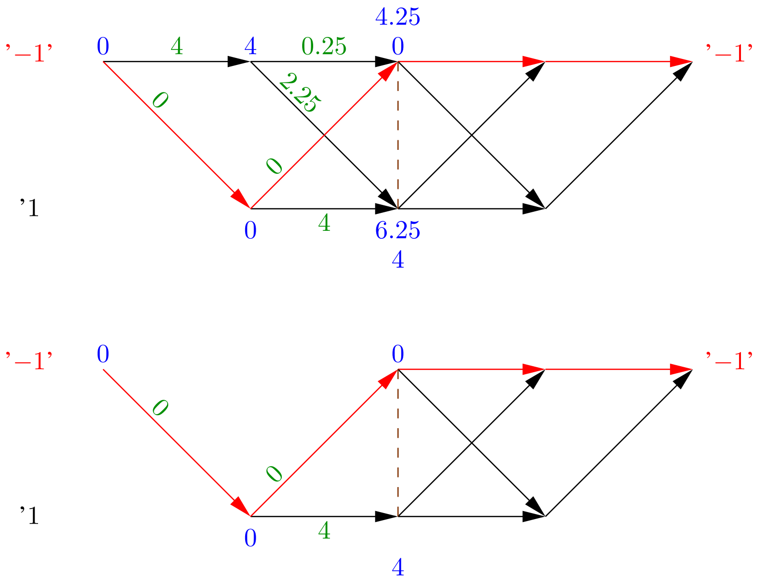
Branch metric (at time n)

$$\Delta J = |y(n) - s_n - 0.25 s_{n-1}|^2$$

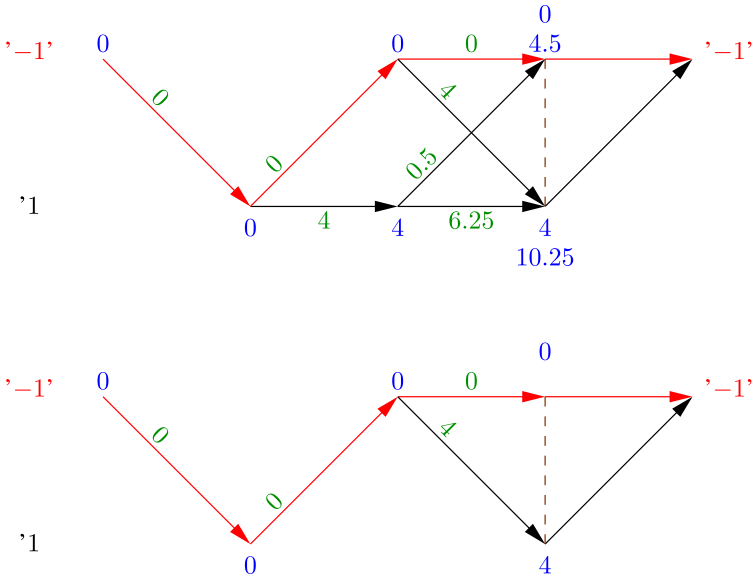
Time 0



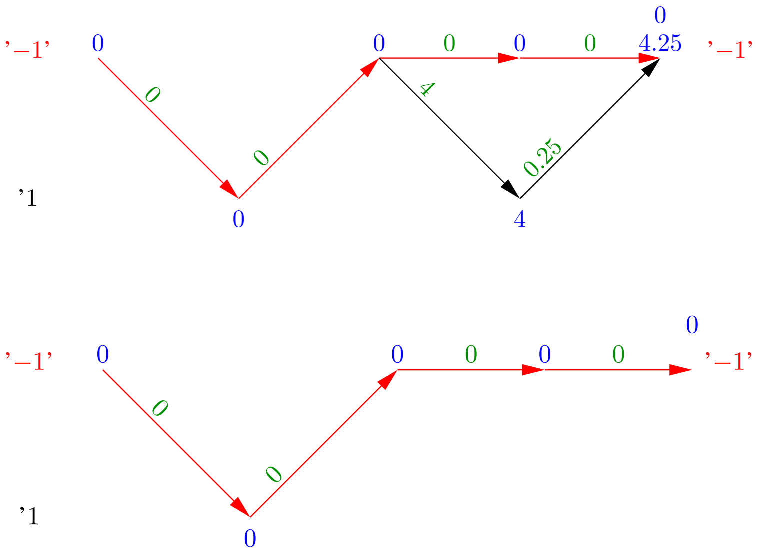
Time 1



Time 2



Time 3



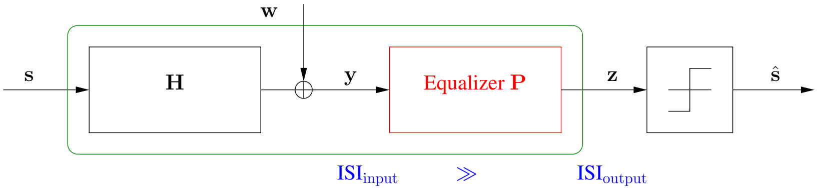
Section 3 : Suboptimal receivers

Principle

Goal

- Viterbi's algorithm **not applicable** in optic systems due to high data rate
- Design a simple receiver (but suboptimal)
- Idea: linear compensation for linear interference

$\Rightarrow$ **linear receivers**



Question: how choosing matrix **P** ?

ZF equalizer

Principle

- Forcing interference to be zero $\Rightarrow$ Zero-Forcing (ZF)
- Mathematically,

$$\mathbf{P}\mathbf{H} = \mathbf{I} \Leftrightarrow \mathbf{P}_{\mathrm{ZF}} = \mathbf{H}^{-1}$$

So

$$\mathbf{z} = \mathbf{s} + \mathbf{w}' \quad \text{with} \quad \mathbf{w}' = \mathbf{H}^{-1}\mathbf{w}$$

Drawback : noise enhancement (we so derive the SNR per component)

$$\mathrm{SNR}_{\mathrm{input}} = \frac{E_s}{2N_0} \frac{\mathrm{trace}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}}\right)}{N_H} \qquad \mathrm{SNR}_{\mathrm{output}} = \frac{E_s}{2N_0} \frac{1}{\frac{\mathrm{trace}(\mathbf{H}^{-1}\mathbf{H}^{-\mathrm{H}})}{N_H}}$$

Thanks to convexity of $x \mapsto 1/x$, one can prove

$$\mathrm{SNR}_{\mathrm{output}} \leq \mathrm{SNR}_{\mathrm{input}}$$

with equality when $\mathbf{H}\mathbf{H}^{\mathrm{H}} = \mathbf{I}$, i.e., unitary matrix (cf. slide 10)

Precompensation: ZF at the transmitter side!

- If $\mathbf{H}$ known at the transmitter side, one can precompensate it!
- How? by sending $\mathbf{x}$ instead of $\mathbf{s}$ with

$$\mathbf{x} = \mathbf{P}\mathbf{s}$$

  where $\mathbf{P}$ corresponds to a linear precoder
- How choosing $\mathbf{P}$? e.g. ZF principle

Drawback:

- No noise enhancement anymore, but
- $\mathbf{P} = \sqrt{a}\mathbf{H}^{-1}$ with $a = N_H/\text{trace}\left(\mathbf{H}^{-1}\mathbf{H}^{-\mathrm{H}}\right)$ (for energy purpose), then

$$\text{SNR}_{\mathrm{w/o\ comp}} = \frac{E_s}{2N_0}\frac{\text{trace}\left(\mathbf{H}\mathbf{H}^{\mathrm{H}}\right)}{N_H} \qquad \text{SNR}_{\mathrm{comp}} = \frac{E_s}{2N_0}\frac{1}{\frac{\text{trace}(\mathbf{H}^{-1}\mathbf{H}^{-\mathrm{H}})}{N_H}}$$

Same SNR issue (just not located at the same place!)

Solution suitable if CSIT and unitary transformation $\Rightarrow$ only CD case

MMSE equalizer

Principle

Choosing $\mathbf{P}$ s.t. $\mathbf{Py}$ close to $\mathbf{s}$

$$\mathbf{P}_{\mathrm{MMSE}} = \arg\min_{\mathbf{P}} \mathbb{E}_{\mathbf{s},\mathbf{w}}[\|\mathbf{Py} - \mathbf{s}\|^2]$$

After algebraic manipulations,

$$\mathbf{P}_{\mathrm{MMSE}} = E_s \left(E_s \mathbf{H}^{\mathrm{H}}\mathbf{H} + 2N_0\mathbf{I}\right)^{-1} \mathbf{H}^{\mathrm{H}}$$

Remarks :

- At high SNR: $\mathbf{P}_{\mathrm{MMSE}} \approx \mathbf{P}_{\mathrm{ZF}}$
- At low SNR: $\mathbf{P}_{\mathrm{MMSE}} \propto \mathbf{H}^{\mathrm{H}}$

DFE equalizer

Principle

- Using previous decision on **s** for removing interference
- Decision Feedback Equalizer (DFE)
- Problem: causality principle in MIMO

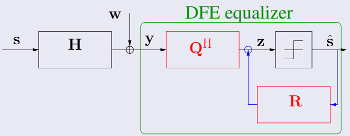
Solution: the so-called QR decomposition

$$\mathbf{H} = \mathbf{QR}$$

with a unitary matrix **Q** and a upper triangular matrix **R**

$$\mathbf{z} = \mathbf{Q}^{\mathrm{H}}\mathbf{y} = \mathbf{Rs} + \mathbf{w}'$$

where $\mathbf{w}'$ is still white Gaussian noise



ZF-DFE: decode first $\hat{s}_{N_H}$, then $\hat{s}_{N_H-1}$ based on $(z_{N_H-1} - r_{N_H-1,N_H}\hat{s}_{N_H})$, and so on

Implementation issue

When N_H is high, implementation issue for matrix inversion

When $\mathbf{H}$ corresponds to ISI,

- equivalence between Toeplitz matrices and filtering
- Implementation through filtering

Example: ZF

$$\mathbf{P}_{\mathrm{ZF}}\mathbf{H} = \mathbf{I} \Longleftrightarrow p_{\mathrm{ZF}} \star h = \delta$$

so

$$p_{\mathrm{ZF}}(e^{2i\pi f}) = \frac{1}{h(e^{2i\pi f})}$$

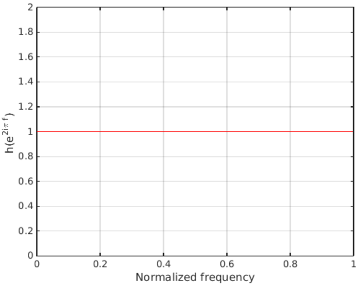
Numerical illustrations

We assume

- $L = 1$ with

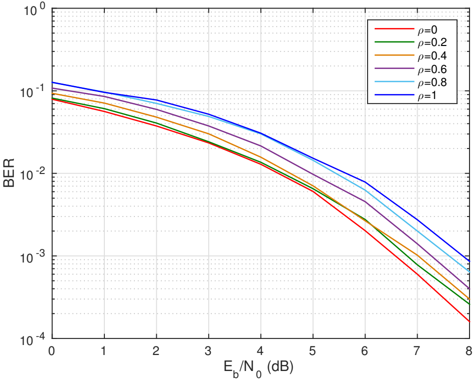
$$h_0 = \frac{1}{\sqrt{1+\rho^2}} \quad \text{and} \quad h_1 = \frac{\rho}{\sqrt{1+\rho^2}}$$

- Matrix implementation of introduced equalizers



The larger ρ is, the stronger ISI is

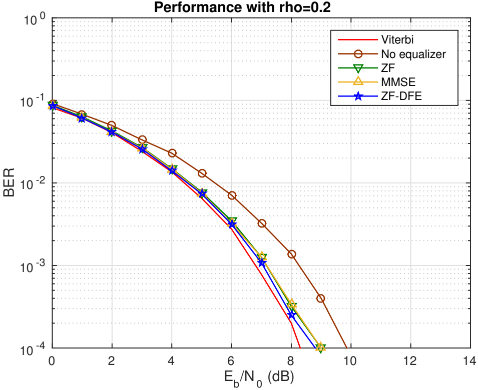
Viterbi's algorithm performance



ISI can not be totally removed

Performance of various equalizers for weak ISI

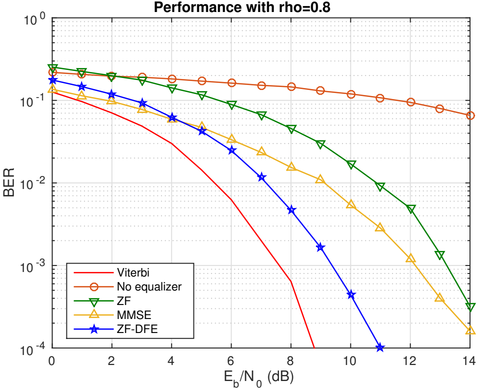
$$\rho = 0.2$$



Close performance between any solution

Performance of various equalizers for strong ISI

$$\rho = 0.8$$



DFE outperforms linear equalizers

Real data

BPSK, SNR $=$ 10dB, and channel with medium ISI ($\rho = 0.5$)

Non equalized

MMSE

Section 4 : OFDM

Refresher: Input/Ouput model

Let

- $x(n)$ be the transmitted signal (may be different from symbols s_n)
- $y(n)$ be the received signal
- $\{h_\ell\}_{\ell=0,\cdots,L}$ be the filter associated with propagation channel

$$y(n) = \sum_{\ell=0}^{L} h_\ell x(n-\ell)$$

Let us consider one block of size N

$\rightarrow$ $\mathbf{y} = [y(N-1), \cdots, y(0)]^{\mathrm{T}}$

$\rightarrow$ $\mathbf{x} = [x(N-1), \cdots, x(0)]^{\mathrm{T}}$

$\rightarrow$ $\tilde{\mathbf{x}} = [x(-1), \cdots, x(-L)]^{\mathrm{T}}$

Matrix model for Input/Output

$$\mathbf{y} = \mathbf{T}_1\mathbf{x} + \mathbf{T}_2\tilde{\mathbf{x}}$$

- $\mathbf{T}_1$: $N \times N$ Toeplitz matrix whose the k-th row is given by
  - $[\mathbf{0}_{k-1}, h_0, h_1, \cdots, h_L, \mathbf{0}_{N-L-k}]$ (if $k \leq N - L$)
  - $[\mathbf{0}_{k-1}, h_0, h_1, \cdots, h_{N-k-1}]$ (if $k > N - L$)
- $\mathbf{T}_2$: $N \times L$ Toeplitz matrix whose the k-th row is given by
  - $\mathbf{0}_L$ (if $k \leq N - L$)
  - $[h_L, h_{L-1}, \cdots, h_{N-k+1}, \mathbf{0}_{N-k}]$ (if $k > N - L$)

In noisy case, we find again

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$

- when null guard interval between blocks: $\mathbf{H} = \mathbf{T}_1$ and $\tilde{\mathbf{x}} = 0$
- when guard interval $\tilde{\mathbf{x}}$ is not empty and linearly depends on current bloc, i.e., $\tilde{\mathbf{x}} = \mathbf{T}_3\mathbf{x}$: $\mathbf{H} = \mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3$

Question: how obtaining an interference-free system at the receiver side (by modifying the transmitter) ?

Lemma 1

Assumption:

- Perfect CSIT (**H** known at the transmitter side)
- Actually, extension of ZF principle described in slide 23

Let $\mathbf{H} = \mathbf{U}^{\mathrm{H}}\mathbf{\Lambda}\mathbf{V}$ be the singular value decomposition (svd) of **H** with **U**, **V** two unitary matrices, and $\mathbf{\Lambda} = \mathrm{diag}(\lambda_0, \cdots, \lambda_{N-1})$

- Instead of sending $\mathbf{x} = \mathbf{s}$, we send $\mathbf{x} = \mathbf{V}^{\mathrm{H}}\mathbf{s}$ (no energy purpose)
- Instead of detecting on **y**, we detect on $\mathbf{z} = \mathbf{U}\mathbf{y}$

$$\Rightarrow \mathbf{z} = \mathbf{\Lambda}\mathbf{s} + \mathbf{w}'$$

with $\mathbf{w}' = \mathbf{U}\mathbf{w}$ still white Gaussian noise

Remarks

- Information located in eigenvectors (not interfer in-between!)
- Eigenvectors usually depend on **H**
- Issue: CSIT assumption unrealistic in wireless and even in optic

Lemma 2

Let $\mathbf{C}$ be a $N \times N$ circulant matrix associated with $\{h_\ell\}_{\ell=0,\cdots,L}$

$$\mathbf{C} = \begin{bmatrix} h_0 & h_1 & \cdots & h_L & 0 & \cdots & 0 \\ 0 & h_0 & h_1 & \cdots & h_L & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ h_1 & \cdots & h_L & 0 & \cdots & 0 & h_0 \end{bmatrix}$$

Properties of C

- Eigenvectors: Fourier vectors (independent of the channel!)
- Eigenvalues: filter responses at Fourier frequencies

$$\mathbf{C} = \mathbf{F}^{\mathrm{H}} \boldsymbol{\Lambda} \mathbf{F}$$

with

- $\mathbf{F}$ FFT matrix (so, $\mathbf{F}^{\mathrm{H}} = \mathbf{F}^{-1}$)
- $\lambda_n = H(e^{2i\pi n/N}) = \sum_{\ell=0}^{L} h_\ell e^{-2i\pi \frac{\ell n}{N}}$

OFDM principle

If $\tilde{\mathbf{x}} = [x(N-1), \cdots, x(N-L)]^{\mathrm{T}}$, then

$$\mathbf{y} = \mathbf{T}_1\mathbf{x} + \mathbf{T}_2\tilde{\mathbf{x}} \Leftrightarrow \mathbf{y} = \mathbf{C}\mathbf{x}$$

Thus we have

$$\mathbf{z} = \mathbf{\Lambda}\mathbf{s}$$

with $\mathbf{z} = \mathbf{F}\mathbf{y}$ and $\mathbf{x} = \mathbf{F}^{-1}\mathbf{s}$

Finally, for the k-th block and the n-th subcarrier, we get

$$z_n^{(k)} = H(e^{2i\pi n/N})s_n^{(k)} \quad \forall n, k$$

Remarks

- cyclic prefix transforms Toeplitz into Circulant (who diagonalizes within a basis independent of the channel, so no required CSIT !)
- OFDM: Orthogonal Frequency Division Multiplexing

A first other way for introducing OFDM

Due to channel, we receive

$$\begin{cases} y(N-1) & = & h_0 x(N-1) + h_1 x(N-2) + \cdots + h_L x(N-L-1) \\ & \vdots & \\ y(0) & = & h_0 x(0) + h_1 x(-1) + \cdots + h_L x(-L) \end{cases}$$

$$y(n) = \sum_{\ell=0}^{L} h_\ell x(n-\ell) \overset{\text{Cyclic prefix}}{\Longrightarrow} y(n) = \sum_{\ell=0}^{L} h_\ell x(n-\ell \bmod N)$$

- Transformation from a convolution into a circular convolution
- If convolution $(\mathbf{T}_1, \mathbf{T}_2)$, $Y(e^{2i\pi n/N}) \neq H(e^{2i\pi n/N})X(e^{2i\pi n/N})$
- If circular convolution $(\mathbf{C})$, $Y(e^{2i\pi n/N}) = H(e^{2i\pi n/N})X(e^{2i\pi n/N})$

A first other way for introducing OFDM

Thanks to cyclic prefix, we finally receive

$$\left\{\begin{array}{lcl} y(N-1) & = & h_0 x(N-1) + h_1 x(N-2) + \cdots + h_L x(N-L-1) \\ & \vdots & \\ y(0) & = & h_0 x(0) + h_1 \cancel{x(-1)} + \cdots + h_L \cancel{x(-L)} \end{array}\right.$$

$$y(n) = \sum_{\ell=0}^{L} h_\ell x(n-\ell) \overset{\text{Cyclic prefix}}{\Longrightarrow} y(n) = \sum_{\ell=0}^{L} h_\ell x(n-\ell \bmod N)$$

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A first other way for introducing OFDM

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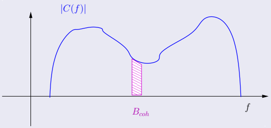
$$\left\{ \begin{array}{lcl} y(N-1) & = & h_0 x(N-1) + h_1 x(N-2) + \cdots + h_L x(N-L-1) \\ & \vdots & \\ y(0) & = & h_0 x(0) + h_1 x(N-1) + \cdots + h_L x(N-L) \end{array} \right.$$

$$y(n) = \sum_{\ell=0}^{L} h_\ell x(n-\ell) \stackrel{\text{Cyclic prefix}}{\Longrightarrow} y(n) = \sum_{\ell=0}^{L} h_\ell x(n - \ell \bmod N)$$

- Transformation from a convolution into a circular convolution
- If convolution $(\mathbf{T}_1, \mathbf{T}_2)$, $Y(e^{2i\pi n/N}) \neq H(e^{2i\pi n/N}) X(e^{2i\pi n/N})$
- If circular convolution $(\mathbf{C})$, $Y(e^{2i\pi n/N}) = H(e^{2i\pi n/N}) X(e^{2i\pi n/N})$

A second other way for introducing OFDM

The historical way



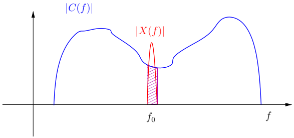
$$y(t) = c(t) \star x(t)$$

with $c(t)$ propagation channel

B_c: coherence bandwidth

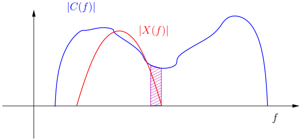
B: signal bandwidth

If $B < B_c$, almost no ISI



$$Y(f) \approx C(f_0)X(f) \Rightarrow y(t) \approx C(f_0)x(t)$$

If $B > B_c$, ISI



$$Y(f) = C(f)X(f) \Rightarrow y(t) = c(t) \star x(t)$$

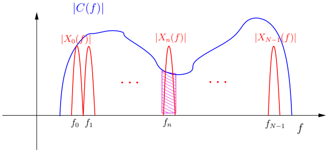
How being always in the configuration $B < B_c$?

A second other way... naive idea

Idea: as $B = 1/T_s$ (symbol period), splitting symbols sequence into N subsequence (with period $T = NT_s$) s.t.

$$\frac{1}{T} < B_c$$

Then each subsequence n transmitted to different subcarriers f_n



On each subcarrier, no ISI

Let $s_n^{(k)} = s_{kN+n}$ be a subsequence, the transmitted signal is

$$x(t) = \sum_{n=0}^{N-1} \sum_{k \in \mathbb{Z}} s_n^{(k)} g(t - kT) e^{2i\pi f_n t}$$

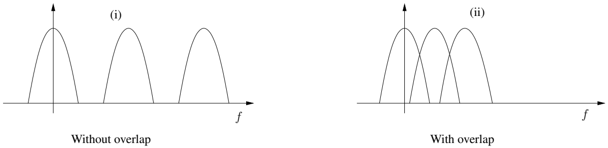
- N: subcarriers number
- $\sum_{n=0}^{N-1} s_n^{(k)} e^{2i\pi f_n t}$: OFDM symbol with period $T = NT_s$

A second other way... orthogonality principle

$$x(t) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{N-1} s_n^{(k)} \Phi_{n,k}(t) \quad \text{with} \quad \Phi_{n,k}(t) = g(t-kT)e^{2i\pi f_n t}$$

Orthogonal subcarriers $\Leftrightarrow \int_{\mathbb{R}} \Phi_{n,k}(t)\overline{\Phi_{n',k'}(t)}dt = \delta_{n,n'}\delta_{k,k'}$

We force $\{\Phi_{n,k}(t)\}_{n,k}$ to be orthonormal basis



Case (ii) : OFDM (Orthogonal Frequency Division Multiplexing)

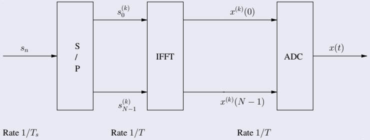
- f_n equally spaced $\Rightarrow f_n = n\Delta f$
- $g(t)$ rectangular function of support $[0, T]$
- **orthogonality iff $\Delta f = 1/T = 1/NT_s$**

A second other way... transmitter description

Bandwidth: almost same spectral efficiency than single carrier

$$B_{tot} = NB_{sc} \approx N\frac{1}{T} = N\frac{1}{NT_s} = \frac{1}{T_s}$$

$$x^{(k)}(m) = x(kT + mT_s) = \underbrace{\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s_n^{(k)} e^{2i\pi\frac{nm}{N}}}_{\text{inverse FFT}}$$



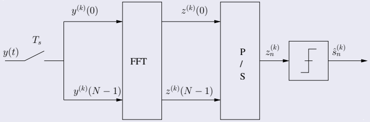
Remarks

- x in time domain. k OFDM block number and m time index (within block)
- s in frequency domain. n frequency index

A second other way... receiver description

$$z^{(k)}(n) = \int_{\mathbb{R}} y(t)\overline{\Phi_{n,k}(t)}dt$$

$$= \frac{\sqrt{T_s}}{\sqrt{N}} \sum_{m=0}^{N-1} y^{(k)}(m) e^{-2i\pi \frac{nm}{N}}$$



with $y^{(k)}(m) = y(kT + mT_s)$

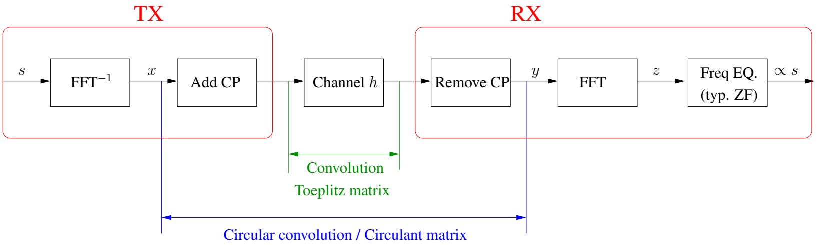
When channel present,

$$y(t) = c(t) \star x(t) = \sum_{k\in\mathbb{Z}} \sum_{n=0}^{N-1} s_n^{(k)} \Psi_{n,k}(t) \quad \text{with} \quad \Psi_{n,k}(t) = c(t) \star \Phi_{n,k}(t)$$

Problem: $\Psi_{n,k}(t)$ is not orthonormal anymore (actually, almost)
Solution: go back to the previously-developed approach

$\Rightarrow$ **Cyclic prefix**

TX/RX Scheme



Cyclic prefix is crucial!

History

- end-50: multicarriers concept (Case (i))
- end-60: orthogonal multicarriers (Case (ii)) $\Rightarrow$ OFDM
- beginning-70: FFT
- mid-80: European project “Eurêka” for DAB
  - cyclic prefix
  - coding and OFDM relationship in wireless context
- beginning-90: First standard based on OFDM (DAB)
- end-90: Very popular standards (ADSL, DVBT, Wifi, LTE, . . .)

OFDM design: two rules

- ***N* should be large enough**

$$L \ll N \Rightarrow LT_s \ll NT_s \Rightarrow T_d \ll NT_s$$

$$N \gg B/B_c \Leftrightarrow \Delta f \ll B_c$$

- ***N* should be small enough**

$$(L+N)T_s \ll T_c \Rightarrow NT_s \ll T_c$$

$$N \ll B/B_d \Leftrightarrow \Delta f \gg B_d$$

but also

- Mis-synchronization of VCO (a few ppm)
- FFT complexity ($\mathcal{O}(N\log(N))$)
- Latency

OFDM design: four examples

	Wifi	ADSL	DVBT	Optic 100G
Carrier freq.	5.2GHz	0.6MHz	700MHz	1550nm
Bandwidth	20MHz	1.1MHz	9.15MHz	5GHz
Sampling period	50ns	0.9μs	0.11μs	0.2ns
Filter length	800ns	135μs	224μs	0.69ns
Filter degree	**16**	**150**	**2036**	**4**
Cyclic prefix	**16**	**32**	**2048**	**8**
Spect. eff. loss	20%	12.5 %	20%	3.125%
# Subcarrier	**64**	**256**	**8192**	**256**
OFDM duration	4μs	256μs	896μs	52.8ns
Subcarrier spac.	312.5kHz	4.31kHz	1.11kHz	19.53 MHz
Coherence band.	1.25MHz	7.4kHz	4.47kHz	1.44GHz
Doppler band.	52Hz (3m/s)	0	52Hz (3m/s)	0

Remark: in ADSL, *Time Equalizer* added for channel shortening

Detection: OFDM-SISO

Formally, we have

$$\mathbf{z} = \mathbf{H}\mathbf{s} + \mathbf{w}$$

with $\mathbf{H} = \mathrm{diag}(H(1), \cdots, \underbrace{H(e^{2i\pi n/N})}_{H(n)}, \cdots, H(e^{2i\pi(N-1)/N}))$

As $\mathbf{H}$ is diagonal (no interference occurs), one can work subcarrier per subcarrier, i.e.,

$$z(n) = H(n)s_n + w(n)$$

with

- $z(n)$ received signal after FFT
- s_n transmitted symbol (before IFFT)
- $H(n)$ filter response at the n-th subcarrier

Optimal detector (application of Section 2)

Threshold detector on $H(n)^{-1}z(n)$

Detection: OFDM-MIMO

Once again, we have

$$\mathbf{z}(n) = \mathbf{H}(n)\mathbf{s}_n + \mathbf{w}(n)$$

with

- $\mathbf{z}(n)$ received signal on two polarizations after FFT
- $\mathbf{s}_n$ transmitted symbols on two polarizations (before IFFT)
- $\mathbf{H}(n)$ $N_H \times N_H$ filter response at the n-th subcarrier

Detectors

- If N_H small enough, then many trees-based decoders
- If CD and PMD, then $\mathbf{H}(n)\mathbf{H}(n)^{\mathrm{H}} = \mathbf{I}$ (unitary matrix)
  - Optimal detector easy (application of Section 2)
  - Threshold detector on $\mathbf{H}(n)^{\mathrm{H}}\mathbf{z}(n)$

Conclusion

- Review on digital receivers
- We omit to discuss about
  - channel estimation
  - synchronization
  - nonlinear impairments
  - MIMO optimization

References

[Viterbi1967] A. Viterbi, Error bounds for convolutional codes and an asymptotic optimum decoding algorithm, IEEE Trans. on Information Theory, 1967

[Tse2005] D Tse, P. Viswanath, Fundamentals of Wireless Communications, Cambridge University Press, 2005

[Goldsmith2005] A. Goldsmith, Wireless Communications, Cambridge University Press, 2005

[VanTrees2013] H. VanTrees, K. Bell, Detection Estimation and Modulation Theory Part I, Wiley, 2013

[Savory2008] S. Savory, Digital filters for coherent optical receivers, Optics Express, 2008