ROSP/DCOM1

Digital Information Processing

Part on ”Synchronization”

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Outline

- Section 0: Refresher on Estimation Theory
- Section 1: Phase estimation
- Section 2: Frequency estimation
- Section 3: Extension to OFDM (Frame timing estimation)

Section 0 : Refresher on Estimation Theory

Introduction

Let x_n be a sequence of data depending on parameter $\boldsymbol{\theta}$

$$x_n = f_n(\boldsymbol{\theta}, ...)$$

e.g. $x_n = \theta + w_n$ *with unknown* θ

Goal

Given $\mathbf{x}_N = \{x_0, \dots, x_{N-1}\}$, retrieve $\boldsymbol{\theta}$

An *estimate* of $\boldsymbol{\theta}$ is a function of the data sequence, we hope, close to the true value of $\boldsymbol{\theta}$

$$\hat{\boldsymbol{\theta}}_N = \hat{\boldsymbol{\theta}}(\mathbf{x}_N)$$

The question is

How finding good estimates according to performance metric capturing the gap between the estimated value and the true one

Detection/Estimation

Detection

θ has a countable/finite number of values

Performance metric

Erreur probability

$$P_e = \text{Prob}(\hat{\theta} \neq \theta)$$

$\Rightarrow$ Optimal detector ?

- Maximum Likelihood (ML) if equilikely

Estimation

θ has a non-countable number of values

Performance metric

Mean Square Error

$$\mathcal{E} = \mathbb{E}[|\hat{\theta} - \theta|^2]$$

$\Rightarrow$ Optimal estimate ?

- often ML

Performance metrics

Notice that $\hat{\theta}(\mathbf{x}_N)$ is a random variable

Bias

$$b_N = \mathbb{E}_X[\hat{\theta}(\mathbf{x}_N)] - \theta$$

Consistency

$$\Pr\left(\mathbf{x}_N : \lim_{N\to\infty} \hat{\theta}(\mathbf{x}_N) = \theta\right) = 1 \Leftrightarrow \hat{\theta}_N \overset{a.s.}{\to} \theta$$

MSE and variance

$$\mathrm{MSE}_N = \mathbb{E}_X[\|\hat{\theta}(\mathbf{x}_N) - \theta\|^2], \quad \mathrm{var}_N = \mathbb{E}_X[\|\hat{\theta}(\mathbf{x}_N) - \mathbb{E}_X[\hat{\theta}(\mathbf{x}_N)]\|^2]$$

- We have $\mathrm{MSE}_N = \|b_N\|^2 + \mathrm{var}_N$
- Estimate is *bias-free* iff $b_N = 0$ for all N
- Estimate is *asymptotically bias-free* iff $\lim_{N\to\infty} b_N = 0$

Example

- Let x_n be a real-valued iid Gaussian process with unknown mean m and unit-variance
- Let $\hat{m}_N$ be the empirical mean estimate

$$\hat{m}_N = \frac{1}{N}\sum_{n=0}^{N-1} x_n$$

Results

- The estimate is bias-free
- Its variance is

$$\mathrm{var}_N = \frac{1}{N}$$

Lower bounds

Result: Cramer-Rao bound (CRB)

Under mild conditions, we have

$$\mathbb{E}_X[(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^{\mathrm{T}}] \geq \mathbf{CRB}_N \ (= \mathbf{F}_N^{-1})$$

with the so-called Fisher Information Matrix (FIM)

$$\begin{aligned} \mathbf{F}_N &= \mathbb{E}_X[(\partial \log p_X(\mathbf{x}_N|\boldsymbol{\theta})/\partial\boldsymbol{\theta})\,(\partial \log p_X(\mathbf{x}_N|\boldsymbol{\theta})/\partial\boldsymbol{\theta})^{\mathrm{T}}] \\ &= -\mathbb{E}_X[\partial^2 \log p_X(\mathbf{x}_N|\boldsymbol{\theta})/(\partial\boldsymbol{\theta})^2] \end{aligned}$$

- $p_X(\mathbf{x}_N|\boldsymbol{\theta})$ is the likelihood
- $\mathrm{MSE}_N \geq \mathrm{trace}(\mathbf{CRB}_N)$
- An estimate whose the MSE achieved the CRB is *efficient*
- An estimate is *asymptotically efficient* iff

$$\frac{\mathrm{MSE}_N}{\mathrm{trace}(\mathbf{CRB}_N)} \to 1, \ \text{when } N \to \infty$$

Example

Let x_n be a real-valued iid Gaussian process with unknown mean m and unit-variance. We consider that N samples are available: $x_0, \cdots, x_{N-1}$

Results

$$F_N = N$$

Consequence: CRB for m is $1/N$

Let us consider the empirical mean estimate

- This estimate offers a MSE equal to $1/N$
- This estimate is thus efficient

Maximum Likelihood (ML) estimator

Definition

$$\hat{\theta}_{N,\mathrm{ML}} = \arg\max_{\boldsymbol{\theta}} p(\mathbf{x}_N|\boldsymbol{\theta})$$

Performances

Under mild conditions, the ML estimate is

- consistent
- asymptotically efficient

Example

$$x_n = f_n(\theta) + w_n, \quad \text{for } n = 0, \ldots, N-1$$

with functions $f_n(.)$ and w_n white zero-mean Gaussian noise

$$\hat{\theta}_{N,\mathrm{ML}} = \arg\min_{\boldsymbol{\theta}} \sum_{n=0}^{N-1} |x_n - f_n(\theta)|^2$$

Here, ML = LS (Least Square)

Section 1 : Phase estimation

Problem statment

$$y(n) = s_n e^{i\phi} + w(n), \quad n = 0, \ldots, N-1$$

- $\{s_n\}$ data sequence
- $w(n)$ white zero-mean circular Gaussian noise with variance $2N_0$
- ϕ the **unknown phase**

Why this model and How handling ϕ

Blind Equalization (e.g. CMA) $\rightsquigarrow$ phase ambiguity (and ISI-free)

- Solution 1: noncoherent modulation carried out: ϕ not required
  - $\rightarrow$ Differential modulation (D-PSK), Orthogonal signals (PPM/FSK)
  - $\rightarrow$ 3dB loss
- Solution 2: coherent modulation carried out: ϕ is required
  - $\rightarrow$ Estimation step for ϕ
  - $\rightarrow$ If not done, $0,13$dB loss if $\phi = 10^o$ or 3dB loss if $\phi = 45^o$ (QPSK)

Data-Aided (DA) estimation

- Training sequence is available

ML estimate

$$p(\mathbf{y}_N|\phi) \propto \exp\left\{-\frac{\sum_{n=0}^{N-1}\left|y(n)-s_n e^{i\phi}\right|^2}{2N_0}\right\}$$

with $\mathbf{y}_N = [y(0), \cdots, y(N-1)]^{\mathrm{T}}$

Result

$$\hat{\phi}_N = \arg\max_{\phi} \underbrace{\sum_{n=0}^{N-1} \Re\left\{\overline{s_n} y(n) e^{-i\phi}\right\}}_{J_N(\phi)}$$

$$\hat{\phi}_N = \arctan\left(\frac{\sum_{n=0}^{N-1} \Im\{\overline{s_n} y(n)\}}{\sum_{n=0}^{N-1} \Re\{\overline{s_n} y(n)\}}\right) = \angle\left\{\sum_{n=0}^{N-1} \overline{s_n} y(n)\right\}$$

Cramer-Rao bound

- Asymptotically bias-free estimate

$$\sum_{n=0}^{N-1} \overline{s_n} y(n) = \left(\sum_{n=0}^{N-1} |s_n|^2 \right) e^{i\phi} + \underbrace{\sum_{n=0}^{N-1} \overline{s_n} w(n)}_{\text{noise term}}$$

- Also asymptotically efficient

Result

$$\mathrm{CRB}_N(\phi) = \frac{N_0}{\sum_{n=0}^{N-1} |s_n|^2}$$

- Decrease in $1/N$
- Decrease in $1/\mathrm{SNR}$

(Stochastic) gradient algorithm

- Actually,

$$J_N(\phi) = \sum_{n=0}^{N-1} j_n(\phi) \quad \text{with} \quad j_n(\phi) = \Re\left\{\overline{s_n} y(n) e^{-i\phi}\right\}$$

- We now work sample per sample

Algorithm

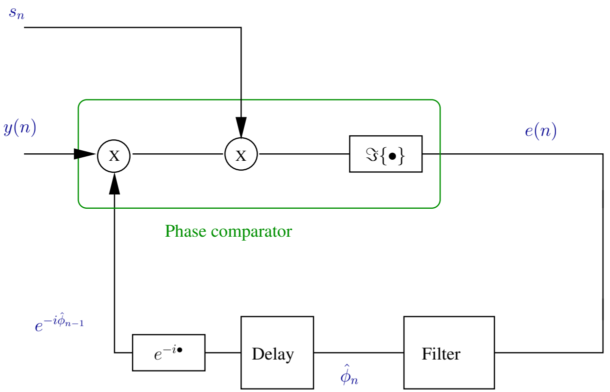
$$\begin{aligned}\hat{\phi}_n &= \hat{\phi}_{n-1} + \mu j_n'(\hat{\phi}_{n-1}) \\ &= \hat{\phi}_{n-1} + \mu \Im\{\overline{s_n} y(n) e^{-i\hat{\phi}_{n-1}}\}\end{aligned}$$

where n is a time/iteration index

Remark

- Phase tracking
- Application to frequency estimation $\phi_n = 2\pi f_0 n + \phi$

Phase Locked Loop (PLL)



Remark

- Small μ: good precision but slow acquisition
- Large μ: poor precision but fast acquisition

Non-Data-Aided (NDA) estimation

- Without training sequence (blind case)

$$\text{ML:} \quad p(\mathbf{y}_N|\phi) = \int p(\mathbf{y}_N|\phi,\mathbf{s}_N)p(\mathbf{s}_N)d\mathbf{s}_N$$

$$\propto \mathbb{E}_{\mathbf{s}}\left[e^{-\frac{\sum_{n=0}^{N-1}\left|y(n)-s_n e^{i\phi}\right|^2}{2N_0}}\right]$$

Result

$$\hat{\phi}_N = \arg\max_{\phi} \sum_{n=0}^{N-1} \log\left(\sum_{m=0}^{P-1} p_m e^{-\frac{|s^{(m)}|^2}{2N_0}} e^{-\frac{\Re\left\{\overline{s^{(m)}}y(n)e^{-i\phi}\right\}}{N_0}}\right)$$

with

- $s^{(m)}$ the m-th symbol of the constellation (of size P)
- p_m the occurrence probability of $s^{(m)}$

A simple case

- BPSK (± 1)
- Equilikely probability

Result

$$\hat{\phi}_N = \arg\max_{\phi} \sum_{n=0}^{N-1} \log\left(\cosh\left(\frac{\Re\left\{y(n)e^{-i\phi}\right\}}{N_0}\right)\right)$$

Adaptive version

$$\begin{aligned}\hat{\phi}_n &= \hat{\phi}_{n-1} + \mu' \Im\left\{d(n)y(n)e^{-i\hat{\phi}_{n-1}}\right\} \\ &= \hat{\phi}_{n-1} + \mu' d(n) \Im\left\{y(n)e^{-i\hat{\phi}_{n-1}}\right\}\end{aligned}$$

with

$$d(n) = \tanh\left(\frac{\Re\left\{y(n)e^{-i\hat{\phi}_{n-1}}\right\}}{N_0}\right) \quad (\in \mathbb{R})$$

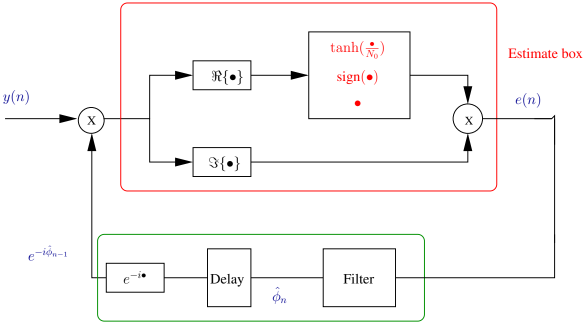
Costas loop

- According to Slide 13, $d(n)$ corresponds to a soft decision on s_n
- We can replace $d(n)$ with another type of decision
  1. hard decision ($\rightsquigarrow$ DD estimation)

$$d(n) = \text{sign}\left(\Re\left\{y(n)e^{-i\hat{\phi}_{n-1}}\right\}\right)$$

  2. no decision ($\rightsquigarrow$ Costas loop [costas1956])

$$d(n) = \Re\left\{y(n)e^{-i\hat{\phi}_{n-1}}\right\}$$



M-th power estimate

We use the fact that s_n is M-th order noncircular $\Leftrightarrow \mathbb{E}[s_n^M] \neq 0$

Constellation	P-PAM	P-PSK	P-QAM
M	2	P	4

By working on $y(n)^M$, we boil down to a phase estimation (with constant data) with non-gaussian noise

$$y(n)^M = \underbrace{\mathbb{E}[s_n^M]}_{U} e^{i(M\phi)} + \varepsilon(n)$$

Result [Viterbi1974]

According to Slide 11,

$$\hat{\phi}_N = \frac{1}{M} \angle \left\{ \sum_{n=0}^{N-1} \overline{U} y(n)^M \right\}$$

Section 2 : Frequency estimation

Problem statment

- Mismatch between VCO at transmitter and receiver
  $\rightsquigarrow$ Carrier Frequency offset (CFO)
- CFO mitigated before any other box (equalization, timing, etc)
  $\rightsquigarrow$ too complicate to design ML-like DA estimate (since joint estimation) [ciblat2008]
- Blind case (no training used even if available)

$$y(n) = \underbrace{\left(\sum_{k=0}^{L} h(k) s_{n-k}\right)}_{a(n)} e^{2i\pi f_0 n} + w(n)$$

- s_n **unknown** data sequence
- $\{h(k)\}$ **unknown filter**
- $w(n)$ white zero-mean circular Gaussian noise with variance $2N_0$
- f_0 **unknown** CFO

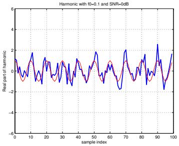
Refresher: pure harmonic retrieval

$$y(n) = ae^{2i\pi f_0 n} + w(n) \quad \text{with unknown} \quad a = |a|e^{i\phi_0}$$

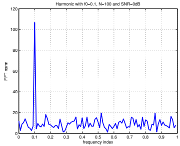
ML: $$\min_{|a|,\phi,f} J(|a|,\phi,f) = \frac{1}{N}\sum_{n=0}^{N-1}\left|y(n) - |a|e^{2i\pi fn + i\phi}\right|^2$$

Result: periodogram maximization

$$\hat{f}_N = \arg\max_f \left|\frac{1}{N}\sum_{n=0}^{N-1} y(n)e^{-2i\pi fn}\right|^2$$



SNR=0dB (time domain)



SNR=0dB (frequency domain)

BPSK

As $U = \mathbb{E}[a^2(n)] \neq 0$ (second-order noncircular), we have

$$z(n) = y^2(n) = Ue^{2i\pi(2f_0)n} + e(n)$$

where $e(n)$ is an additive noise neither Gaussian nor stationary

Remark

- Harmonic retrieval in multiplicative and additive noise

  $\Updownarrow$

  Harmonic retrieval in (nonstandard) additive noise
- Periodogram based on $y^2(n)$ instead of $y(n)$

$$\hat{f}_N = \arg\max_f \left| \frac{1}{N} \sum_{n=0}^{N-1} y(n)^2 e^{-2i\pi(2f)n} \right|^2$$

PSK or QAM

P-PSK (1983)

$$\mathbb{E}[a(n)^P] \neq 0 \Leftrightarrow \hat{f}_N = \arg\max_f \left\| \frac{1}{N} \sum_{n=0}^{N-1} y^P(n) e^{-2i\pi(Pf)n} \right\|^2$$

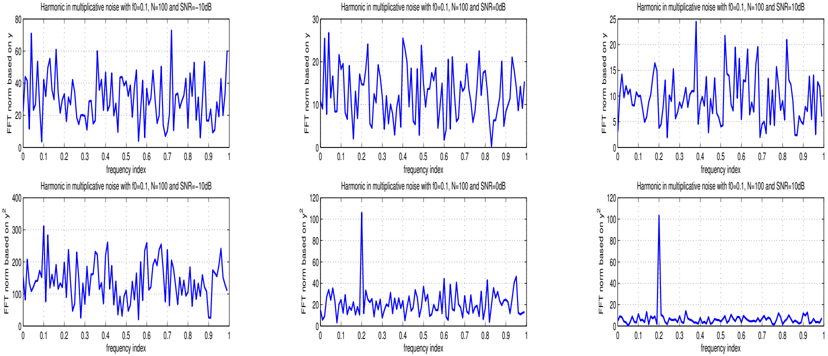
P-QAM [yang2003]

$$\mathbb{E}[a(n)^4] \neq 0 \Leftrightarrow \hat{f}_N = \arg\max_f \left\| \frac{1}{N} \sum_{n=0}^{N-1} y^4(n) e^{-2i\pi(4f)n} \right\|^2$$

$\Rightarrow$ once again, *M*-th power estimate

Simulations

- $a(n)$ belongs to BPSK
- $N = 100$

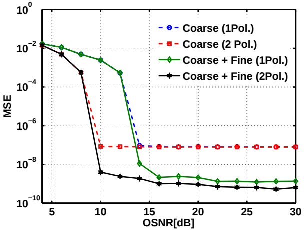


SNR=−10dB SNR=0dB SNR=10dB

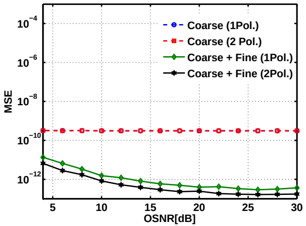
MSE

Periodogram maximized into two steps

- *coarse* step to detect the peak
- *fine* step to refine estimation around the detected peak



$N = 256$



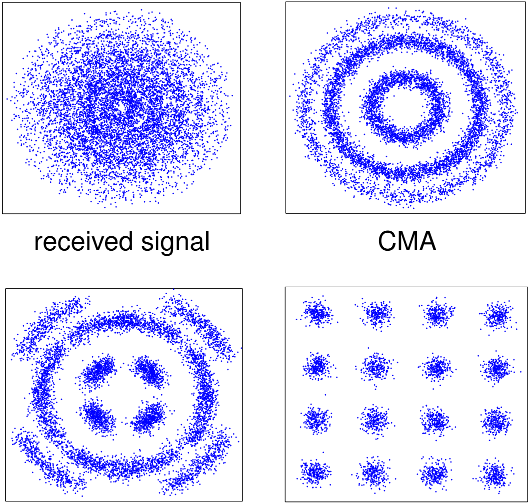
$N = 4096$

Remark

For weak SNR or small N, coarse step may fail $\Rightarrow$ outliers effect

BER

- 16-QAM, 2 polarizations, 100Gbits/s, OSNR=20dB



received signal

CMA

coarse CFO and CMA

fine CFO and CMA

Section 3 : OFDM extension

Problem statment

- (constant) **Phase:** as done in single carrier

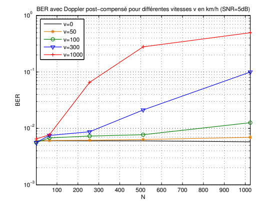
- **Frequency:**

$$\mathbf{y} = \mathbf{\Delta C x} \text{ avec } \mathbf{\Delta} = \mathrm{diag}(1, e^{2i\pi f_0}, \cdots, e^{2i\pi f_0 (N-1)})$$

If flat fading ($\mathbf{C} = \mathbf{Id}$)

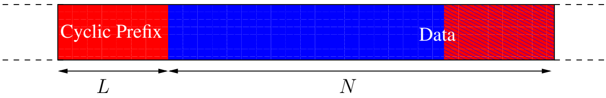
$$\mathbf{z}_{\text{OFDM}} = \mathbf{F \Delta F}^{-1}\mathbf{s}$$
$$\mathbf{z}_{\text{SC}} = \mathbf{\Delta s}$$

No InterCarrier Interference in SC



- **Timing:** frame beginning detection $\Rightarrow$ new problem

Timing estimation



Correlation with cyclic prefix:
$y(n)$ and $y(n+N)$ are identical if n belongs to Cyclic Prefix

$$\hat{\tau}_0 = \arg \max_{\tau \in \{0,\cdots,N-1\}} \frac{1}{L} \sum_{n=0}^{L-1} y(n+\tau) y(n+\tau+N)$$

Structured OFDM symbol **`[schmidl&cox1997]`:**
Assume OFDM is split into two identical halves : $y(n)$ and $y(n+N/2)$ are identical if n belongs to the first half

$$\hat{\tau}_0 = \arg \max_{\tau \in \{0,\cdots,N-1\}} \frac{1}{N/2} \sum_{n=0}^{N/2-1} y(n+\tau) y(n+\tau+N/2)$$

<u>Drawback:</u> rate divided by two

Frequency estimation

- Assume OFDM is split into two identical halves [schmidl&cox1997]
- If n belongs to the first half, $y(n+N/2) = e^{2i\pi f_0 (N/2)} y(n)$

Assuming $\hat{\tau}_0 = 0$ (for sake of simplicity), we have

$$y(n)y(n+N/2) = e^{2i\pi f_0 (N/2)} |y(n)|^2$$

then

Result

$$\hat{f}_0 = \frac{1}{N/2} \angle \left\{ \sum_{n=0}^{N/2-1} y(n)y(n+N/2) \right\}$$

References

[vantrees2968] H.L. Van Trees, "Detection, Estimation, and Modulation Theory", Part 1, 1968

[costas1956] J. Costas, “Synchronous Communications”, Proceedings of the IRE, Dec. 1956

[viterbi1974] A. Viterbi, "Non-linear estimation of PSK-modulated carrier phase with application to burst digital transmissions", IEEE Trans. on Information Theory, Jul. 1983

[yang2003] Y. Wang, E. Serpedin, and P. Ciblat, “Optimal blind nonlinear least-squares carrier phase and frequency offset estimation for general QAM modulations”, IEEE Trans. on Wireless Communications, Sep. 2003

[schmidl&cox1997] T. Schmidl and D. Cox, “Robust frequency and timing synchronization for OFDM”, IEEE Trans. on Communications, Dec. 1997

[mengali1997] U. Mengali and A. d’Andrea, "Synchronisation techniques for digital receivers", Plenum Press, 1997

[meyr1998] H. Meyr et al., " Digital communications receivers", Wiley, 1998

[selmi2011] M. Selmi, “Advanced signal processing tools for QAM-based optical fiber communications”, PHD thesis, Telecom ParisTech, 2011

[ghogho2014] M. Ghogho, P. Ciblat, and A. Swami, “Synchronization (chap 2, p.9-94), Communications and Radar Signal Processing (vol.2)”, Academic Press Library in Signal Processing, Elsevier, 2014