Chapter 2

Exponential distribution

J'ai une mémoire admirable, j'oublie tout. [1]

Alphonse Allais (1854 – 1905).

We start with the definition and the main properties of the exponential distribution, which is key to the study of Poisson and Markov processes.

2.1. Definition

We say that a non-negative random variable X has the exponential distribution with parameter $\lambda > 0$ if:

$$\mathrm{P}(X > t) = e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+.$$

The density of this distribution is given by:

$$f(t) = \lambda e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+.$$

The mean and variance of X are respectively given by:

$$\mathrm{E}(X) = \int_0^\infty t f(t)\,\mathrm{d}t = \frac{1}{\lambda}, \quad \mathrm{var}(X) = \int_0^\infty t^2 f(t)\,\mathrm{d}t - \mathrm{E}(X)^2 = \frac{1}{\lambda^2}.$$

1. *I have an admirable memory, I forget everything*.

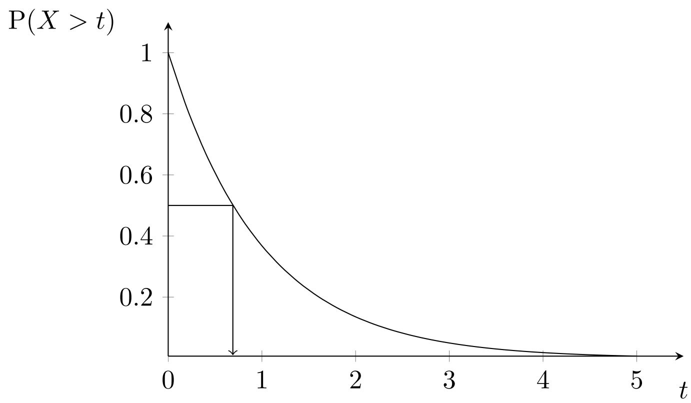

Figure 2.1: Exponential distribution with parameter $\lambda = 1$ and half-life.

The exponential distribution is used for instance in physics to represent the lifetime of a particle, the parameter λ representing the rate at which the particle ages. The *half-life* of the particle is defined as the time t such that $\mathrm{P}(X > t) = 1/2$, that is $t = \ln(2)/\lambda$, as illustrated by figure 2.1.

2.2. Discrete analogue

The exponential distribution is in continuous time what the geometric distribution is in discrete time. A positive integer random variable X has the geometric distribution with parameter $p \in (0, 1]$ if:

$$\mathrm{P}(X = n) = p(1-p)^{n-1}, \quad \forall n \geq 1,$$

or, equivalently, if:

$$\mathrm{P}(X > n) = (1-p)^n, \quad \forall n \in \mathbb{N}.$$

The mean and variance of X are respectively given by:

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} np(1-p)^{n-1} = \frac{1}{p},$$

$$\mathrm{var}(X) = \sum_{n=1}^{\infty} n^2 p(1-p)^{n-1} - \mathrm{E}(X)^2 = \frac{1-p}{p^2}.$$

Thus if p represents the probability of winning the lottery, X gives the distribution of the number of attempts necessary to win. When p is low, the geometric distribution is close to the exponential distribution, as illustrated by figure 2.2.

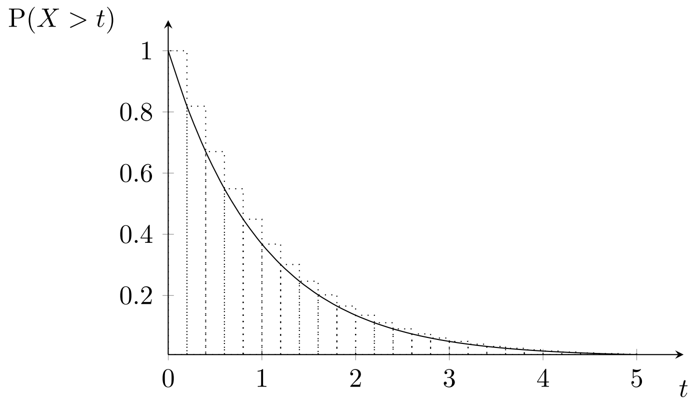

Figure 2.2: Approximation of the exponential distribution by the geometric distribution.

Formally, denote by $X^{(\tau)}$ a geometric random variable with parameter $p^{(\tau)} = \lambda\tau$, where λ is a fixed, positive parameter, and τ a sufficiently small time step. When τ tends to zero, the real random variable $X^{(\tau)}\tau$ tends in distribution to an exponential random variable with parameter λ:

$$P(X^{(\tau)}\tau > t) = (1 - p^{(\tau)})^{\lfloor \frac{t}{\tau} \rfloor} \to e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+.$$

2.3. An amnesic distribution

The geometric distribution is *memoryless*: the number of attempts necessary to win the lottery is independent of the past attempts. This amnesia property is also satisfied by the exponential distribution and writes:

$$P(X > s + t \mid X > s) = P(X > t), \quad \forall s, t \in \mathbb{R}_+.$$

This is illustrated by figure 2.3: conditionally on the event $X > s$, the random variable $X - s$ has an exponential distribution with parameter λ.

Denoting by $F(t) = P(X > t)$ the inverse cumulative distribution function of the random variable X, and observing that for all $s \in \mathbb{R}_+$ such that $F(s) > 0$,

$$P(X > s + t \mid X > s) = \frac{F(s + t)}{F(s)},$$

the amnesia property is equivalent to the functional equation:

$$F(s + t) = F(s)F(t), \quad \forall s, t \in \mathbb{R}_+.$$

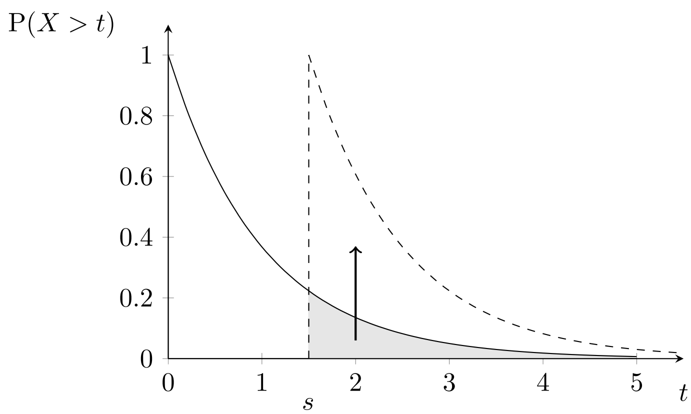

Figure 2.3: Memory-less distribution: the random variable forgets its past.

The exponential functions are the only solutions to this equation. Since $F(0) = 1$ and F is decreasing, there exists a constant $\lambda > 0$ such that:

$$F(t) = e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+.$$

A consequence of this amnesia property is that an exponentially distributed random variable can be described by its behaviour at time $t = 0$. Thus, if X represents the life-time of a particle, this particle "dies" at constant rate λ, independently of its age:

$$\mathrm{P}(X \leq t) = 1 - e^{-\lambda t} = \lambda t + o(t).$$

2.4. Minimum of exponential variables

Let $X_1, X_2, \ldots, X_K$ be K independent exponential random variables, of respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_K$. We denote by λ the sum of these parameters. The minimum X of these random variables satisfies:

$$\begin{aligned}
\mathrm{P}(X > t, X = X_1) &= \mathrm{P}(X_1 > t, X_2 \geq X_1, \ldots, X_K \geq X_1), \\
&= \int_t^\infty \lambda_1 e^{-\lambda_1 s} e^{-\lambda_2 s} \ldots e^{-\lambda_K s}\, \mathrm{d}s, \\
&= \int_t^\infty \lambda_1 e^{-\lambda s}\, \mathrm{d}s, \\
&= \frac{\lambda_1}{\lambda} e^{-\lambda t}, \quad \forall t \in \mathbb{R}_+.
\end{aligned}$$