Locally Sensitive Hashing

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Locally sensitive hashing (LSH) is an approach to searching approximate nearest neighbors in high dimension. The reader can consult the chapter 3 of the textbook of Stanford course on Mining of Massive Datasets for more details on LSH.

1 Principle

Let \( H = \{ h : \mathbb{R}^d \to \{1, \ldots, m\} \} \) be a set of hash functions.

The hash scheme \( H \) is said to be locally sensitive if there exist distances \( d_1 < d_2 \) and probabilities \( p_1 > p_2 \) such that for all \( x, y \in \mathbb{R}^d \):

\[
\begin{align*}
    d(x, y) &\leq d_1 \quad \Rightarrow \quad \Pr(h(x) = h(y)) \geq p_1 \\
    d(x, y) &\geq d_2 \quad \Rightarrow \quad \Pr(h(x) = h(y)) \leq p_2
\end{align*}
\]

where \( h \) is chosen uniformly at random in \( H \).

The idea is that close samples have likely the same hash value (or signature). Note that this property is satisfied whenever \( \Pr(h(x) = h(y)) \) decreases with \( d(x, y) \).

The concatenation of locally sensitive hash functions provide locally sensitive hash functions.

If \( H \) is a locally sensitive hash scheme, then for any \( N < \text{card}(H) \), the hash scheme \( H' = \{(h_1, \ldots, h_N) \in H^N \} \) is locally sensitive.

To prove this result, consider \( x, y \in \mathbb{R}^d \) and \( (h_1, \ldots, h_N) \in H' \):

\[
\Pr((h_1, \ldots, h_N)(x) = (h_1, \ldots, h_N)(y)) = \Pr(h_1(x) = h_1(y)) \ldots \Pr(h_N(x) = h_N(y)) = \Pr(h(x) = h(y))^N.
\]

2 Hash tables

For any hash function \( h : \mathbb{R}^d \to \{1, \ldots, m\} \), a hash table can be built to index a dataset \( x_1, \ldots, x_n \in \mathbb{R}^d \).

The hash table associated with the dataset \( x_1, \ldots, x_n \in \mathbb{R}^d \) is indexed by \( j \in \{1, \ldots, m\} \).

The bucket \( j \) contains all data \( x_i \) (or corresponding indices \( i \)) such that \( h(x_i) = j \).

\(^1\)The book is available online at [http://www.mmds.org](http://www.mmds.org)
For searching the nearest neighbors of a target \( x \), one looks at all data samples in bucket \( j = h(x) \). If several hash tables are built (for different hash functions), the corresponding buckets can be considered in increasing order of size (the smaller buckets, the more specific the corresponding data samples).

## 3 Hash functions

Finally, we present some usual locally sensitive hash functions.

**Bit sampling.** For binary features, the simplest LSH scheme consists in looking at a single (random) bit.

The Bit sampling scheme is \( \mathcal{H} = \{ h(1), \ldots, h(d) \} \) with \( h(j)(x) = x_j \in \{0,1\} \) for all \( j = 1, \ldots, d \).

This hash scheme is locally sensitive for the **Hamming distance**, because for any \( x, y \in \{0,1\}^d \) and any hash function \( h \) chosen uniformly at random in \( \mathcal{H} \),

\[
P(h(x) = h(y)) = \frac{1}{d} \sum_{j=1}^{d} 1\{x_j = y_j\} = 1 - \frac{d(x,y)}{d},
\]

where \( d(x,y) \) is the Hamming distance between \( x \) and \( y \) (number of distinct bits). Thus \( P(h(x) = h(y)) \) decreases with \( d(x,y) \). By concatenation, we get a rich family of locally sensitive hash schemes.

**MinHash.** A popular LSH scheme is MinHash. For any permutation \( \sigma \) of the \( d \) features, we define:

\[
h_\sigma(x) = \min_{j:x_j=1} \sigma(j).
\]

This is the rank of the first bit equal to 1 when the components of \( x \) are read in the order \( \sigma \). Let \( S_d \) be the set of all permutations of \( \{1, \ldots, d\} \).

The MinHash scheme is \( \mathcal{H} = \{ h_\sigma, \sigma \in S_d \} \).

The MinHash scheme is locally sensitive for the **Jaccard distance**, because for any \( x, y \in \{0,1\}^d \),

\[
P(h_\sigma(x) = h_\sigma(y)) = \left( \min_{j:x_j=1} \sigma(j) = \min_{j:y_j=1} \sigma(j) \right) = \frac{\sum_{j=1}^{d} 1\{x_j = 1 \text{ and } y_j = 1\}}{\sum_{j=1}^{d} 1\{x_j = 1 \text{ or } y_j = 1\}} = s(x,y),
\]

where \( s(x,y) \) is the Jaccard similarity between \( x \) and \( y \) (fraction of equal features among expressed features).

A variant of MinHash is 1-bit MinHash, defined by the hash functions \( h_\sigma \mod 2 \). This hash scheme is also locally sensitive for the Jaccard distance, since:

\[
P(h_\sigma(x) = h_\sigma(y) \mod 2) = P(h_\sigma(x) = h_\sigma(y)) + \frac{1}{2} P(h_\sigma(x) \neq h_\sigma(y)) = \frac{1 + s(x,y)}{2}.
\]

Like bit sampling, these hash functions must be concatenated to form interesting hash schemes.
Sign random projection. For any vector \( z \in \mathbb{R}^d \), let:

\[
h_z(x) = 1_{\{z^T x > 0\}}.
\]

If \( z \) is a standard Gaussian vector, we get a LSH scheme.

The Sign Random Projection scheme is \( \mathcal{H} = \{ h_z \text{ with } z \sim \mathcal{N}(0, I_d) \} \).

The Sign Random Projection scheme is locally sensitive for the cosine similarity, because for any \( x, y \in \mathbb{R}^d \),

\[
P(h_z(x) = h_z(y)) = 1 - \frac{x^T y}{\pi},
\]

where \( x^T y \in [0, \pi] \) is the angle between \( x \) and \( y \). In particular, \( P(h_z(x) = h_z(y)) \) increases with the cosine similarity.

Concatenating \( N \) such hash functions gives efficient LSH schemes. This can be considered as the \( 2^N \) discretization of the vector space spanned by the \( N \) random vectors \( z_1, \ldots, z_N \). The fact that this random projection preserves the relative Euclidean distances between data samples for sufficiently large \( N \) (of order \( \log n \) for \( n \) data samples) is known as the Johnson–Lindenstrauss lemma (see the Appendix).

Appendix

The Johnson–Lindenstrauss lemma

Let \( z \sim \mathcal{N}(0, I_d) \) be some standard Gaussian vector. The projection over \( z \) tends to preserve Euclidean distances, in the sense that for any \( x, y \in \mathbb{R}^d \),

\[
(z^T x - z^T y)^2 = (x - y)^T zz^T (x - y).
\]

Taking the expectation, we get:

\[
E((z^T x - z^T y)^2) = (x - y)^T E(zz^T) (x - y) = ||x - y||^2,
\]

showing that \( (z^T x - z^T y)^2 \) is an unbiased estimator of the square Euclidean distance between \( x \) and \( y \).

Now consider \( N \) i.i.d. random vectors \( z_1, \ldots, z_N \sim \mathcal{N}(0, I_d) \). Then the projection over the vector space spanned by \( z_1, \ldots, z_N \) also preserves the relative Euclidean distances. Denoting by \( Z = (z_1, \ldots, z_N) \) the matrix formed by these vectors, we get:

\[
||Z^T x - Z^T y||^2 = (x - y)^T ZZ^T (x - y) = (x - y)^T \left( \sum_{i=1}^{N} z_i z_i^T \right) (x - y),
\]

so that

\[
E(||Z^T x - Z^T y||^2) = (x - y)^T \left( \sum_{i=1}^{N} E(z_i z_i^T) \right) (x - y) = N||x - y||^2.
\]

The square Euclidean distances are preserved in expectation, up to the multiplicative constant \( N \). The Johnson–Lindenstrauss lemma consist in bounding the deviation with respect to this expected value by concentration inequalities [3, 1]. In particular, it is shown that the relative Euclidean distances between \( n \) data samples \( x_1, \ldots, x_n \) are preserved up to some relative error \( \epsilon \) provided \( N \) is of order \( O\left(\frac{\log n}{\epsilon^2}\right) \).
References

