Geometry of Spatial Bipolar Fuzzy Sets Based on Bipolar Fuzzy Numbers and Mathematical Morphology

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Abstract. We propose in this paper new tools for dealing with bipolar fuzzy spatial information: particular geometrical objects are defined, as well as measures such as cardinality and perimeter, represented as bipolar fuzzy numbers. A definition of distance from a point to a bipolar fuzzy set is introduced as well. These definitions are based on mathematical morphology operators, recently proposed in the framework of bipolar fuzzy sets.

1 Introduction

Bipolarity has not been much exploited in the spatial domain yet, although it has many features to manage imprecise and incomplete information that could be interesting in this domain. As highlighted e.g. in [1,2], positive information represents what is guaranteed to be possible, for instance because it has already been observed or experienced, while negative information represents what is impossible or forbidden, or surely false. This paper aims at introducing new geometrical tools for dealing with bipolar fuzzy spatial information. After recalling some definitions in Section 2, we propose to extend the notion of bipolar fuzzy number to define particular geometrical objects such as points, disks and rectangles in Section 3. Geometrical measures such as cardinality (Section 4) and perimeter based on gradient (Section 5) are then proposed. They are defined as bipolar fuzzy numbers, in order to reflect the bipolar and fuzzy nature of the objects. Finally, we introduce a definition of distance from a point to a bipolar fuzzy set in Section 6. These definitions are based on mathematical morphology operators, recently proposed in the framework of bipolar fuzzy sets [3,4].

2 Background: Bipolar Fuzzy Numbers and Spatial Bipolar Fuzzy Sets

Interval-valued fuzzy numbers and intervals have been defined in [5,6]. Similarly, we define a bipolar fuzzy number as a pair of fuzzy sets μ and ν such that μ and $1-\nu$ are fuzzy numbers and $\forall \alpha \in \mathbb{R}\,(or\,\mathbb{N}), \mu(\alpha)+\nu(\alpha) \leq 1$. This definition can be relaxed by allowing $1-\nu$ to be a fuzzy interval (i.e. its core is an interval).

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If both μ and $1-\nu$ are fuzzy intervals, then (μ, ν) will be called bipolar fuzzy interval.

Let us now move to spatial information and let $\mathcal{S}$ denote the spatial domain. It could typically be $\mathbb{R}^n$ or $\mathbb{Z}^n$. Here we consider a finite bounded domain $\mathcal{S}$. A bipolar fuzzy set on $\mathcal{S}$ is defined by a pair of functions (μ, ν) such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. Note that a bipolar fuzzy set is formally equivalent to an intuitionistic fuzzy set [7] or an interval-valued fuzzy set [8], where the interval at each point x is $[\mu(x), 1-\nu(x)]$. For each point x, $\mu(x)$ defines the degree to which x belongs to the bipolar fuzzy set (positive information) and $\nu(x)$ the non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness. Semantically, a bipolar fuzzy set is not one physical object, but may represent information coming from different sources: for instance the positive part may represent observed or preferred positions for a spatial object, while the negative part may represent constraints on this position, and $1-\mu-\nu$ indifferent or neutral positions.

Let $\mathcal{B}$ denote the set of bipolar fuzzy sets on $\mathcal{S}$, $(\mathcal{B}, \preceq)$ is a complete lattice for the partial order defined as: $(\mu_1, \nu_1) \preceq (\mu_2, \nu_2)$ iff $\forall x \in \mathcal{S}, \mu_1(x) \leq \mu_2(x)$ and $\nu_1(x) \geq \nu_2(x)$. The supremum and the infimum are denoted by $\wedge$ and $\vee$, respectively.

An equivalent of the extension principle writes [5,6] $((\mu_1, \nu_1) \circledast (\mu_2, \nu_2))(\gamma) = \vee_{\gamma=\alpha\circledast\beta}(\mu_1, \nu_1)(\alpha) \wedge (\mu_2, \nu_2)(\beta)$, where $\circledast$ denotes any operation. This principle can in particular be applied to define operations on fuzzy numbers or intervals.

On the lattice $(\mathcal{B}, \preceq)$, a dilation is defined as an operation that commutes with the supremum and an erosion as an operation that commutes with the infimum, as shown in our previous work [3,4]. Particular forms, invariant under translation and involving a bipolar fuzzy structuring element, have also been detailed in this work, along with their properties.

3 Bipolar Fuzzy Points, Spheres and Parallelepipeds

We propose to use the notion of bipolar fuzzy number to derive particular geometrical bipolar fuzzy sets on $\mathcal{S}$.

Bipolar fuzzy points, disks (in 2D) or spheres (in 3D) can be defined by applying a rotation invariance principle, while rectangles or parallelepipeds can be defined based on a Cartesian product.

Definition 1. *Let (μ_1, ν_1) be a bipolar fuzzy number, and let $d(x_0, x)$ denote the distance between two points x_0 and x in $\mathcal{S}$ (Euclidean distance, or a discrete version of it for instance). A bipolar fuzzy point is defined as: $\forall x \in \mathcal{S}, (\mu_{x_0}, \nu_{x_0})(x) = (\mu_1(d(x_0, x)), \nu_1(d(x_0, x)))$.*

Definition 2. *Let (μ_1, ν_1) be a bipolar fuzzy interval, and let $d(x_0, x)$ denote the distance between two points x_0 and x in $\mathcal{S}$. A bipolar fuzzy disk (in 2D) or sphere (in 3D) is defined as: $\forall x \in \mathcal{S}, (\mu_D, \nu_D)(x) = (\mu_1(d(x_0, x)), \nu_1(d(x_0, x)))$.*

Definition 3. *Let (μ_i, ν_i) be bipolar fuzzy intervals defined on each axis of the coordinate frame. A bipolar fuzzy rectangle (in 2D) or parallelepiped* *(in 3D) is defined as the Cartesian product of these bipolar fuzzy intervals:* $\forall x \in \mathcal{S}, (\mu_R, \nu_R)(x) = \wedge_i((\mu_i, \nu_i)(x_i)) = (\min_i(\mu_i(x_i)), \max_i(\nu_i(x_i)))$ *where* $x_1, x_2(, x_3)$ *denote the coordinates of* x $(x = (x_1, x_2(, x_3))^t)$ *and* $\wedge$ *denotes the conjunction of bipolar fuzzy sets.*

These definitions extend the notion of fuzzy point [9], fuzzy disk [10] and fuzzy rectangle [10] to the bipolar case. Note that μ_D is a convex fuzzy disk, and ν_D the complement of a convex fuzzy disk. Relaxing the convexity assumption would lead to more general bipolar fuzzy disks and spheres. As for rectangles, the conjunction is expressed as a minimum of the positive parts and the maximum of the negative parts. The positive part is exactly a fuzzy rectangle.

Proposition 1. *Definitions 1, 2 and 3 actually provide bipolar fuzzy sets in* $\mathcal{S}$.

Proposition 2. *If the bipolar fuzzy numbers or intervals involved in Definitions 1–3 are not bipolar (i.e.* $\nu_i = 1 - \mu_i$*), then these definitions provide non-bipolar fuzzy sets and are consistent with the existing definitions in the fuzzy case. If moreover* μ_i *is crisp, then the defined sets are crisp and are points, disks and rectangles in the classical sense.*

An example of bipolar fuzzy disk is shown in Figure 1.

Fig. 1. A bipolar fuzzy disk with its positive part μ (left) and its negative part ν (middle). The indetermination (or neutral area) $\pi = 1 - \mu - \nu$ is shown on the right.

4 Cardinality, Surface and Volume

Let $(\mu, \nu) \in \mathcal{B}$ be a bipolar fuzzy set defined in the spatial domain $\mathcal{S}$. The cardinality of intuitionistic or interval-valued fuzzy sets has been introduced e.g. in [11] as an interval: $[\sum_{x \in \mathcal{S}} \mu(x), \sum_{x \in \mathcal{S}}(1 - \nu(x))]$, with the lower bound representing the classical cardinality of the fuzzy set defining the positive part (the least certain cardinality), and the upper bound the cardinality of the complement of the negative part (i.e. the whole not impossible region is considered, leading to the largest possible cardinality). The length of the interval reflects the indetermination encoded by the bipolar representation. Several authors have used a similar approach, based on interval representations of the cardinality.

When dealing with fuzzy sets, it may be more interesting to consider the cardinality as a fuzzy number, instead as a crisp number, for instance using the extension principle: $|\mu|(n) = \sup\{\alpha \in [0, 1] : |\mu_\alpha| = n\}$, where μ_α denotes α-cuts, defining the degree to which the cardinality of μ is equal to n.

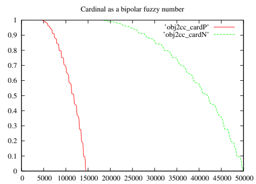


Fig. 2. Cardinality of the bipolar fuzzy set of Figure 3 represented as a bipolar fuzzy number (the negative part, represented by the upper curve is inverted)

Here we propose a similar approach for defining the cardinality of a bipolar fuzzy set as a bipolar fuzzy number, which contrasts with the previously interval-based approaches [4].

Definition 4. *Let* $(\mu, \nu) \in \mathcal{B}$. *Its cardinality is defined as:* $\forall n, |(\mu, \nu)|(n) = (|\mu|(n), 1 - |1 - \nu|(n))$.

Proposition 3. *The cardinality introduced in Definition 4 is a bipolar fuzzy number on* $\mathbb{N}$ *(with* $\forall n, |\mu|(n) + (1 - |1 - \nu|(n)) \leq 1$*).*

In the spatial domain, the cardinality can be interpreted as the surface (in 2D) or the volume (in 3D) of the considered bipolar fuzzy set.

An example is shown in Figure 2, for the bipolar fuzzy object displayed in Figure 3. For this example, the cardinality computed as an interval would provide $[11000, 40000]$, which approximately corresponds to the 0.5-level of the bipolar fuzzy number.

5 Gradient and Perimeter

A direct application of erosion and dilation is the morphological gradient, which extracts boundaries of objects by computing the difference between dilation and erosion, as introduced in [4] for bipolar fuzzy sets.

Definition 5. *Let* (μ, ν) *be a bipolar fuzzy set. We denote its dilation by a bipolar fuzzy structuring element by* (δ^+, δ^-) *and its erosion by* $(\varepsilon^+, \varepsilon^-)$. *We define the bipolar fuzzy gradient as:* $\nabla(\mu, \nu) = (\min(\delta^+, \varepsilon^-), \max(\delta^-, \varepsilon^+))$ *which is the set difference, expressed as the conjunction between* (δ^+, δ^-) *and the negation* $(\varepsilon^-, \varepsilon^+)$ *of* $(\varepsilon^+, \varepsilon^-)$.

Proposition 4. *The bipolar fuzzy gradient has the following properties: (i) Definition 5 defines a bipolar fuzzy set; (ii) If the dilation and erosion are defined using t-representable bipolar t-norms and t-conorms (see [3,4] for details), we have:* $\nabla(\mu, \nu) = (\min(\delta_{\mu_B}(\mu), \delta_{\mu_B}(\nu)), \max(\varepsilon_{1-\nu_B}(\nu), \varepsilon_{1-\nu_B}(\mu)))$. *Moreover, if* (μ, ν) *is not bipolar (i.e.* $\nu = 1 - \mu$*), then the positive part of the gradient is equal to* $\min(\delta_{\mu_B}(\mu), 1 - \varepsilon_{\mu_B}(\mu))$, *which is exactly the morphological gradient in the fuzzy case.*

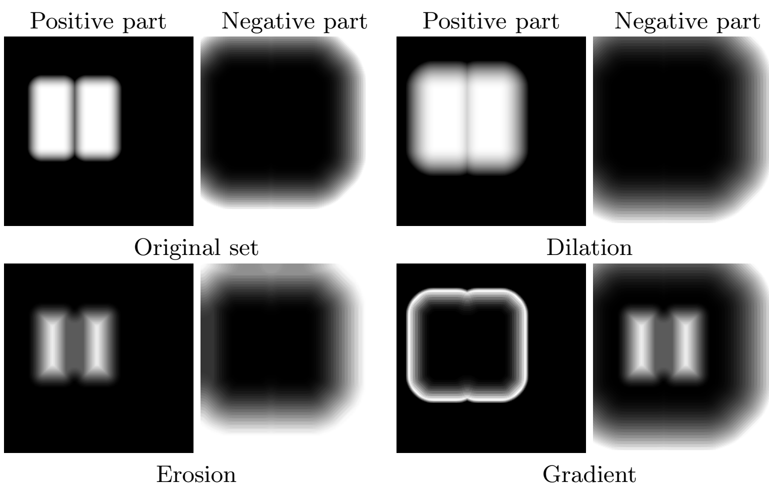


Fig. 3. Gradient of a bipolar fuzzy object

An illustration is displayed in Figure 3. It illustrates both the imprecision (through the fuzziness of the gradient) and the indetermination (through the indetermination between the positive and the negative parts). The object is here somewhat complex, and exhibits two different parts, that can be considered as two connected components to some degree. The positive part of the gradient provides a good account of the boundaries of the union of the two components, which amounts to consider that the region between the two components, which has lower membership degrees, actually belongs to the object. The positive part has the expected interpretation as a guaranteed position and extension of the contours. The negative part shows the level of indetermination in the gradient: the gradient could be larger as well, and it could also include the region between the two components.

Now the perimeter (in 2D) or surface (in 3D) of a bipolar fuzzy set can be derived from the notions of cardinality and gradient. It is then a bipolar fuzzy number. As for the cardinality, this representation is suitable to account for both fuzziness and indetermination. This is a richer representation than a

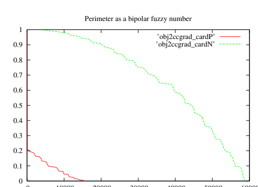


Fig. 4. Perimeter of the bipolar fuzzy set shown in Figure 3 represented as a bipolar fuzzy number (the negative part is inverted), and computed as the cardinality of the gradient

simple number, that could be obtained for instance as a weighted sum of the cardinalities of the cuts, as done in the fuzzy case in [10].

Definition 6. *Let (μ, ν) be a bipolar fuzzy set. Its perimeter (or surface) is defined as the bipolar fuzzy number $|\nabla(\mu, \nu)|$, where the gradient $\nabla(\mu, \nu)$ is given in Definition 5 and the cardinality $|.|$ in Definition 4.*

An example is shown in Figure 4.

6 Distance from a Point to a Bipolar Fuzzy Set

While there is a lot of work on distances and similarity between interval-valued fuzzy sets or between intuitionistic fuzzy sets (see e.g. [12,13]), none of the existing definitions addresses the question of the distance from a point to a bipolar fuzzy set, nor includes the spatial distance in the proposed definitions. As in the fuzzy case [14], we propose to define the distance from a point to a bipolar fuzzy set using a morphological approach. In the crisp case, the distance from a point x to a set X is equal to n iff x belongs to the dilation of size n of X (the dilation of size 0 being the identity), but not to dilations of smaller size (it is sufficient to test this condition for $n-1$ in the discrete case). The transposition of this property to the bipolar fuzzy case leads to the following novel definition, using bipolar fuzzy dilations introduced in [3].

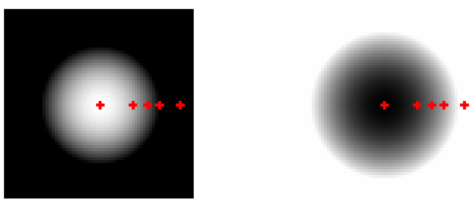
Definition 7. *The distance from a point x of $\mathcal{S}$ to a bipolar fuzzy set (μ, ν) $(\in \mathcal{B})$ is defined as: $d(x, (\mu, \nu))(0) = (\mu(x), \nu(x))$ and $\forall n \in \mathbb{N}^*, d(x, (\mu, \nu))(n) = \delta^n_{(\mu_B, \nu_B)}(x) \wedge c(\delta^{n-1}_{(\mu_B, \nu_B)}(x))$, where c is a complementation (typically the standard negation $c(a, b) = (b, a)$ is used) and $\delta^n_{(\mu_B, \nu_B)}$ denotes n iterations of the dilation, using the bipolar fuzzy set (μ_B, ν_B) as structuring element.*

In order to clarify the meaning of this definition, let us consider the case where the structuring element is not bipolar, i.e. $\nu_B = 1 - \mu_B$. Then the dilation writes (see [3,4] for details): $\delta_{(\mu_B, 1-\mu_B)}(\mu, \nu) = (\delta_{\mu_B}(\mu), \varepsilon_{\mu_B}(\nu))$, where $\delta_{\mu_B}(\mu)$ is the fuzzy dilation of μ by μ_B and $\varepsilon_{\mu_B}(\nu)$ is the fuzzy erosion of ν by μ_B (see [15] for fuzzy mathematical morphology). The bipolar degree to which the distance from x to (μ, ν) is equal to n then writes: $d(x, (\mu, \nu))(n) = (\delta^n_{\mu_B}(\mu) \wedge \varepsilon^{n-1}_{\mu_B}(\nu), \varepsilon^n_{\mu_B}(\nu) \vee \delta^{n-1}_{\mu_B}(\mu))$, i.e. the positive part is the conjunction of the positive part of the dilation of size n (i.e. a dilation of the positive part of the bipolar fuzzy object) and the negative part of the dilation of size $n-1$ (i.e. an erosion of the negative part of the bipolar fuzzy object), and the negative part is the disjunction of the negative part of the dilation of size n (erosion of ν) and the positive part of the dilation of size $n-1$ (dilation of μ).

Proposition 5. *The distance introduced in Definition 7 has the following properties: (i) it is a bipolar fuzzy set on $\mathbb{N}$; (ii) it reduces to the distance from a point to a fuzzy set, as defined in [14], if (μ, ν) and (μ_B, ν_B) are not bipolar (hence the consistency with the classical definition of the distance from a* *point to a set is achieved as well); (iii) the distance is strictly equal to 0 (i.e.* $d(x,(\mu,\nu))(0) = (1,0)$ *and* $\forall n \neq 0, d(x,(\mu,\nu))(n) = (0,1)$*) iff* $\mu(x) = 1$ *and* $\nu(x) = 0$*, i.e.* x *completely belongs to the bipolar fuzzy set.*

An example is shown in Figure 5. The results are in agreement with what would be intuitively expected. The positive part of the bipolar fuzzy number is put towards higher values of distances when the point is moved to the right of the object. After a number n of dilations, the point completely belongs to the dilated object, and the value to which the distance is equal to n', with $n' > n$, becomes $(0,1)$. Note that the indetermination in the membership or non-membership to the object (which is truly bipolar in this example) is also reflected in the distances.

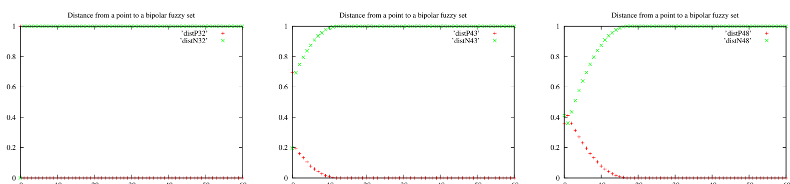
These distances can be easily compared using the extension principle given in Section 2, providing a bipolar degree $d_{\leq}$ to which a distance is less than another one. For the examples in Figure 5, we obtain for instance : $d_{\leq}[d(x_1,(\mu,\nu)) \leq d(x_2,(\mu,\nu))] = [0.69, 0.20]$ where x_i denotes the ith point from left to right in the figure. In this case, since x_1 completely belongs to (μ,ν), the degree



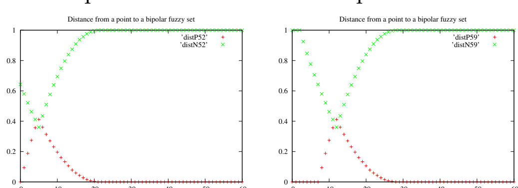
Bipolar fuzzy object:

positive part negative part

Test points in red (numbered 1..5 from left to right)



point 1 point 2 point 3



point 4 point 5

Fig. 5. A bipolar fuzzy set and the distances from 5 different points to it, represented as bipolar fuzzy numbers (positive part: lower curve, negative part: upper curve)

to which its distance is less than the distance from x_2 to (μ, ν) is equal to $[\sup_a d^+(a), \inf_a d^-(a)]$, where d^+ and d^- denote the positive and negative parts of $d(x_2, (\mu, \nu))$. As another example, we have $d_{\leq}[d(x_5, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.03, 0.85]$, reflecting that x_5 is clearly not closer to the bipolar fuzzy set (μ, ν) than x_2.

7 Conclusion

We have shown in this paper how the set of operations on bipolar fuzzy objects (or equivalently interval-valued or intuitionistic fuzzy sets) can be enhanced with new geometrical features, having nice properties. This enriches existing tools developed for image thresholding or edge detection based on intuitionistic or interval-valued fuzzy sets [16,17,13] or for mathematical morphology [3,4,18]. Extensions to other types of operations will be the aim of future work.

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