Similarity between Hypergraphs Based on Mathematical Morphology

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Abstract. In the framework of structural representations for applications in image understanding, we establish links between similarities, hypergraph theory and mathematical morphology. We propose new similarity measures and pseudo-metrics on lattices of hypergraphs based on morphological operators. New forms of these operators on hypergraphs are introduced as well. Some examples based on various dilations and openings on hypergraphs illustrate the relevance of our approach.

Keywords: Hypergraphs, similarity, pseudo-metric, valuation, mathematical morphology on hypergraphs.

1 Introduction

The notion of similarity plays a very important role in various fields of applied sciences. Classification is an example [6], and other examples such as indexing, retrieval or matching demonstrate the usefulness of the concept of similarity [7], with typical applications in image processing and image understanding. A recent trend in these domains is to rely on structural representations of the information (images for instance). Beyond the classical graph representations, and the associated notion of graph similarity, hypergraphs (in which edges can have any cardinality and are then called hyperedges), introduced in the 1960s [23], have recently proved useful. This concept has developed rapidly and has become both a powerful and well-structured mathematical theory for modeling complex situations. This theory is now widely used in sciences as diverse as chemistry, physics, genetics, computer science, psychology... [23], most of them requiring the notions of comparison and similarity measure. In image applications, most similarity measures rely on features computed locally, or among the vertices of an hyperedge, and therefore do not completely exploit the structure of the hypergraph at this level [10,11,14,15].

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In this paper, we propose new tools for defining similarity measures and metrics based on mathematical morphology. In order to deal with structured information, mathematical morphology has been developed on graphs [8,17,21,22], triangulated meshes [16], and more recently on simplicial complexes [9] and hypergraphs [2,3], where preliminary notions of dilation-based similarity were introduced. In this paper, we propose to study similarities on lattices and more specifically on lattices of hypergraphs. We define some of them based on valuations on hypergraphs and mathematical morphology operators. They are illustrated on various types of lattices of hypergraphs, by also introducing new morphological operators, showing the interest of the proposed definitions for achieving robustness with respect to small variations of the compared hypergraphs. This paper is organized as follows. In Section 2 we recall some definitions on hypergraphs and lattices of hypergraphs on which morphological operators are defined. In Section 3, we show some general results on similarities, valuations and pseudo-metrics. Similarity and pseudo-metrics based on mathematical morphology are then defined in Section 4, with a number of illustrative examples.[1]

2 Background and Notations

Basic Concepts on Hypergraphs [5]. A *hypergraph* H denoted by $H = (V, E = (e_i)_{i \in I})$ on a finite set V is a family (which can be a multi-set) $(e_i)_{i \in I}$, (where I is a finite set of indices) of subsets of V called *hyperedges*. Let $(e_j)_{j \in \{1,2,\dots l\}}$ be a sub-family of hyperedges of E. The set of vertices belonging to these hyperedges is denoted by $v(\cup_{j \in \{1,2,\dots l\}} e_j)$, and $v(e)$ denotes the set of vertices forming the hyperedge e. If $\cup_{i \in I} v(e_i) = V$, the hypergraph is without *isolated vertex* (a vertex x is isolated if $x \in V \setminus \cup_{i \in I} v(e_i)$). The set of isolated vertices is denoted by $V_{\setminus E}$. By definition the *empty hypergraph* is the hypergraph $H_\emptyset$ such that $V = \emptyset$ and $E = \emptyset$. A hypergraph is called *simple* if $\forall (i,j) \in I^2, v(e_i) \subseteq v(e_j) \Rightarrow i = j$. The *incidence graph* of a hypergraph $H = (V, E)$ is a bipartite graph $IG(H)$ with a vertex set $S = V \sqcup E$ (where $\sqcup$ stands for the disjoint union), and where $x \in V$ and $e \in E$ are adjacent if and only if $x \in v(e)$. Conversely, to each bipartite graph $\Gamma = (V_1 \sqcup V_2, A)$, we can associate two hypergraphs: a hypergraph $H = (V, E)$, where $V_1 = V$ and $V_2 = E$ and its dual $H^* = (V^*, E^*)$, where $V_1 = E^*$ and $V_2 = V^*$. This notion is useful to prove some results in Section 3.

Mathematical Morphology on Hypergraphs. In [3], we introduced mathematical morphology on hypergraphs. The first step was to define complete lattices on hypergraphs. Then the whole algebraic apparatus of mathematical morphology applies [4,12,13,18,19] and is not recalled here.

We denote the universe of hypergraphs by $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V}$ the set of vertices (that we assume to be finite) and $\mathcal{E}$ the set of hyperedges. The powersets of $\mathcal{V}$

[1] Proofs are omitted because of lack of space.

and $\mathcal{E}$ are denoted by $\mathcal{P}(\mathcal{V})$ and $\mathcal{P}(\mathcal{E})$, respectively. We denote a hypergraph by $H = (V, E)$ with $V \subseteq \mathcal{V}$ and $E \subseteq \mathcal{E}$. As developed in [3], several complete lattices can be built on $(\mathcal{V}, \mathcal{E})$. Let us denote by $(\mathcal{T}, \preceq)$ any of these lattices. We denote by $\wedge$ and $\vee$ the infimum and the supremum, respectively. The least element is denoted by $0_\mathcal{T}$ and the largest element by $1_\mathcal{T}$. Here we will use three examples of complete lattices: $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$, $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$ (which are simply lattices over the power set of vertices and the power set of edges, respectively), and $\mathcal{T}_3 = (\{H\}, \preceq)$ where $\{H = (V, E)\}$ denotes a set of hypergraphs defined on $(\mathcal{V}, \mathcal{E})$ such that $\forall e \in E, v(e) \subseteq V$, and the partial ordering is defined as $\forall (H_1 = (V_1, E_1), H_2 = (V_2, E_2)) \in \mathcal{T}_3^2, H_1 \preceq H_2 \Leftrightarrow V_1 \subseteq V_2 \textit{ and } E_1 \subseteq E_2$ [3]. As shown in [3], we have $H_1 \wedge H_2 = (V_1 \cap V_2, E_1 \cap E_2)$ and $H_1 \vee H_2 = (V_1 \cup V_2, E_1 \cup E_2)$. Examples of dilations on these lattices can be found in [3]. In Section 4, we provide further examples, along with the adjoint erosions, as well as examples of openings.

3 Similarity, Valuation and Pseudo-Metric

3.1 Similarity and Pseudo-metric

A similarity on a set $\mathcal{T}$ is defined as a function from $\mathcal{T} \times \mathcal{T}$ into $[0, 1]$ such that $\forall (x, y) \in \mathcal{T}^2, s(x, y) = s(y, x)$ and $s(x, x) = 1$. We will consider in particular the case where $\mathcal{T}$ is a lattice defined on hypergraphs. From a similarity s, a semi-pseudo-metric can be defined as $\forall (x, y) \in \mathcal{T}^2, d(x, y) = 1 - s(x, y)$. If moreover s satisfies $\forall (x, y, z) \in \mathcal{T}^3, s(x, z) + s(z, y) - 1 \leq s(x, y)$, then d is a pseudo-metric[2].

Proposition 1. *Let w be a positive, monotonous (increasing) real-valued function defined on a lattice $\mathcal{T}$.*

(a) If $\forall (x, y, z) \in \mathcal{T}^3, w(x \wedge y) \leq w(x \wedge z) + w(z \wedge y)$ and $w(x \vee y) \geq \max(w(x \vee z), w(z \vee y))$, then the function d_1 defined as $\forall (x, y) \in \mathcal{T}^2, d_1(x, y) = \frac{w(x \wedge y)}{w(x \vee y)}$ if $w(x \vee y) \neq 0$, and 0 otherwise, is a pseudo-metric.

(b) If $\forall (x, y, z) \in \mathcal{T}^3, w(x \wedge y) \geq w(x \wedge z) + w(z \wedge y)$ and $w(x \vee y) \leq \min(w(x \vee z), w(z \vee y))$, then the function d_2 defined as $\forall (x, y) \in \mathcal{T}^2, d_2(x, y) = 1 - \frac{w(x \wedge y)}{w(x \vee y)}$ if $w(x \vee y) \neq 0$, and 0 otherwise, is a pseudo-metric.

Note that the conditions involved in this proposition are quite strong. In particular, they do not hold for simple valuations such as the cardinality on a graded lattice (see Section 3.2).

Proposition 2. *Under condition (b) above $d(x, y) = w(x \vee y) - w(x \wedge y)$ defines a pseudo-metric.*

[2] For a pseudo-metric, we have $d(x, x) = 0$ but we may have $d(x, y) = 0$ for $x \neq y$, and for a semi-metric the triangular inequality does not necessarily hold. So a semi-pseudo-metric satisfies $\forall (x, y) \in \mathcal{T}^2, d(x, y) = d(y, x), d(x, x) = 0$.

3.2 Valuation on a Lattice $(\mathcal{T}, \preceq)$ and Pseudo-metric

Definition 1. *[1] A valuation w on a lattice $(\mathcal{T}, \preceq)$ is defined as a real-valued function such that: $\forall (x, y) \in \mathcal{T}^2, w(x) + w(y) = w(x \wedge y) + w(x \vee y)$. A valuation is increasing if $\forall (x, y) \in \mathcal{T}^2, x \preceq y \Leftrightarrow w(x) \leq w(y)$.*

In the following we consider only increasing valuations. We have then $\forall x \in \mathcal{T}, w(0_\mathcal{T}) \leq w(x) \leq w(1_\mathcal{T})$, and $\forall (x, y) \in \mathcal{T}^2, w(x \wedge y) \leq w(x \vee y)$. A pseudo-metric can be derived as follows [1].

Theorem 1. *[1] Let w be an increasing valuation on $(\mathcal{T}, \preceq)$. Then d, defined by $\forall (x, y) \in \mathcal{T}^2, d(x, y) = w(x \vee y) - w(x \wedge y)$ is a pseudo-metric. The following inequality also holds: $\forall (a, x, y) \in \mathcal{T}^3, d(a \vee x, a \vee y) + d(a \wedge x, a \wedge y) \leq d(x, y)$.*

Note that this result requires weaker assumptions than the condition (b) used in Proposition 2.

Let us consider the lattice $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ [3]. The cardinality defines an increasing valuation: $\forall V \subseteq \mathcal{V}, w(V) = |V|$. We have $w(V) = 0 \Leftrightarrow V = \emptyset = 0_\mathcal{T}$ and $w(1_\mathcal{T}) = |\mathcal{V}|$. In this case, d is a metric (in particular we have $d(V, V') = 0 \Leftrightarrow V = V'$), and can be expressed as: $\forall (V, V') \in \mathcal{P}(\mathcal{V})^2, d(V, V') = |V \cup V'| - |V \cap V'| = |V| + |V'| - 2|V \cap V'| = |V \Delta V'|$.

Proposition 3. *The lattice $\mathcal{T}_3 = (\{H\}, \preceq)$ is modular.*

Note that if the hypergraphs are supposed to be without isolated vertices, the double partial ordering reduces to inclusion between hyperedge sets and $\mathcal{T}_3$ is isomorphic to $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$ and hence modular (this is derived from the fact that this lattice is distributive). Here we consider the more general case where V can contain isolated vertices.

As shown in [1], on any modular lattice, the height function (i.e. assigning to every $x \in \mathcal{T}$ the least upper bound of the lengths of the chains from $0_\mathcal{T}$ to x) defines a valuation w, leading to a graded lattice. This valuation is strictly monotonous. An interesting property is that $\forall (x, y) \in \mathcal{T}^2$, if y covers x (i.e. $x \prec y$ and $\nexists z \in \mathcal{T}, x \prec z \prec y$) then $w(y) = w(x) + 1$.

Proposition 4. *On $\mathcal{T}_3 = (\{H\}, \preceq)$, the valuation defined by the height function is equal to: $\forall H = (V, E) \in \mathcal{T}_3, w(H) = |V| + |E|$.*

4 Mathematical Morphology and Similarity between Hypergraphs

4.1 Similarity and Dilation

If A and B are $m \times m$ matrices, we denote by $A \circ B$ their entry-wise product, i.e. the matrix whose $m_{i,j}$ entry is $a_{i,j} b_{i,j}$. It is called the *Schur or Hadamard product* of A and B. It is known that if A and B are positive definite, then so is $A \circ B$.

Theorem 2. *Let S be a set and s a similarity on S such that $s(x) \in [0;1] \cap \mathbb{Q}_+$, for all $x \in S$ (where $\mathbb{Q}_+$ denotes the set of positive rational numbers). Let us assume that s can be written as:*

$$\forall (u_i, u_j) \in S^2, i, j \in \{1, \ldots, m\},\ s(u_i, u_j) = \left(\frac{x_{i,j}}{x_i + x_j - x_{i,j}} \right)$$

with $x_{i,j} = x_{j,i}$, $x_{i,i} = x_i$ and $x_j \geq x_{i,j}$ Then the matrix $M(s) = (s(u_i, u_j))_{i,j \in \{1,\ldots,m\}}$ can be written as the Hadamard product of two matrices A and B verifying the following properties:

1. *the matrix A is a semi-positive definite symmetric Cauchy matrix (i.e. having the following form: $A = (a_{ij})_{i,j} = \left(\frac{1}{x_i + x_j}\right)_{i,j}$; $\quad x_i + y_j \neq 0$);*
2. *the matrix B is a matrix defined by the following process: there is a simple hypergraph $H = (V, E)$ with $|E| = m$ and a dilation from $(\mathcal{P}(E), \subseteq)$ into $(\mathcal{P}(V), \subseteq)$ such that $B = (|\delta(e_i) \cap \delta(e_j)|)_{i,j \in \{1,\ldots,m\}}$.*

From this result it is easy to show the following result.

Corollary 1. *Let S be a set and s a similarity on S defined as above. Let us assume that s can be written as:*

$$\forall (u_i, u_j) \in S^2, i, j \in \{1, \ldots, m\},\ s(u_i, u_j) = \left(\frac{2x_{i,j}}{x_i + x_j} \right)$$

with $x_{i,i} = x_i$. Then the matrix $M(s) = (s(u_i, u_j))_{i,j \in \{1,\ldots,m\}}$ can be written as the Hadamard product of two matrices A and B verifying the following properties:

1. *the matrix A is a Cauchy matrix;*
2. *the matrix B is a matrix defined by the following process: there is a simple hypergraph $H = (V, E)$ with $|E| = m$ and a dilation on E such that $B = (|\delta(e_i) \cap \delta(e_j)|)_{i,j \in \{1,\ldots,m\}}$.*

4.2 Similarity from a Valuation and a Morphological Operator

Let us consider any lattice of hypergraphs $(\mathcal{T}, \preceq)$, an increasing valuation w and a morphological operator ψ defined on this lattice. In this section, we generalize ideas suggested in [3] in the particular case where the lattice was the power set of vertices, w was the cardinality and ψ was a dilation.

Definition 2. *Let $(\mathcal{T}, \preceq)$ be a lattice of hypergraphs, w an increasing valuation on this lattice such that $w(x) = 0$ iff $x = 0_{\mathcal{T}}$, and ψ a morphological operator from $(\mathcal{T}, \preceq)$ into $(\mathcal{T}, \preceq)$ such that $\psi(x) = 0_{\mathcal{T}} \Rightarrow x = 0_{\mathcal{T}}$. We define a real-valued function s as: $\forall (x, y) \in \mathcal{T}^2 \setminus (0_{\mathcal{T}}, 0_{\mathcal{T}}), s(x, y) = \frac{w(\psi(x) \wedge \psi(y))}{w(\psi(x) \vee \psi(y))}$ and $s(0_{\mathcal{T}}, 0_{\mathcal{T}}) = 1$.*

Proposition 5. *The function s introduced in Definition 2 is a similarity.*

In a similar way as in Theorem 1, we introduce a pseudo-metric defined from w and ψ.

Proposition 6. *Let w and ψ defined on $(\mathcal{T}, \preceq)$ as in Definition 2. The real-valued function d_ψ defined as: $\forall (x,y) \in \mathcal{T}^2, d_\psi(x,y) = w(\psi(x) \vee \psi(y)) - w(\psi(x) \wedge \psi(y))$ is a pseudo-metric. In the particular case where $\mathcal{T}$ is the power set of the set of vertices or of hyperedges (with $\preceq$ equal to $\subseteq$), and the valuation is the cardinality, then d_ψ is a metric.*

Note that again this result requires weaker assumptions than the condition (b) used in Proposition 2.

The similarity s and the pseudo-metric d_ψ are linked by the following relation: $\forall (x,y) \in \mathcal{T}^2 \setminus (0_\mathcal{T}, 0_\mathcal{T}), 1 - s(x,y) = \frac{d_\psi(x,y)}{w(\psi(x) \vee \psi(y))}$ and $1 - s(0_\mathcal{T}, 0_\mathcal{T}) = d_\psi(0_\mathcal{T}, 0_\mathcal{T}) = 0$. The similarity is then a normalized version of d_ψ. If moreover $w \circ \psi$ satisfies the condition (b) of Proposition 1, then this normalized version is a pseudo-metric.

We also have the following additional properties.

Proposition 7. *Let w and ψ defined on $(\mathcal{T}, \preceq)$ as in Definition 2, and d and d_ψ as in Theorem 1 and Proposition 6.*

- *Two elements of the lattice that are equivalent up to ψ have a zero distance: $\forall (x,y) \in \mathcal{T}^2, \psi(x) = \psi(y) \Rightarrow d_\psi(x,y) = 0$.*
- *If ψ is a morphological filter (i.e. increasing and idempotent), then $\forall (x,y) \in \mathcal{T}^2, x \preceq y \Rightarrow d_\psi(x,y) = w(\psi(y)) - w(\psi(x))$, and $d_{\psi\psi} = d_\psi$.*
- *If ψ is moreover anti-extensive (i.e. ψ is an opening), then $\forall x \in \mathcal{T}, d(x, \psi(x)) = w(x) - w(\psi(x))$. If ψ is extensive (i.e. ψ is a closing), then $\forall x \in \mathcal{T}, d(x, \psi(x)) = w(\psi(x)) - w(x)$.*
- *Let us denote by $Inv(\psi)$ the set of invariants by ψ (i.e. $x \in Inv(\psi) \Leftrightarrow \psi(x) = x$). We have: $\forall (x,y) \in Inv(\psi)^2, d_\psi(x,y) = d(x,y)$.*

The interest of the definitions and results of this section is that similarity and metrics are defined up to a transformation, which makes the results robust to variations of hypergraphs encoded by this transformation. The case where ψ is a filter is then of particular interest.

4.3 Example for a Dilation on $(\mathcal{P}(E), \subseteq)$

Let us first consider the simple example introduced in [3]. For any hypergraph (V, E), we define a dilation δ on $(\mathcal{P}(E), \subseteq)$ as: $\forall A \subseteq E, \delta(A) = \{e \in E \mid v(A) \cap v(e) \neq \emptyset\}$ where $v(A) = \cup_{e' \in A} v(e')$. Let $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ be two hypergraphs without empty hyperedge and δ_{E_1} and δ_{E_2} dilations defined on the set of hyperedges of H_1 and H_2, as above. We define a similarity function s by: $\forall A_1 \subseteq E_1, \forall A_2 \subseteq E_2, s(A_1, A_2) = \frac{|\delta_{E_1}(A_1) \cap \delta_{E_2}(A_2)|}{|\delta_{E_1}(A_1) \cup \delta_{E_2}(A_2)|}$, which corresponds to the similarity introduced in Definition 2 for $w = |.|$ and $\psi = \delta$.

Let us consider an example where hypergraphs are defined to represent image information. Vertices are pixels of the image, and hyperedges are subsets of pixels. Let us assume that the two images have the same support, and hence the corresponding hypergraphs have the same set of vertices. Let us denote them by $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$. In this example, the hyperedge were built from color and connectivity relations as follows: we define a neighborhood of each pixel x as $\Gamma_{\alpha,\beta}(x) = \{x' \mid d_C(\mathcal{I}(x), \mathcal{I}(x')) < \alpha \text{ and } d_N(x, x') \leq \beta\}$, were d_C denotes a distance in the color space (or gray scale), I denotes the color of the intensity function, d_N denotes the distance in the spatial domain and α and β are two parameters to tune the extent of the neighborhood. The set of hyperedges is then defined as the set of $\Gamma_{\alpha,\beta}(x)$ for all $x \in V$. A weighted average similarity can be defined as: $s(H_1, H_2) = \frac{1}{2}(\frac{1}{\sum_{e\in E_1} |\delta_{E_1}(e)|} \sum_{e\in E_1} s(e, E_2)|\delta_{E_1}(e)| + \frac{1}{\sum_{e'\in E_2} |\delta_{E_2}(e')|} \sum_{e'\in E_2} s(e', E_1)|\delta_{E_2}(e')|)$ where $s(e, E_2) = \max_{e'\in E_2} \frac{|\delta_{E_1}(e)\cap\delta_{E_2}(e')|}{|\delta_{E_1}(e)\cup\delta_{E_2}(e')|}$ and a similar expression for $s(e', E_1)$. In the example in Figure 1(a), the similarity between the left image and its modification with an additional line is equal to 0.96. The figure on the right illustrates the dissimilarity between the two images. The dilation leads to more robustness to small and non relevant variations in the images (without the dilation, the similarity would be 0.94). Similarly, the similarity is computed between registered x-ray images of normal (b) and pathological (c) lungs, highlighting the pathological region (d). Its value is 0.75 (and 0.61 without dilation).

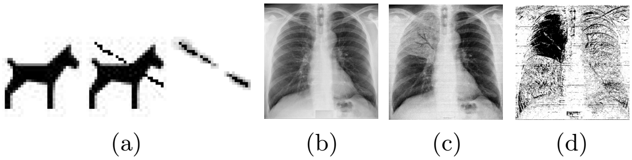

Fig. 1. (a) An image and a modified version where a line has been introduced. The image on the right illustrates the dissimilarity (darkest grey levels). The global similarity is 0.96. Similarity (d) between normal (b) and pathological (c) lungs.

Another example is illustrated in Figure 2, where two images exhibiting some differences are compared. The comparison is illustrated in four sub-images. The similarity is equal to 1 in the top left part, to 0.75 in the top right part, to 0.98 in the bottom left part and to 0.97 in the bottom right part. Again this fits what could be intuitively expected. The global similarity, computed over the whole images, is equal to 0.93. The subdivision (even very simple here) allows us to better localize the differences.

Fig. 2. Two images with some differences, and dissimilarity image

4.4 Example for an Opening on $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$

Let us now consider $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ and $\mathcal{T}_2 = (\mathcal{P}(\mathcal{E}), \subseteq)$. As in [3], we define a dilation from $\mathcal{T}_2$ into $\mathcal{T}_1$ as: $\forall e \in E, B_e = \delta(\{e\}) = \{x \in \mathcal{V} \mid \exists e' \in \mathcal{E}, x \in v(e')$ and $v(e) \cap v(e') \neq \emptyset\} = \cup\{v(e') \mid v(e') \cap v(e) \neq \emptyset\}$, and the dilation of any subset of $\mathcal{E}$ is defined using the sup-generating property. The adjoint erosion ε, from $\mathcal{T}_1$ into $\mathcal{T}_2$ is given by: $\forall V \in \mathcal{P}(\mathcal{V}),\ \varepsilon(V) = \cup\{E \in \mathcal{P}(\mathcal{E}) \mid \forall e \in E,\ \delta(\{e\}) \subseteq V\} = \{e \in \mathcal{E} \mid \forall e' \in \mathcal{E},\ v(e') \cap v(e) \neq \emptyset \Rightarrow v(e') \subseteq V\}$.

Proposition 8. *The opening $\gamma = \delta\varepsilon$ is defined from $\mathcal{T}_1$ into $\mathcal{T}_1$ and is expressed as: $\forall V \in \mathcal{P}(\mathcal{V}), \gamma(V) = \cup\{v(e') \mid \exists e \in \varepsilon(V), v(e') \cap v(e) \neq \emptyset\} = \cup\{B_e \mid v(e) \subseteq V, B_e \subseteq V\}$.*

It is the set of vertices of the hyperedges whose neighbors (as defined by B_e) are in V and vertices of these neighbors. The example in Figure 3 illustrates that vertices that belong to "incomplete" hyperedges (i.e. for which the set of vertices is not completely included in V) are removed. This can be used for filtering hypergraphs by keeping only vertices of complete hyperedges (e_4 and e_5 here), which can be interesting for indexing and retrieval purposes (vertices from incomplete hyperedges being then considered as noise).

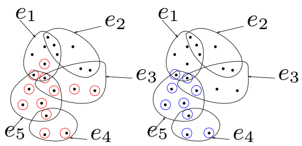

Fig. 3. Example of an opening from $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ into $\mathcal{T}_1 = (\mathcal{P}(\mathcal{V}), \subseteq)$ [3]. The red circled vertices on the left represent V. Its opening is shown in blue on the right.

When computing the similarity $s(V, V') = \frac{|\gamma(V) \cap \gamma(V')|}{|\gamma(V) \cup \gamma(V')|}$, it is clear that if V' differs from V only by vertices from incomplete hyperedges, then $s(V, V') = 1$. The similarity is then robust to noise vertices. In particular $s(V, \gamma(V)) = 1$ since $\gamma(\gamma(V)) = V$. Other examples are shown in Figure 4, which have the same opening as in Figure 3 (right). Hence all these subsets of vertices have a similarity equal to 1 (i.e. they are equivalent up to γ and only differ by their isolated vertices).

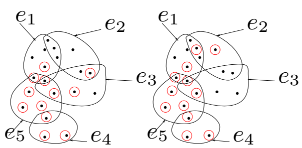

Fig. 4. Two other subsets of $\mathcal{V}$ having the same opening (shown in blue on the right in Figure 3), i.e. vertices of e_4 and e_5

Let us now consider another example of opening.

Proposition 9. *The operator* γ' *from* $\mathcal{T}_1$ *into* $\mathcal{T}_1$ *defined as* $\forall V \in \mathcal{P}(\mathcal{V}), \gamma'(V) = \cup\{v(e) \mid v(e) \subseteq V\}$ *is an opening.*

This opening keeps all vertices of complete hyperedges, i.e. the ones that are "well connected" in the hypergraph. Invariants of γ' are the subsets V that contain only vertices of complete hyperedges. An example is illustrated in Figure 5. The subset V is shown in red on the left and its opening $\gamma'(V) = v(e_5)$ in blue on the right. Note that for this example we have $\gamma(V) = \emptyset$, thus illustrating the difference between γ and γ'.

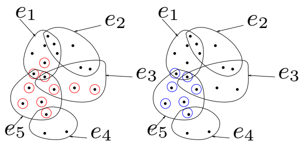

Fig. 5. Subset V (in red) and its opening $\gamma'(V)$ (in blue)

Again this makes the similarity robust to vertices which belong to incomplete hyperedges. We have $s(V, V') = 0$ iff $V \cap V'$ is the set of noise vertices.

If we consider a binary version of the similarity, i.e. V and V' are equivalent iff $\gamma'(V) = \gamma'(V')$, then equivalence classes are built of subsets of $\mathcal{V}$ which contain the vertices of the same complete hyperedges. In particular V and $\gamma'(V)$ belong to the same equivalence class. Using this equivalence relation can be useful for robust indexing and retrieval, for robust entropy definition, etc.

4.5 Example on $\mathcal{T}_3 = (\{H\}, \preceq)$

Let us now consider the most interesting case where $\mathcal{T}$ is the lattice of hypergraphs $\mathcal{T}_3 = (\{H\}, \preceq)$. We again consider opening.

Proposition 10. *The operator* γ_1 *defined for each hypergraph* $H = (V, E)$ *in* $\mathcal{T}_3$ *by* $\gamma_1(H) = (\cup_{e \in E} v(e), E) = (V \setminus V_{\setminus E}, E)$ *is an opening.*

Let us now consider the dilation introduced in [3] on this lattice. The canonical decomposition of H, from its sup generating property, is expressed as: $H = (\vee_{e \in E}(v(e), \{e\})) \vee (\vee_{x \in V_{\setminus E}}(\{x\}, \emptyset))$. From this decomposition, a dilation is defined as: $\forall x \in V_{\setminus E},\ \delta(\{x\}, \emptyset) = (\{x\}, \emptyset)$, for isolated vertices, and for elementary hypergraphs associated with hyperedges: $\forall e \in E,\ \delta(v(e), \{e\}) = (\cup\{v(e') \mid v(e') \cap v(e) \neq \emptyset\}, \{e' \in \mathcal{E} \mid v(e') \cap v(e) \neq \emptyset\})$. The dilation of any H is then derived from its decomposition and from the commutativity of dilation with the supremum.

In the particular case where H has no isolated vertices, then it is sufficient to consider the hyperedges (since the set of vertices is automatically equal to $\cup_{e \in E} v(e)$), and δ can be written in a simpler form as $\delta(\{e\}) = B_e = \{e' \in \mathcal{E} \mid v(e) \cap v(e') \neq \emptyset\}$, and $\delta(E) = \cup_{e \in E}\delta(\{e\}) = \{e' \in E \mid \exists e \in E, v(e') \cap v(e) \neq \emptyset\}$. An example is illustrated in Figure 6.

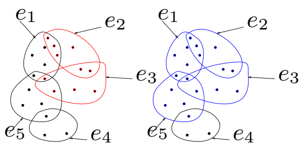

Fig. 6. The figure on the left represents $\mathcal{V}$ (vertices represented as points) and $\mathcal{E}$ (hyperedges represented as closed lines). The red lines indicate the hyperedges of H. The vertices of H are the points enclosed in these lines. The blue lines on the right represent the hyperedges of $\delta(H)$ and its vertices are the points enclosed in these lines. For this example, $\varepsilon(H)$ and $\gamma_2(H)$ are empty.

Proposition 11. *Let us consider hypergraphs without isolated vertices. The adjoint erosion of δ is given by: $\forall E \in \mathcal{P}(\mathcal{E}), \varepsilon(E) = \cup\{E' \in \mathcal{P}(\mathcal{E}) \mid \delta(E') \subseteq E\} = \{e' \in \mathcal{E} \mid B_e \subseteq E\}$ and $\varepsilon(H) = (\cup_{e\in\varepsilon(E)} v(e), \varepsilon(E))$. The opening $\gamma_2 = \delta\varepsilon$ is then $\gamma_2(E) = \cup_{B_e \subseteq E} B_e$, and $\gamma_2(H) = (\cup_{e\in\gamma_2(E)} v(e), \gamma_2(E))$.*

This result is similar to Proposition 8 on the lattices built on vertices.

In Figure 6, the erosion of H shown in red is empty, and the opening is empty as well. In Figure 7, the erosion of H is equal to $(v(e_1), \{e_1\})$ and the opening is $\gamma_2(H) = H$. Another example shows the filtering effect of this opening.

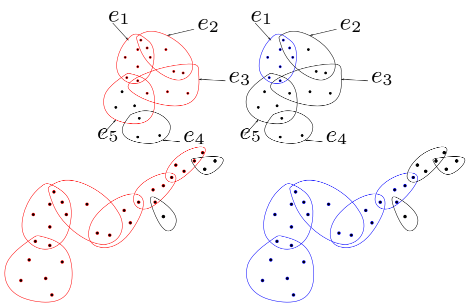

Fig. 7. Top: H is represented on the left using the same conventions as in Figure 6. Its erosion is shown on the right and $\gamma_2(H) = H$. Bottom: H and its opening $\gamma_2(H)$.

If we now consider the more general case where the hypergraphs have isolated vertices, since these are preserved by dilation, they are also preserved in the adjoint erosion and in the derived opening. These isolated vertices do not induce any change when using γ_2 or not for computing the similarity or the distance. For instance if H' is equal to H plus k additional isolated vertices, then $d(H, H') = d_{\gamma_2}(H, H') = k$. Let us now consider as a valuation on $\mathcal{T}_3$ the height. As shown in Proposition 4, we have $\forall H = (V, E) \in \mathcal{T}_3, w(H) = |V| + |E|$. We have: $d(H, \gamma_1(H)) = |V_{\setminus E}|$, which is the number of isolated vertices in H (the distance evaluates the amount of "noise" in H if isolated vertices are interpreted as noise vertices). If H and H' differ only by isolated vertices, then $d_{\gamma_1}(H, H') = 0$. If we consider now γ_2, then the general results expressed in Proposition 7 hold, along with the associated interpretation. Let us give a few simple examples: For the first example of H depicted in Figure 7, $d(H, \gamma_2(H)) = d(H, H) = 0$. For the second example of H, $d(H, \gamma_2(H)) = 4 + 1 = 5$. Two hypergraphs H_1 and H_2 having the same opening by γ_2 are displayed in Figure 8. Hence $d_{\gamma_2}(H_1, H_2) = 0$. Now if k isolated vertices are added to one of the two hypergraphs, their opening will stay the same up to these isolated vertices, and $d_{\gamma_2}(H_1, H_2) = k$.

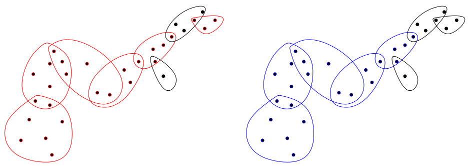

Fig. 8. Two hypergraphs H_1 (left) and H_2 (right). Their openings are $\gamma_2(H_1) = \gamma_2(H_2) = H_2$ and $d_{\gamma_2}(H_1, H_2) = 0$.

5 Conclusion

The proposed framework offers new tools for defining similarity measures and pseudo-metrics, which are robust to variations (encoded by morphological operators) of hypergraphs. They can be incorporated in existing systems for hypergraph-based feature selection, indexing, retrieval, matching. As an example, let us consider the equivalence relation on any lattice of hypergraphs $\mathcal{T}$ defined by $\forall (x, y) \in \mathcal{T}^2, xRy \Leftrightarrow \psi(x) = \psi(y)$ where ψ is a morphological operator on $\mathcal{T}$. This equivalence relation induces a partition of $\mathcal{T}$, denoted by $\mathcal{T} = \cup_i T_i$. A discrete probability distribution can then be defined as $p_i = \frac{|T_i|}{|\mathcal{T}|}$ from which an entropy of $\mathcal{T}$ (up to ψ) can be derived. This defines a new entropic criterion that can be used in feature selection methods such as [24]. Future work aims at exploring other examples of morphological operators in the proposed framework (for instance as the ones defined in [9] on simplicial complexes), and weaker forms of valuations, by considering the sub- or supra-modular cases [20].

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