# Distance to bipolar information from morphological dilation

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**Abstract**

In this paper we propose definitions of distances to bipolar information. This notion is important to find a solution that matches bipolar queries, to exhibit recommendations satisfying preferences of an agent while respecting her constraints, to find an agreement between several agents. All these situations require to be able to compare candidate solutions, via their distance to some bipolar information, and several evaluation measures are proposed as well. We base distances and their comparison on mathematical morphology operators, in particular dilations and skeleton by influence zones.

**Keywords**: Bipolar information, distance, mathematical morphology, dilation, SKIZ, comparison and evaluation of solutions.

## 1. Introduction

Distance is an important component of problems in databases (for queries for instance), information systems, recommendation systems, etc. In this paper we propose definitions of distances to bipolar information. Indeed, bipolar information has become a major research topic in contemporary information processing [1, 2]. It has motivated work in several directions both from theoretical aspects and applicative perspectives (see for instance the papers in [2]).

In several information processing situations, one may have to assess the distance of an element (a candidate solution to a query for instance) to a bipolar information (representing the query). A similar situation occurs if bipolar information represents preferences (positive information) and constraints (negative information). Distances are then useful for selecting solutions in various mono-agent and multi-agent problems. For instance, in a recommendation system, we may want to propose a solution to an agent according to her desires and constraints. A query can also be expressed in a bipolar form, and then distances can be used to evaluate the matching of an element of the database with the query. In such situations, a natural choice is always to choose among the candidates having a minimal distance to the bipolar information. Ideally a good solution should minimize the distance to the positive information and maximize the distance to the negative information (this idea is typically used in classification, pattern recognition or query systems, based on examples and counter-examples, see e.g. [3]). In a multi-agent context, one typical problem is to search for a good compromise between agents' preferences. If the agents are equally considered, a natural choice is to select a solution which is at equal distance to all preferences, i.e. as close as possible to all of them. If the agents are not equally considered, we may want to satisfy as well as possible the agent with the highest priority. If several solutions are found to be equivalent, then a choice among them can be performed using the information provided by the second agent, and so forth. Such situations typically occur in recommendation systems for a group of persons, or in negotiation.

Let us assume that bipolar information is represented by a pair $(\mu, \nu)$, where $\mu$ represents the positive information and $\nu$ the negative information, under a consistency constraint [1], which guarantees that the positive information is compatible with the constraints or rules expressed by the negative information. From a formal point of view, bipolar information can be represented in different settings, depending on the application domain, leading to different forms of $\mu$ and $\nu$, which are all mathematically equivalent [2, 4].

In this paper, we want to define $d(x, (\mu, \nu))$ for $x$ belonging to a space $\mathcal{S}$ on which $\mu$ and $\nu$ are defined (or $d(\omega, (\varphi, \psi))$ for $\omega$ a given world, and $\varphi$ and $\psi$ well-formed formulas representing the preferences and what is forbidden or impossible in a logical setting[1]). Note that $x$ and $\omega$ are elements (worlds) of the considered space, and are neither bipolar nor fuzzy. This problem has not yet been addressed, although it has numerous potential applications as suggested above. Once distances are computed, they have often to be compared. Comparison can concern the distances of different elements to some bipolar information, or the distances of one (or several) elements to several pieces of bipolar information.

The proposed setting is within the framework of mathematical morphology [5] on complete lattices and its extensions to fuzzy sets (see e.g. [6, 7, 8, 9], among others) and to bipolar fuzzy sets (see e.g. [4, 10, 11]), and exploits links between distances and dilations. A short reminder on morphological dila-

[1]The consistency constraint is then expressed as $\varphi \wedge \psi \models \bot$

tions and erosions is presented in Section 2. Two main contributions are detailed next: defining distances is addressed in Section 3, and comparing or evaluating potential solutions via their distance to the bipolar information is addressed in Section 4.

## 2. Dilation and erosion

Let $(\mathcal{T}, \preceq_\mathcal{T})$ be a complete lattice, $\vee$ the supremum and $\wedge$ the infimum[2].

**Definition 1** *[5, 12] A dilation is an operator $\delta$ on $\mathcal{T}$ which commutes with the supremum: $\forall (x_i) \in \mathcal{T}$, $\delta(\vee_i x_i) = \vee_i \delta(x_i)$. An erosion is an operator $\varepsilon$ on $\mathcal{T}$ which commutes with the infimum: $\forall (x_i) \in \mathcal{T}$, $\varepsilon(\wedge_i x_i) = \wedge_i \varepsilon(x_i)$.*

Such operators are called algebraic dilation and erosion. An important property is that they are increasing with respect to $\preceq_\mathcal{T}$.

**Definition 2** *An adjunction on $(\mathcal{T}, \preceq_\mathcal{T})$ is a pair of operators $(\varepsilon, \delta)$ such that $\forall (x, y) \in \mathcal{T}^2$, $\delta(x) \preceq_\mathcal{T} y \Leftrightarrow x \preceq_\mathcal{T} \varepsilon(y)$.*

If $(\varepsilon, \delta)$ is an adjunction, then $\varepsilon$ is an algebraic erosion and $\delta$ an algebraic dilation.

In the particular case of the lattice of subparts of $\mathbb{R}^n$ or $\mathbb{Z}^n$, denoted by $\mathcal{S}$ in the following, endowed with inclusion as partial ordering, adding a property of invariance under translation leads to the particular following forms (called morphological dilation and erosion):

$$\forall X \subseteq \mathcal{S}, \delta_B(X) = \{x \in \mathcal{S} \mid \check{B}_x \cap X \neq \emptyset\},$$

$$\forall X \subseteq \mathcal{S}, \varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\},$$

where $B$ is a subset of $\mathcal{S}$ called structuring element, $B_x$ denotes its translation at point $x$ and $\check{B}$ its symmetrical with respect to the origin of space. Alternatively, $B$ can be considered as a binary relation between elements of $\mathcal{S}$.

Definitions 1 and 2 are general and apply to any complete lattice. Let us consider the lattice $(\mathcal{F}, \preceq_\mathcal{F}, {}^\mathcal{F}\vee, {}^\mathcal{F}\wedge, 0_\mathcal{F}, 1_\mathcal{F})$ of fuzzy sets $\mathcal{F}$ defined on some domain $\mathcal{S}$. We identify fuzzy sets (elements of $\mathcal{F}$) and membership functions (functions from $\mathcal{S}$ into $[0, 1]$) in the following.

**Definition 3** *[6, 7] The dilation of a fuzzy set $\mu$ by a fuzzy structuring element $\nu$ (considered as a binary relation) is defined as:*

$$\forall x \in \mathcal{S}, {}^\mathcal{F}\delta_\nu(\mu)(x) = {}^\mathcal{F}\vee_{y \in \mathcal{S}} t(\nu(y, x), \mu(x)),$$

*where $t$ is a t-norm. The erosion is defined from its residual implication $I$ as:*

$$\forall x \in \mathcal{S}, {}^\mathcal{F}\varepsilon_\nu(\mu)(x) = {}^\mathcal{F}\wedge_{y \in \mathcal{S}} I(\nu(x, y), \mu(x)).$$

[2]The lattice is also denoted by $(\mathcal{T}, \preceq_\mathcal{T}, \vee, \wedge, 0_\mathcal{T}, 1_\mathcal{T})$, where $0_\mathcal{T}$ and $1_\mathcal{T}$ are the smallest and largest elements.

Let us now consider the lattice of bipolar fuzzy sets $(\mathcal{B}, \preceq_\mathcal{B}, {}^\mathcal{B}\wedge, {}^\mathcal{B}\vee, 0_\mathcal{B}, 1_\mathcal{B})$, where a bipolar fuzzy set is defined by two membership functions $\mu$ and $\nu$, representing positive and negative information, such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$.

**Definition 4** *[4, 10] Dilation and erosion of a bipolar fuzzy set $(\mu, \nu)$ by a bipolar fuzzy structuring element $(\mu_B, \nu_B)$ (relation on $\mathcal{S} \times \mathcal{S}$) are then defined as: $\forall x \in \mathcal{S}$,*

$${}^\mathcal{B}\delta_{(\mu_B, \nu_B)}(\mu, \nu)(x) = {}^\mathcal{B}\vee_{y \in \mathcal{S}} \top((\mu_B, \nu_B)(y, x), (\mu, \nu)(y)),$$

$${}^\mathcal{B}\varepsilon_{(\mu_B, \nu_B)}(\mu, \nu)(x) = {}^\mathcal{B}\wedge_{y \in \mathcal{S}} I((\mu_B, \nu_B)(x, y), (\mu, \nu)(y)),$$

*where $\top$ is a bipolar t-norm, $I$ its residual implication, and ${}^\mathcal{B}\vee$ and ${}^\mathcal{B}\wedge$ are defined according to the choice of $\preceq_\mathcal{B}$ (e.g. Pareto, lexicographic...).*[3]

Details on bipolar connectives can be found in [13].

In propositional logics and its fuzzy or bipolar extensions, the equivalence between a formula $\varphi$ and its sets of models $[\![\varphi]\!]$, up to the syntactic equivalence, allows using the previous definitions on sets or fuzzy sets of models, thus working at the semantic level [14]. Dilation and erosion of a formula are then defined via their models as follows: $[\![\delta(\varphi)]\!] = \delta([\![\varphi]\!])$ and $[\![\varepsilon(\varphi)]\!] = \varepsilon([\![\varphi]\!])$, or similar expressions for $(\varphi, \psi)$ representing positive and negative information (since no confusion can occur, the same notations are used for operations on formulas and operations on sets of models).

Since the properties of morphological operators are mostly derived from the algebraic framework of complete lattices and residuated lattices when using $\top$ and $I$, dilations and erosions have the same properties in all the above settings. Details can be found in the mentioned references. Among the important ones, let us mention the increasingness of dilation and erosion, the iterativity property (in particular $n$ dilations of size 1 are equivalent to one dilation of size $n$, denoted $\delta^n$, with $\delta^0 = Id$). Moreover in this paper, we consider structuring elements such that $\forall x \in \mathcal{S}, (\mu_B, \nu_B)(x, x) = (1, 0)$ (i.e. the relation defining the structuring element is strictly reflexive) in order to have extensive dilations.

## 3. Defining distances to a bipolar information

In this section we consider some bipolar information, denoted by $(\mu, \nu)$, defined over some space $\mathcal{S}$. These notations encompass different settings. In particular in a logical framework, $\mathcal{S}$ will be a set of worlds $\Omega$ and $\mu$ and $\nu$ will be membership functions defining the (possibly fuzzy) sets of models of formulas $[\![\varphi]\!]$ and $[\![\psi]\!]$ for positive and negative information, respectively. In the following, we suggest several ways to define the distance $d(x, (\mu, \nu))$ of an element $x$ of $\mathcal{S}$ to the bipolar information.

[3]We will use the same notations ${}^\mathcal{B}\wedge$ and ${}^\mathcal{B}\vee$ in $\mathcal{B}$ and in $\mathcal{L} = \{(a, b) \in [0, 1]^2 \mid a + b \leq 1\}$.

The proposed definitions differ in the manner positive and negative parts of information are taken into account. Their common feature is that they all rely on mathematical morphology, which is an original aspect. Other interesting definitions could be based on the extension principle or other methods, but are out of the scope of this paper.

### 3.1. Joint account of positive and negative information

A first approach was proposed in [4, 15], where the distance is defined from dilations of $(\mu, \nu)$ considered jointly, as a whole. In the crisp case, the distance of a point $x$ to a set $X$ is obtained by the minimal size of the dilation of $X$ such that $x$ is included in the dilated set, where dilations are performed using balls of a distance on $\mathcal{S}$ as structuring elements. The extension to the bipolar case leads to the following definition.

**Definition 5** *Let $(\mu, \nu) \in \mathcal{B}$ be the representation of a bipolar information. Let ${}^{\mathcal{B}}\delta^n$ denote the dilation of size $n$, as introduced in Section 2. In the discrete case, the distance of any element $x \in \mathcal{S}$ to $(\mu, \nu)$ is defined as: $d(x, (\mu, \nu))(0) = (\mu, \nu)(x)$, and $\forall n \in \mathbb{N}^*$,*

$$d(x, (\mu, \nu))(n) = {}^{\mathcal{B}}\delta^n(\mu, \nu)(x) \; {}^{\mathcal{B}}\wedge N({}^{\mathcal{B}}\delta^{n-1}(\mu, \nu))(x)$$

*where $N$ denotes a bipolar complementation. This extends to the continuous case as: $\forall n \in \mathbb{R}^{+*}$,*

$$d(x, (\mu, \nu))(n) =$$

$${}^{\mathcal{B}}\wedge_{n'<n}({}^{\mathcal{B}}\delta^n(\mu, \nu)(x) \; {}^{\mathcal{B}}\wedge N({}^{\mathcal{B}}\delta^{n'}(\mu, \nu))(x)).$$

In this definition, positive and negative parts of information are taken into account simultaneously, in a way that may depend on the chosen partial ordering. For instance, using Pareto ordering, the negative and positive parts play symmetrical roles, while a priority is given to one of them (typically to the negative part) if a lexicographic ordering is used. Note that if no imprecision has to be taken into account (i.e. $\mu$ and $\nu$ take only values 0 and 1), then these two orderings are equivalent. In this case, the ordering does not make any difference in the way positive and negative parts of information are taken into account, and this should be performed at a different level, in the morphological operations and distance definitions.

**Proposition 1** *[4] The distance $d(x, (\mu, \nu))$ in Definition 5 is a (fuzzy) bipolar number. It reduces to a classical distance to a set or a fuzzy set if $(\mu, \nu)$ and $(\mu_B, \nu_B)$ are not bipolar. It is strictly equal to 0 if and only if $(\mu, \nu)(x) = (1, 0)$.*

An example in the spatial domain can be found in [4], and is reproduced in Figure 1. The results are in agreement with what would be intuitively expected. The positive part of the bipolar fuzzy number is put towards higher values of distances when the point is moved to the right of the object. After a number $n$ of dilations, the point completely belongs to the dilated object, and the value to which the distance is equal to $n'$, with $n' > n$, becomes $(0, 1) = 0_{\mathcal{B}}$. Note that the indetermination in the membership or non-membership to the object (which is truly bipolar in this example) is also reflected in the distances.

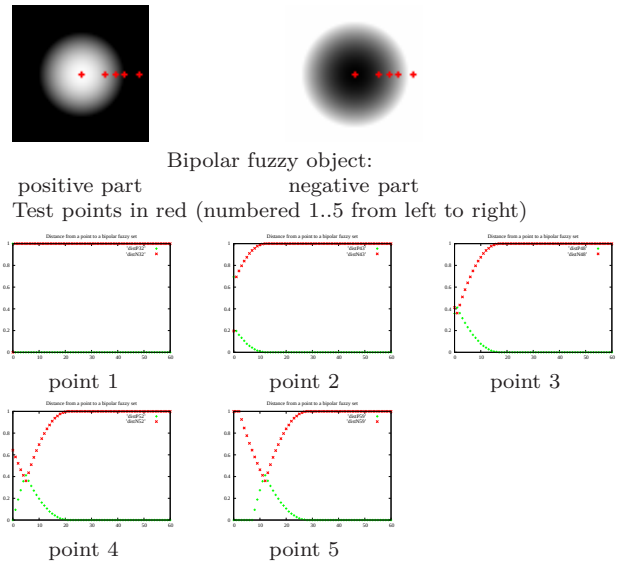


Figure 1: A bipolar fuzzy set (membership degrees are encoded as grey levels) and the distances from 5 different points to it, represented as bipolar fuzzy numbers (the positive part is shown in green and the negative part in red).

### 3.2. Conditioning by the negative information

We now propose a new definition, with an asymmetric account of positive and negative information. More precisely: for the positive information, we keep the metric feature, expressed as dilations while we propose to take the negative information into account using only topology (via conditioning). This results in the following definition. It is given in the discrete case only. Extension to the continuous case is straightforward, as for Definition 5.

**Definition 6** *Let $(\mu, \nu) \in \mathcal{B}$ be the representation of a bipolar information. Let ${}^{\mathcal{F}}\delta^n$ denote the dilation of size $n$, as introduced in Section 2. The distance of any element $x \in \mathcal{S}$ to $(\mu, \nu)$ is defined as: $d(x, (\mu, \nu))(0) = \mu(x) \wedge c(\nu)(x)$ and $\forall n \in \mathbb{N}^*$,*

$$d(x, (\mu, \nu))(n) = {}^{\mathcal{F}}\delta^n(\mu)(x) \wedge c({}^{\mathcal{F}}\delta^{n-1}(\mu))(x) \wedge c(\nu)(x)$$

*where $c$ is a fuzzy complementation.*

This definition provides a (fuzzy) number. The negative part does not undergo any morphological transformation, which is well suited when constraints are considered as strict.

It is interesting to note that successive dilations provide a ranking of the elements $x$, from which a

distance is derived. This defines a stratification of $\mathcal{S}$, based on the following rank function:

$$\forall x \in \mathcal{S}, r(x) = \min\{n \in \mathbb{N} \mid x \in {}^{\mathcal{F}}\delta^n(\mu)\,{}^{\mathcal{F}}\wedge c(\nu)\}.$$

This can be used to choose a solution satisfying an agent's preference. If the agent is ready to extend her preferences (for instance to get an agreement with another agent), then she may choose a higher rank solution. An example of such a situation will be provided in Section 4.2.

### 3.3. Relaxing the negative information

In the same line as above, with an asymmetric account of positive and negative information, but in order to provide more flexibility in the way negative information is considered, we propose to include a metric feature for negative information too. The idea is to erode the constraints, so that the conditioning is less strict.

**Definition 7** *Let $(\mu, \nu) \in \mathcal{B}$ be the representation of a bipolar information. Let ${}^{\mathcal{F}}\delta^n$ denote the dilation of size $n$, as introduced in Section 2. The distance of any element $x \in \mathcal{S}$ to $(\mu, \nu)$ is defined as: $d(x, (\mu, \nu))(0) = \mu(x) \wedge c(\nu)(x)$ and $\forall n \in \mathbb{N}^*$,*

$$d(x, (\mu, \nu))(n) = {}^{\mathcal{F}}\delta^n(\mu)(x) \wedge c({}^{\mathcal{F}}\delta^{n-1}(\mu))(x) \wedge c({}^{\mathcal{F}}\varepsilon^k(\nu))(x).$$

This definition provides an evaluation of the distance as a (fuzzy) number. The negative part of the information can be eroded "at the same speed" as the positive part is dilated (and then $k = n$ in Definition 7), or at a different speed. Typically we may want to allow some limited flexibility and erode constraints slower (for instance, we may erode the negative information less rapidly than dilating the positive information, e.g. $k = n/2$). This additional flexibility allows the constraints to be relaxed, which may lead to a potentially larger set of solutions in problems such as negotiations or recommendation systems.

### 3.4. Geodesic dilations

It may also be useful to guarantee that constraints are satisfied at each step of the dilations. In the following definition, conditioning is then performed at each dilation step, which corresponds to a geodesic dilation in the discrete case, denoted $\delta^n_{c(\nu)}(\mu)$ and computed as follows: $\delta^0_{c(\nu)}(\mu) = \mu \wedge c(\nu)$, $\delta^1_{c(\nu)}(\mu) = \delta^1(\delta^0_{c(\nu)}(\mu)) \wedge c(\nu), \cdots, \delta^n_{c(\nu)}(\mu) = \delta^1(\delta^{n-1}_{c(\nu)}(\mu)) \wedge c(\nu)$, where $\delta^1$ denotes the dilation by an elementary structuring element (of size 1).

**Definition 8** *Let $(\mu, \nu) \in \mathcal{B}$ be the representation of a bipolar information. Let ${}^{\mathcal{F}}\delta^n$ denote the dilation of size $n$, as introduced in Section 2. The distance of any element $x \in \mathcal{S}$ to $(\mu, \nu)$ is defined as: $d(x, (\mu, \nu))(0) = \mu(x) \wedge c(\nu)(x)$ and $\forall n \in \mathbb{N}^*$,*

$$d(x, (\mu, \nu))(n) = {}^{\mathcal{F}}\delta^n_{c(\nu)}(\mu)(x) \wedge c({}^{\mathcal{F}}\delta^{n-1}_{c(\nu)}(\mu))(x).$$

This definition provides an evaluation of the distance as a (fuzzy) number[4]. It guarantees that at each step of dilation, constraints are satisfied. The whole path from $(\mu, \nu)$ to $x$ can be followed while avoiding $\nu$. This is useful when agents are ready to extend progressively their preferences, while following the path of successive potential solutions, all satisfying the constraints.

The example in Figure 2 illustrates Definitions 6 and 8. Conditioning at the end leads to $d(x, (\mu, \nu)) = d(y, (\mu, \nu))$ ($x$ and $y$ are reached after 3 dilations), while computing a geodesic distance (i.e. conditioning at each step of dilation) provides different results ($y$ is still reached after 3 dilations, while 5 are needed to reach $x$).

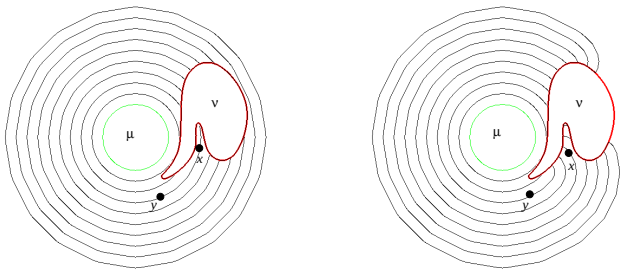


Figure 2: Illustration of Definitions 6 (left) and 8 (right).

### 3.5. Separate account of positive and negative information

Finally, instead of defining the distance as one number (whether it be crisp, fuzzy or bipolar), we propose to separate completely the positive and negative parts of the information, and define two distances, to each of them.

**Definition 9** *Let $(\mu, \nu) \in \mathcal{B}$ be the representation of a bipolar information. The distance of any element $x \in \mathcal{S}$ to $(\mu, \nu)$ is defined as: $d(x, (\mu, \nu)) = (d(x, \mu), d(x, \nu))$, where $d(x, \mu)$ and $d(x, \nu)$ are distances to sets or fuzzy sets.*

In this definition, where distances to $\mu$ and $\nu$ are computed separately (see e.g. [16] for dilation-based fuzzy distances), the evaluation is provided as two (fuzzy) numbers, that can be interpreted as pros and cons.

## 4. Comparison and evaluation of solutions

Once $d(x, (\mu, \nu))$ is defined and can be computed for each $x$, we can compare the obtained values for different elements $x$. This is the aim of this section. We consider two typical problems where such comparisons are useful:

[4]Note that geodesic distances are actually generalized distances since they can take infinite values (for instance if $\nu(x) = 1$, then $d(x, (\mu, \nu))$ is infinite with this definition).

1. "mono-agent" situation: given one $(\mu, \nu)$ defining preferences and constraints, find a best solution that satisfies the preferences while respecting the constraints;
2. "multi-agent" situation: given a set $\{(\mu_i, \nu_i), i \in I\}$ of preferences and constraints expressed for $|I|$ agents, find a solution that satisfies as well as possible all agents' preferences while respecting all constraints.

We propose several approaches to evaluate and compare solutions in these two situations. In the following, if no ambiguity can occur, superscripts indicating the underlying lattice are suppressed to simplify equations.

### 4.1. Mono-agent case

We first consider the mono-agent situation, where bipolar information is represented by $(\mu, \nu) \in \mathcal{B}$.

*Comparison of distances.* In order to assess whether $x$ is better than $x'$, i.e. if $x$ is closest to $(\mu, \nu)$ than $x'$ is, the simplest way is to compare $d(x, (\mu, \nu))$ and $d(x', (\mu, \nu))$. If distances are evaluated as numbers, the comparison is direct. If they are evaluated as fuzzy numbers or bipolar (fuzzy) numbers, then the comparison can be performed by computing the degree to which a distance is lower than another one, thus defining a fuzzy number or a bipolar number $\Delta_<(d(x, (\mu, \nu)), d(x', (\mu, \nu)))$. This has been defined in the fuzzy case in [17] and in the bipolar fuzzy case in [4], either using dilations or using the extension principle[5]. Let us describe the most general case. Using dilations, the idea is that a point $x$ is closer than another one $x'$ to a bipolar information $(\mu, \nu)$ if less dilations of $(\mu, \nu)$ are needed to reach $x$ than to reach $x'$.

**Definition 10** *Let $(\mu, \nu)$ in $\mathcal{B}$ and two elements $x$ and $x'$ of $\mathcal{S}$. The degree to which $d(x, (\mu, \nu))$ is stricly less than $d(x', (\mu, \nu))$ is defined from dilations as: $\Delta^{dil}_<(d(x, (\mu, \nu)) < d(x', (\mu, \nu))) = \vee_{n \in \mathbb{N}} N(I(\delta^n(\mu, \nu)(x), \delta^n(\mu, \nu)(x')))$, where $I$ is a bipolar implication. The degree to which $d(x, (\mu, \nu))$ is less or equal than $d(x', (\mu, \nu))$ is defined from dilations as: $\Delta^{dil}_\leq(d(x, (\mu, \nu)) \leq d(x', (\mu, \nu))) = \wedge_{n \in \mathbb{N}} I(\delta^n(\mu, \nu)(x'), \delta^n(\mu, \nu)(x))$.*

*The degree to which $d(x, (\mu, \nu))$ is stricly less than $d(x', (\mu, \nu))$ is defined from the extension principle as: $\Delta^{ext}_<(d(x, (\mu, \nu)) < d(x', (\mu, \nu))) = \vee_{n<n'} \top(d(x, (\mu, \nu))(n), d(x', (\mu, \nu))(n'))$, where $\top$ is a bipolar t-norm, and the degree to which it is less or equal is defined in a similar way.*

**Proposition 2** *We have $\Delta^{ext}_< \preceq \Delta^{dil}_<$.*

For the examples in Figure 1, we obtain for instance: $\Delta^{ext}_\leq[d(x_1, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.69, 0.20]$ where $x_i$ denotes the i$^{th}$ point from left to right in the figure. In this case, since $x_1$ completely belongs to $(\mu, \nu)$, the degree to which its distance is less than the distance from $x_2$ to $(\mu, \nu)$ is equal to $[\sup_a d^+(a), \inf_a d^-(a)]$, where $d^+$ and $d^-$ denote the positive and negative parts of $d(x_2, (\mu, \nu))$. As another example, we have $\Delta^{ext}_\leq[d(x_5, (\mu, \nu)) \leq d(x_2, (\mu, \nu))] = [0.03, 0.85]$, reflecting that $x_5$ is clearly not closer to the bipolar fuzzy set $(\mu, \nu)$ than $x_2$.

*Comparison of a distance and a model.* We may also want to evaluate a solution with respect to a model, or a constraint (i.e. what we expect as a distance). This can be done by computing a similarity between $d$ and the model, or by pattern matching.

Similarity can be defined in many different ways, and it is out of the scope of this paper to review similarity measures between fuzzy or bipolar numbers. As an example, let us mention the possible use of Definition 10 as $\top(\Delta_\leq(d(x, (\mu, \nu)) \leq d(x', (\mu, \nu))), \Delta_\leq(d(x', (\mu, \nu)) \leq d(x, (\mu, \nu))))$ where $\top$ is a bipolar t-norm.

A bipolar pattern matching can be defined from a bipolar degree of inclusion of one bipolar fuzzy set into the other, and from a bipolar degree of intersection between them, as a direct extension of fuzzy pattern matching. Let $\alpha$ and $\beta$ be two bipolar fuzzy sets (e.g. defined in $\mathbb{R}^+$ if they represent bipolar distances). A degree of inclusion of $\alpha$ in $\beta$ is defined as $\wedge_x I(\alpha(x), \beta(x))$ where $I$ is a bipolar implication, and a degree of intersection is defined as $\vee_x \top(\alpha(x), \beta(x))$ where $\top$ is a bipolar conjunction (these notions are directly used for defining bipolar erosions and dilations as in Section 2).

*Case of separate distances.* Let us now consider Definition 9 where distances to $\mu$ and to $\nu$ are evaluated separately. We then have to compare $(d(x, \mu), d(x, \nu))$ and $(d(x', \mu), d(x', \nu))$. Typically a good solution should correspond to a low value of $d(x, \mu)$ and to a high value of $d(x, \nu)$. If distances to $\mu$ and to $\nu$ are considered as evaluations of pros and cons respectively, then the whole comparison apparatus proposed in [19], where several orderings enjoying good properties are proposed, can be used. Note that this work applies mainly when evaluations are provided as numbers. Extensions to fuzzy numbers could however be derived. Interestingly, here the bipolarity is not taken into account in the computation of distances, but in the evaluation or comparison of solutions.

### 4.2. Multi-agent case

We now move to the multi-agent case, where we assume that each agent $i$ ($i \in I$) expresses preferences and constraints as $(\mu_i, \nu_i) \in \mathcal{B}$. A candidate solution $x \in \mathcal{S}$ is then evaluated as a set $\{d(x, (\mu_i, \nu_i)), i \in I\}$ or $(\{d(x, \mu_i), i \in$

[5] In the fuzzy case, comparison could also be based on the various existing methods in fuzzy sets and possibility theory for comparing fuzzy numbers [18].

$I\}, \{d(x, \nu_i), i \in I\}$). The aim of this section is to provide ways to combine these evaluations and to compare candidate solutions. If the pieces of bipolar information representing the agents' preferences and constraints are consistent, then solutions can be searched for in their conjunction. This will be briefly addressed at the end of this section. If they are not consistent, then some compromise has to be found. This case is more interesting and some solutions are proposed next.

*Fusion.* A first way to proceed is to combine the agent's preferences and constraints using a fusion or aggregation operator, and then compute the distance to the result of the fusion. Similarly, the distance evaluations can be combined using a fusion operator. We then come up with the mono-agent case, and we do not further investigate this approach here.

*Using influence zones and SKIZ.* The skeleton by influence zones (or SKIZ) is a powerful notion in mathematical morphology [5] to define the best separation between objects. Let $\{X_j\}$ be a set of disjoint objects. The influence zone of $X_j$ is defined as the set of points that are strictly closer to $X_j$ than to all other objects. The SKIZ (or generalized Voronoi diagram) is the complement of the union of influence zones (i.e. the set of points equidistant of at least two objects). This notion has been extended to the fuzzy case in [17]. For the purpose of this paper, this notion is interesting since it provides the best compromise in the sense of distances, if all agents are equally considered. Let us first propose an extension of the SKIZ to the bipolar case.

**Definition 11** *Let $(\mu_i, \nu_i) \in \mathcal{B}, \forall i \in I$. The influence zone of $(\mu_i, \nu_i)$ is defined as a bipolar fuzzy set as follows: $\forall x \in \mathcal{S}$,*

$$IZ(\mu_i, \nu_i)(x) = \wedge_{j \neq i} \Delta_<(d(x, (\mu_i, \nu_i)) < d(x, (\mu_j, \nu_j)))$$

*with $\Delta_<$ as in Definition 10.*

*The SKIZ is defined as: $SKIZ(\{(\mu_i, \nu_i), i \in I\}) = N(\vee_{i \in I} IZ(\mu_i, \nu_i))$, where $N$ is a bipolar complementation.*

**Proposition 3** *As in the fuzzy case, a larger $IZ$ is obtained when using $\Delta_<^{dil}$ than when using $\Delta_<^{ext}$.*

Now, let us suggest several options using this notion: (i) compute the SKIZ using $d(x, (\mu, \nu))$ as in Definition 11; (ii) compute the SKIZ using only $\mu$ (using its fuzzy version [17]) and then condition by $c(\nu)$ to guarantee that the solutions satisfy the constraints; (iii) compute the SKIZ using the geodesic dilations, i.e. using the distances in Definition 8; (iv) compute the SKIZ using dilations and associated ranking as suggested in Section 3.2.

Let us illustrate the last option on a non-fuzzy bipolar example (Figure 3). In this case, the influence zones are simply expressed as $IZ(\mu_i, \nu_i) = \{x \in \mathcal{S} \mid \forall j \neq i, r_i(x) < r_j(x)\}$, where $r_i$ is the rank function introduced in Section 3.2, derived from the dilation of $\mu_i$, conditioned by $\nu_i$.

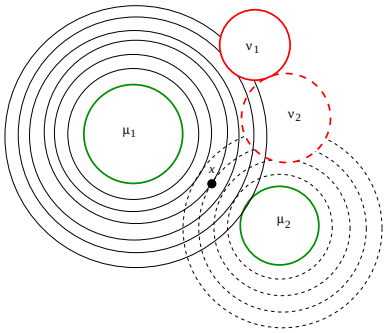


Figure 3: Successive conditional dilations (leading to ranking). The element $x$ is a good compromise between agent preferences, respecting their constraints.

**Proposition 4** *If we interpret distances to the agents' preferences (conditionally to the negative information) as a utility function, the obtained solutions are then Pareto optimal, in the sense that an agent cannot increase its utility without decreasing the one of the other.*

This approach corresponds to negotiation situations where the agents are equally ready to extend their preferences in order to get a solution that leads to the same degree of satisfaction (measured by the rank) for all agents.

*Using successive rankings.* Here the agents are not equally considered, but they are ordered, by some priorities. Finding a solution satisfying the agents' preferences while respecting the constraints of each agent and the priority between them can be performed in successive steps: finding the best solution for one agent; rank further these solutions according to the preferences and constraints of the next one; etc.

An example is illustrated in Figure 4. Two agents have preferences and constraints represented (here in a 2D domain) by $(\mu_1, \nu_1)$ and $(\mu_2, \nu_2)$ (crisp bipolar here). Considering first agent 1, from successive dilations of $\mu_1$ conditionally to $\nu_1$, $x$ is the preferred solution among the four displayed points, while $y, z, t$ are all equivalent, and worse than $x$. Now considering agent 2, $x$ should be excluded since it does not satisfy the constraints $\nu_2$, and $y, z, t$ can be further ranked according to their distance to $\mu_2$: $t$ is better than $y$ which is better than $z$.

*Separate evaluations.* If the distances to $\mu$ and $\nu$ are evaluated separately (Definition 9), for $a \in \mathcal{S}$ we define a set of pros $A^+ = \{d(a, \mu_i), i \in I\}$ and a set of cons $A^- = \{d(a, \nu_i), i \in I\}$. Similarly for $b \in \mathcal{S}$ we define $B^+$ and $B^-$. The problem then amounts to compare $A = (A^+, A^-)$ and $B = (B^+, B^-)$.

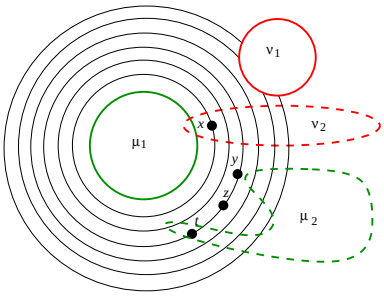


Figure 4: Successive rankings, according to agent 1 and then to agent 2.

Again one of the methods in [19] can be applied, either directly if evaluations are provided as numbers, or based on their extensions to fuzzy numbers.

*Consistent preferences.* Let us now briefly mention the case where agents' preferences are consistent. In the case where $\wedge_i \mu_i \neq \emptyset$, a few additional possibilities can be proposed: (i) compute the last non empty erosion of the conjunction, as suggested in [20] for abductive reasoning, (ii) compute a median set, that can be defined as the SKIZ of $(\mu_1, \nu_1) \wedge (\mu_2, \nu_2)$ and $N((\mu_1, \nu_1) \vee (\mu_2, \nu_2))$ for two agents [17, 21]. This provides a larger set of solutions than just considering the conjunction, that takes a more fair account of each agent's preferences (see example in [21] for mediation in a logical framework); (iii) use the *Discri* order of [19]: if there is no imprecision, all solutions in the conjunction are equivalent; (iv) ranking in the conjunction as a function of the distance to $\vee_i \nu_i$. Note that by construction, $\wedge_i \mu_i$ has no intersection with the $\nu_i$ so any solution in the conjunction satisfies the constraints. The suggestions above allow refining this set of solutions.

### 4.3. Examples in propositional logics

In this section we detail a simple example to illustrate the interest of some of our proposals, when bipolar information is expressed in a logical framework. Examples showing the interest of bipolar modeling in a logical setting, in particular for combining agents' preferences, can be found e.g. in [4, 22].

Let us consider three propositional symbols ($PS = \{a, b, c\}$). The elements of $\Omega$ are represented as the vertices of a cube in the following figures. Figure 5 illustrates the dilation of two agents' preferences (here $[\![\varphi_1]\!] = \{\neg a \neg bc\}$ and $[\![\varphi_2]\!] = \{ab\neg c\}$) to find an agreement between them, conditionally to the constraints ($[\![\psi_1]\!] = [\![\psi_2]\!] = \{\neg abc, a\neg bc\}$). The elementary structuring element is defined as a ball of radius 1 of the Hamming distance between worlds. While the initial preferences do not contain any common model, after two dilations potential solutions have models in $\{a\neg b\neg c, \neg ab\neg c, \neg a\neg b\neg c\}$.

Figure 6 illustrates an example where constraints are not strict and can be eroded. Models of $\varphi$ are

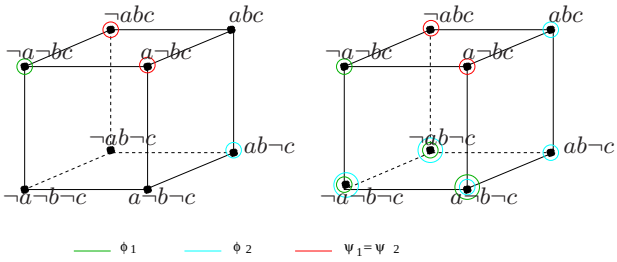


Figure 5: Left: $\varphi_1, \varphi_2, \psi_1 = \psi_2$. Right: $\delta^2(\varphi_1) \wedge \neg\psi_1$, $\delta^2(\varphi_2) \wedge \neg\psi_2$.

circled in green and models of $\psi$ in red. We have $[\![\delta(\varphi) \wedge \neg\varepsilon(\psi)]\!] = \{\neg a\neg bc, a\neg bc, \neg abc, \neg a\neg b\neg c\}$. Note that $a\neg b\neg c$ cannot be reached by geodesic dilations of $\varphi$ conditionally to $\psi$, while it is reached by a dilation of size 2 if $\psi$ is eroded. The ranking also reflects this situation, and it can change if we accept to relax the constraints by eroding $\psi$. The ranking is useful for partial ordering of solutions, retrieval as a ranked list, stratified recommendation systems (recommended choices are partially ordered), etc.

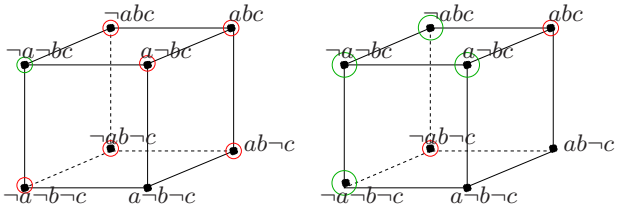


Figure 6: $\delta(\varphi) \wedge \neg\varepsilon(\psi)$.

Let us finally consider an example in the fuzzy case, illustrated in Figure 7. Here the sets of models are fuzzy subsets of $\Omega$. Membership degrees to these sets of models are represented using colors. Table 1 provides the values of successive conditional dilations and distances to $(\varphi, \psi)$ (which are then fuzzy numbers) for the elements of $\Omega$.

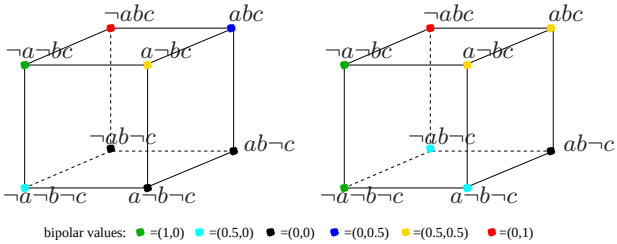


Figure 7: Left: $(\varphi, \psi)$. Right: dilation of size 1 of $\varphi$ conditionally to $\psi$ (see Table 1).

## 5. Conclusion

We proposed several ways to define distances to bipolar information, and methods for evaluating and comparing potential solutions according to these distances. Applications for decision making in mono- or multi-agent contexts are suggested, for information retrieval, queries in databases, negotiations. Future work aims at investigating more properties of the proposed definitions of distances and of

| | $abc$ | $\neg abc$ | $a\neg bc$ | $ab\neg c$ | $\neg a\neg bc$ | $\neg ab\neg c$ | $a\neg b\neg c$ | $\neg a\neg b\neg c$ |
|---|---|---|---|---|---|---|---|---|
| $\varphi$ | 0 | 0 | 0.5 | 0 | 1 | 0 | 0 | 0.5 |
| $\psi$ | 0.5 | 1 | 0.5 | 0 | 0 | 0 | 0 | 0 |
| $\delta^1_{c(\psi)}(\varphi)$ | 0.5 | 0 | 0.5 | 0 | 1 | 0.5 | 0.5 | 1 |
| $\delta^2_{c(\psi)}(\varphi)$ | 0.5 | 0 | 0.5 | 0.5 | 1 | 1 | 1 | 1 |
| $\delta^3_{c(\psi)}(\varphi)$ | 0.5 | 0 | 0.5 | 1 | 1 | 1 | 1 | 1 |
| $\delta^4_{c(\psi)}(\varphi)$ | 0.5 | 0 | 0.5 | 1 | 1 | 1 | 1 | 1 |
| $d(0) = \delta^0_{c(\psi)}(\varphi)$ | 0 | 0 | 0.5 | 0 | 1 | 0 | 0 | 0.5 |
| $d(1) = \delta^1_{c(\psi)}(\varphi) \wedge c(\delta^0_{c(\psi)}(\varphi))$ | 0.5 | 0 | 0.5 | 0 | 0 | 0.5 | 0.5 | 0.5 |
| $d(2) = \delta^2_{c(\psi)}(\varphi) \wedge c(\delta^1_{c(\psi)}(\varphi))$ | 0.5 | 0 | 0.5 | 0.5 | 0 | 0.5 | 0.5 | 0 |
| $d(3) = \delta^3_{c(\psi)}(\varphi) \wedge c(\delta^2_{c(\psi)}(\varphi))$ | 0.5 | 0 | 0.5 | 0.5 | 0 | 0 | 0 | 0 |
| $d(4) = \delta^4_{c(\psi)}(\varphi) \wedge c(\delta^3_{c(\psi)}(\varphi))$ | 0.5 | 0 | 0.5 | 0 | 0 | 0 | 0 | 0 |

Table 1: Values of conditional dilations and fuzzy distances $d(n) = d(\omega, (\varphi, \psi))(n)$ for the example in Figure 7.

the comparison methods, and at exploring the suggested applications as well as new ones, including the choice of the most appropriate definition.

## References


[1] D. Dubois, S. Kaci, and H. Prade. Bipolarity in Reasoning and Decision, an Introduction. In *International Conference on Information Processing and Management of Uncertainty, IPMU'04*, pages 959–966, Perugia, Italy, 2004.

[2] D. Dubois and H. Prade (Eds.). Special issue on bipolar representations of information and preference. *International Journal of Intelligent Systems*, 23(8-10), 2008.

[3] M. De Calmès, D. Dubois, E. Hullermeier, H. Prade, and F. Sedes. Flexibility and fuzzy case-based evaluation in querying: An illustration in an experimental setting. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 11(1):43–66, 2003.

[4] I. Bloch. Mathematical morphology on bipolar fuzzy sets: general algebraic framework. *International Journal of Approximate Reasoning*, 53:1031–1061, July 2012.

[5] J. Serra. *Image Analysis and Mathematical Morphology*. Academic Press, London, 1982.

[6] I. Bloch and H. Maître. Fuzzy Mathematical Morphologies: A Comparative Study. *Pattern Recognition*, 28(9):1341–1387, 1995.

[7] I. Bloch. Duality vs. Adjunction for Fuzzy Mathematical Morphology and General Form of Fuzzy Erosions and Dilations. *Fuzzy Sets and Systems*, 160:1858–1867, 2009.

[8] B. De Baets. Fuzzy Morphology: a Logical Approach. In B. Ayyub and M. Gupta, editors, *Uncertainty in Engineering and Sciences: Fuzzy Logic, Statistics and Neural Network Approach*, pages 53–67. Kluwer Academic, 1997.

[9] M. Nachtegael and E. E. Kerre. Classical and Fuzzy Approaches towards Mathematical Morphology. In E. E. Kerre and M. Nachtegael, editors, *Fuzzy Techniques in Image Processing*, Studies in Fuzziness and Soft Computing, chapter 1, pages 3–57. Physica-Verlag, Springer, 2000.

[10] I. Bloch. Dilation and Erosion of Spatial Bipolar Fuzzy Sets. In *International Workshop on Fuzzy Logic and Applications WILF 2007*, volume LNAI 4578, pages 385–393, Genova, Italy, jul 2007.

[11] P. Sussner, M. Nachtegael, T. Mélange, G. Deschrijver, E. Esmi, and E. Kerre. Interval-Valued and Intuitionistic Fuzzy Mathematical Morphologies as Special Cases of L-Fuzzy Mathematical Morphology. *Journal of Mathematical Imaging and Vision*, 43(1):50–71, 2012.

[12] H. J. A. M. Heijmans and C. Ronse. The Algebraic Basis of Mathematical Morphology – Part I: Dilations and Erosions. *Computer Vision, Graphics and Image Processing*, 50:245–295, 1990.

[13] G. Deschrijver, C. Cornelis, and E. Kerre. On the Representation of Intuitionistic Fuzzy t-Norms and t-Conorms. *IEEE Transactions on Fuzzy Systems*, 12(1):45–61, 2004.

[14] I. Bloch and J. Lang. Towards Mathematical Morpho-Logics. In *8th International Conference on Information Processing and Management of Uncertainty in Knowledge based Systems IPMU 2000*, volume III, pages 1405–1412, Madrid, Spain, 2000.

[15] I. Bloch. A Contribution to the Representation and Manipulation of Fuzzy Bipolar Spatial Information: Geometry and Morphology. In *Workshop on Soft Methods in Statistical and Fuzzy Spatial Information*, pages 7–25, Toulouse, France, sep 2008.

[16] I. Bloch. On Fuzzy Distances and their Use in Image Processing under Imprecision. *Pattern Recognition*, 32(11):1873–1895, 1999.

[17] I. Bloch. Fuzzy Skeleton by Influence Zones - Application to Interpolation between Fuzzy Sets. *Fuzzy Sets and Systems*, 159:1973–1990, 2008.

[18] D. Dubois and H. Prade. Ranking fuzzy numbers in the setting of possibility theory. *Information Sciences*, 30(3):183–224, 1983.

[19] D. Dubois, H. Fargier, and J.F. Bonnefon. On the qualitative comparison of decisions having positive and negative features. *Journal of Artificial Intelligence Research*, 32(1):385–417, 2008.

[20] I. Bloch, R. Pino-Pérez, and C. Uzcátegui. Explanatory Relations based on Mathematical Morphology. In *ECSQARU 2001*, pages 736–747, Toulouse, France, sep 2001.

[21] I. Bloch, R. Pino-Pérez, and C. Uzcategui. Mediation in the Framework of Morphologic. In *European Conference on Artificial Intelligence ECAI 2006*, pages 190–194, Riva del Garda, Italy, 2006.

[22] S. Benferhat, D. Dubois, S. Kaci, and H. Prade. Bipolar Possibility Theory in Preference Modeling: Representation, Fusion and Optimal Solutions. *Information Fusion*, 7:135–150, 2006.