A fuzzy extension of explanatory relations based on mathematical morphology

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Abstract

In this paper, we build upon previous work defining explanatory relations based on mathematical morphology operators on logical formulas in propositional logics. We propose to extend such relations to the case where the set of models of a formula is fuzzy, as a first step towards morphological fuzzy abduction. The membership degrees may represent degrees of satisfaction of the formula, preferences, vague information for instance related to a partially observed situation, imprecise knowledge, etc. The proposed explanatory relations are based on successive fuzzy erosions of the set of models, conditionally to a theory, while the maximum membership degree in the results remains higher than a threshold. Two explanatory relations are proposed, one based on the erosion of the conjunction of the theory and the formula to be explained, and the other based on the erosion of the theory, while remaining consistent with the formula at least to some degree. Extensions of the rationality postulates introduced by Pino-Perez and Uzcategui are proposed. As in the classical crisp case, we show that the second explanatory relation exhibits stronger properties than the first one.

Keywords: Propositional logic with fuzzy models, fuzzy mathematical morphology, explanatory relations, fuzzy abduction.

1. Introduction

In this paper we focus on the approximate flavor of abduction, considering it as an approximate reasoning process. We propose to make an explicit account of imprecision and uncertainty related to this process, via fuzzy representations, in logics where the semantic part handles fuzzy sets of models.

Among the different ways to define abduction (see e.g. [1] or [2, 3] for fuzzy abduction), we focus on the search for minimal and consistent explanations of an observation, relying on the axiomatic approach proposed in [4]. Explicit explanatory relations satisfying the rationality axioms identified in [4] have been proposed in [5, 6], based on operators from the mathematical morphology framework, in particular erosions. We extend this work by considering fuzzy sets of models, and mathematical morphology operators on them. This provides concrete and explicit explanatory relations, which contrasts with most works, where they are implicitly defined via a set of axioms or properties. The proposed approach enjoys interesting properties in terms of rationality properties, and in terms of flexibility both in knowledge representation and in the proposed explanatory relations. In particular it offers the possibility of a tunable compromise between specialization and generalization of the solution.

This paper is organized as follows. In Section 2 we specify the logic considered in this paper (i.e. having fuzzy sets of models). Mathematical morphology operators are then defined on these fuzzy sets in Section 3. In particular morphological erosions are detailed, since there are the basis of the proposed explanatory relations. Two such relations are proposed in Section 4, extending the work in [5] to the fuzzy case. Rationality postulates, as proposed in [4], are expressed in the considered fuzzy context and the two explanatory relations are examined under their light in Section 5.

2. Propositional logics with fuzzy sets of models

Definition 1 *Let us denote by PS a finite set of propositional symbols, and let $a \in PS$. We consider a language generated by PS and the following connectives:* $\varphi ::= a \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \leftrightarrow \varphi) \mid (\varphi \& \varphi) \mid \bot$

In this paper we consider a fuzzy version of propositional logic, by associating with any well formed formula φ a set of models $[\![\varphi]\!]$ that is a fuzzy set, i.e. $[\![\varphi]\!] \in \mathcal{F}$, where $\mathcal{F}$ denotes the set of fuzzy subsets of the set of worlds. This can be achieved in different ways. Here we suggest two simple ones by considering different evaluation functions.

2.1. The basic fuzzy logic BL

BL is the basic fuzzy logic [7]. Let us consider an evaluation function μ, assigning to each propositional variable a a truth value in $\mu(a) \in [0, 1]$, and a continuous t-norm $\star$ with its residuum $\Longrightarrow$. Then we have:

- $\mu(\bot)=0$,
- $\mu(\varphi \to \psi) = (\mu(\varphi) \implies \mu(\psi))$,
- $\mu(\varphi \& \psi) = (\mu(\varphi) \star \mu(\psi))$.

This extends to the other connectives by the following equivalences:

- $\varphi \wedge \psi \Leftrightarrow \varphi \& (\varphi \to \psi)$,
- $\varphi \vee \psi \Leftrightarrow ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$,
- $\neg \varphi \Leftrightarrow \varphi \to \bot$,
- $\varphi \leftrightarrow \psi \Leftrightarrow (\varphi \to \psi) \& (\psi \to \varphi)$.

The truth function of $\wedge$ is then the minimum and the one of $\vee$ is the maximum, regardless of the choice of the t-norm (and its residuum).

A world is defined as the set generated by a given evaluation μ over the finite propositional variables. That is $\omega = \{(\mu(a_i)) \mid i \in \{1, \cdots, |PS|\}\}$ (it can be considered as a point in a $|PS|$-dimensional space, and this representation will be used in the illustrative examples). Let us denote by $\mathcal{M}$ the set of all possible truth evaluations μ, by Φ the set all well-formed formulas generated by the language considered in Definition 1, and by Ω the set of all possible worlds. Considering the semantic mapping defined above, for each well-formed formula $\varphi \in \Phi$ a degree of satisfaction $\mu_\varphi(\omega)$ is associated with each world $\omega \in \Omega$ (i.e. the degree to which $\omega \models \varphi$). The set of models of φ is then $[\![\varphi]\!] = \{(\omega, \mu_\varphi(\omega)) \mid \omega \in \Omega\}$.

2.2. Propositional logic with fuzzy evaluation

Let us now consider the simple case where an evaluation function e assigns to each propositional variable a a truth value $e(a) \in \{0, 1\}$. Then & (conjunction) and $\wedge$ connectives coincide. Furthermore the size of the set of worlds reduces to $2^{|PS|}$ (instead of $[0,1]^{|PS|}$ as in the previous case). Fuzziness can be considered at the reasoning level by defining a membership function μ_φ over the crisp set Ω. This allows for more flexibility by introducing prior knowledge in the reasoning process while in the case of the logic BL, such a flexibility is reduced to the choice of the t-norm.

The membership function μ_φ can be defined in different ways. In this paper we first define crisp subsets of Ω representing the core Ker (where membership values are equal to 1) and the support $Supp$ (where membership values are non zero) of $[\![\varphi]\!]$, with $Ker(\varphi) \subseteq Supp(\varphi)$. Then for $\omega \in Supp(\varphi) \setminus Ker(\varphi)$, its membership is defined as a value in $]0, 1[$, for instance as a decreasing function of a distance measure between ω and $Ker(\varphi)$.

As previously, for each $\omega \in \Omega$, $\mu_\varphi(\omega)$ denotes the degree to which $\omega \models \varphi$. The set of models is defined in the same way: $[\![\varphi]\!] = \{(\omega, \mu_\varphi(\omega)) \mid \omega \in \Omega\}$.

2.3. Towards explanations

Once μ_φ is defined for all well-formed formulas φ, all what follows applies regarless of its definition. Let us denote by $\preceq$ the usual partial ordering on fuzzy sets or equivalently on their membership functions, endowing $(\mathcal{F}, \preceq)$ with a complete lattice structure. Supremum and infimum are denoted by $\vee$ and $\wedge$, respectively (max and min in the finite case).

In the following, we will mainly deal with the semantic part of the logic, and define operators on $\mathcal{F}$. However, it is equivalent to reason on formulas (up to the syntactic equivalence), using the following relations:

- $\vdash \varphi \leftrightarrow \psi \Leftrightarrow \forall \omega \in \Omega, \mu_\varphi(\omega) = \mu_\psi(\omega)$;
- $\vdash \varphi \to \psi \Leftrightarrow \mu_\varphi \preceq \mu_\psi$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$ (which justifies that the same symbol $\wedge$ is used for formulas and for the fuzzy sets of models);
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$.
- In the case of the logic BL: $[\![\varphi \& \psi]\!] = [\![\varphi]\!] \star [\![\psi]\!]$.

Let us now consider a background theory Σ (a consistent set of formulas). Inferring an explanation of an observation φ, i.e. performing abduction, consists in finding a formula γ such that $\Sigma \cup \{\gamma\} \vdash \varphi$. The aim of this paper is to define *preferred* explanations, denoted $\varphi \rhd \gamma$, when the set of models of φ is fuzzy. We propose to define such explanatory relations from mathematical morphology operators. The fact that φ holds under the theory Σ (i.e. $\Sigma \vdash \varphi$) is denoted by $\vdash_\Sigma \varphi$. We look for explanations such that $\Sigma \cup \{\gamma\}$ is consistent, and that are *preferred* according to rationality postulates.

2.4. Examples

As a first example, let us consider a three-valued language version of BL, with two propositional variables $\{a, b\}$, where a truth value in $\{0, 0.5, 1\}$ is assigned to each variable. Let us take the Łukasiewicz t-norm for & and its residual implication:

- $\mu(\varphi \& \psi) = \max\{0, \mu(\varphi) + \mu(\psi) - 1\}$,
- $\mu(\varphi \to \psi) = \min\{1, 1 - \mu(\varphi) + \mu(\psi)\}$.

Their associated truth table is provided in Table 1, along with the membership values of $[\![\varphi]\!]$ for two examples of formulas: $(a \to b) \vee (a \& b)$ and $(a \to b) \wedge (a \& b)$. A graphical representation of $a \to b$ is provided in Figure 1.

a	b	$a \to b$	$a \& b$	$(a \to b) \vee (a \& b)$	$(a \to b) \wedge (a \& b)$
0	0	1	0	1	0
0	0.5	1	0	1	0
0	1	1	0	1	0
0.5	0	0.5	0	0.5	0
0.5	0.5	1	0	1	0
0.5	1	1	0.5	1	0.5
1	0	0	0	0	0
1	0.5	0.5	0.5	0.5	0.5
1	1	1	1	1	1

Table 1: Three-valued Łukasiewicz logic.

As a second example, let us consider three propositional symbols ($PS = \{a, b, c\}$), crisply instantiated. The elements of Ω are represented as the vertices of a cube in Figure 1, and their membership degrees to the sets of models of formulas are represented using colors. For instance in this example $\mu_\varphi(abc) = 1$, $\mu_\varphi(\neg abc) = \mu_\varphi(ab\neg c) = \mu_\varphi(\neg ab\neg c) = 0.8$, $\mu_\varphi(a\neg bc) = \mu_\varphi(\neg a\neg bc) = \mu_\varphi(a\neg b\neg c) = 0.4$, $\mu_\varphi(\neg a\neg b\neg c) = 0$.

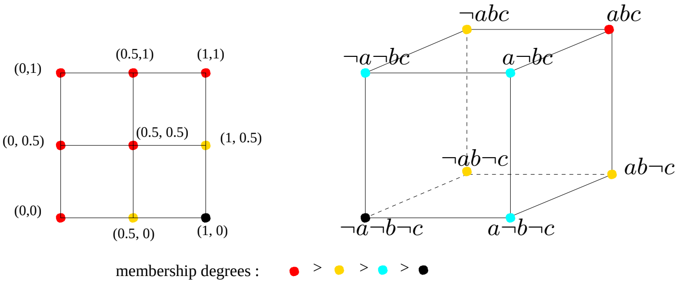

Figure 1: Graphical representation of the models of $a \to b$ in BL (left) and μ_φ in the second example (right). Membership degrees are represented by colors.

3. Fuzzy mathematical morphology on fuzzy sets of models

The main idea is, as proposed in [5] for (crisp) propositional logic, to define explanatory relations from morphological operators, in particular erosions. This allows on the one hand deriving explicit formulations of explanations, and on the other hand defining formally the notion of *preferred* explanation, based on some minimality principles. In this section, we remind the basics of fuzzy mathematical morphology, expressed on $\mathcal{F}$. More details on mathematical morphology and fuzzy mathematical morphology can be found e.g. in [8, 9, 10] and [11, 12, 13, 14, 15, 16, 17], respectively.

3.1. Definitions and properties

The following definitions and properties are derived from the general algebraic framework of mathematical morphology [18, 19].

Definition 2 *In the complete lattice $(\mathcal{F}, \preceq)$, an* erosion *ε is defined as an operator that commutes with the infimum and a* dilation *δ as an operator that commutes with the supremum, i.e. for any family (φ_i) (with $\forall i, [\![\varphi_i]\!] \in \mathcal{F}$):*

$$\varepsilon(\wedge_i [\![\varphi_i]\!]) = \wedge_i \varepsilon([\![\varphi_i]\!]), \quad (1)$$
$$\delta(\vee_i [\![\varphi_i]\!]) = \vee_i \varepsilon([\![\varphi_i]\!]). \quad (2)$$

Let δ be a dilation and ε an erosion, from $(\mathcal{F}, \preceq)$ into $(\mathcal{F}, \preceq)$. These operators induce similar operations on formulas and $\delta(\varphi)$ and $\varepsilon(\varphi)$ are defined via their models as follows: $[\![\delta(\varphi)]\!] = \delta([\![\varphi]\!])$ and $[\![\varepsilon(\varphi)]\!] = \varepsilon([\![\varphi]\!])$ (since no confusion can occur, the same notations are used for operations on formulas and operations on sets of models). We then have for any family (φ_i): $\vdash \varepsilon(\wedge_i\varphi_i) \leftrightarrow \wedge_i\varepsilon(\varphi_i)$ and $\vdash \delta(\vee_i\varphi_i) \leftrightarrow \vee_i\delta(\varphi_i)$.

The membership functions of $[\![\delta(\varphi)]\!]$ and $[\![\varepsilon(\varphi)]\!]$ are denoted by $\mu_{\delta(\varphi)}$ and $\mu_{\varepsilon(\varphi)}$, respectively.

More specifically, particular forms of operators involve a particular fuzzy set, called structuring element. A structuring element can be defined equivalently as a "neighborhood" $V(\omega)$ of each world, or as a binary relation $B(\omega, \omega')$ between worlds, with $\forall(\omega, \omega') \in \Omega^2, \mu_{V(\omega)}(\omega') = \mu_B(\omega, \omega')$, i.e. $V(\omega) = B(\omega, .)$. In the following, without loss of generality, we consider the structuring element as a binary relation, i.e. a fuzzy set of $\Omega \times \Omega$. The set of structuring elements is denoted by $\mathcal{B}$. Dilation is then defined as a degree of intersection and erosion as a degree of inclusion [12].

Definition 3 *Let t be a t-norm and I its residual implication. Let $B \in \mathcal{B}$ a structuring element, with membership function μ_B (i.e. a fuzzy binary relation between worlds). The* morphological erosion *of φ by B is defined as:*

$$\forall \omega \in \Omega, \mu_{\varepsilon_B(\varphi)}(\omega) = \bigwedge_{\omega' \in \Omega} I(\mu_B(\omega, \omega'), \mu_\varphi(\omega')). \quad (3)$$

The morphological dilation *of φ by B is defined as:*

$$\forall \omega \in \Omega, \mu_{\delta_B(\varphi)}(\omega) = \bigvee_{\omega' \in \Omega} t(\mu_B(\omega', \omega), \mu_\varphi(\omega')). \quad (4)$$

This definition extends the morpho-logic operators introduced in [20]. The connectives can be chosen as $\star$ for the conjunction and $\Longrightarrow$ for its residual implication, as in Section 2.1, which amounts to rely on the structure of the residuated lattice $(\mathcal{F}, \vee, \wedge, \star, \Longrightarrow, \bot, \top)$.

Proposition 1 *The operators introduced in Definition 3 are algebraic erosions and dilations, i.e. ε_B commutes with the infimum and δ_B commutes with the supremum.*

In the particular case where B is crisp, then Equations 3 and 4 become:

$$\forall \omega \in \Omega, \mu_{\varepsilon_B(\varphi)}(\omega) = \bigwedge_{\omega' \in \Omega | B(\omega, \omega')=1} \mu_\varphi(\omega'). \quad (5)$$

$$\forall \omega \in \Omega, \mu_{\delta_B(\varphi)}(\omega) = \bigvee_{\omega' \in \Omega | B(\omega', \omega)=1} \mu_\varphi(\omega'). \quad (6)$$

It has been proved in [13] that the conditions on t and I (i.e. being a t-norm and its residual implication) are required to have all usual properties of mathematical morphology (including adjunction and properties of the compositions $\varepsilon\delta$ and $\delta\varepsilon$). However most properties also hold in the fuzzy case with weaker assumptions on t and I. If duality with respect to complementation is also required, then I should also be derived from the dual t-conorm of t. Note that this additional condition strongly reduces the possible choices for t and I, and only Łukasiewicz operators (up to a bijection on the membership degree) can then be used.

Since the proposed explanatory relations rely on erosions using structuring elements, we remind here only the main properties of these operators, that will be used in the following (see [12, 13] for more details on fuzzy mathematical morphology and its properties):

Proposition 2 *Let ε_B be an erosion on $(\mathcal{F}, \preceq)$ by a structuring element B, defined from an implication I as in Definition 3, and its equivalent on formulas. The following properties hold:*

- *independence of the syntax (since definitions are provided via the sets of models): if $\vdash \varphi \leftrightarrow \psi$ then $\vdash \varepsilon_B(\varphi) \leftrightarrow \varepsilon_B(\psi)$ for any structuring element B;*
- *compatibility with the binary case: if B and $[\![\varphi]\!]$ are crisp, then the definitions are equivalent to the ones originally proposed in the crisp case in [20];*
- *increasingness with respect to φ: $\forall(\varphi, \psi) \in \Phi^2$, if $\vdash \varphi \rightarrow \psi$ (i.e. $\mu_\varphi \preceq \mu_\psi$), then $\forall B \in \mathcal{B}, \vdash \varepsilon_B(\varphi) \rightarrow \varepsilon_B(\psi)$ (i.e. $\mu_{\varepsilon_B(\varphi)} \preceq \mu_{\varepsilon_B(\psi)}$);*
- *decreasingness with respect to B: $\forall(B, B') \in \mathcal{B}^2$, if $\mu_B \preceq \mu'_B$, then $\forall \varphi \in \Phi, \vdash \varepsilon_{B'}(\varphi) \rightarrow \varepsilon_B(\varphi)$ (i.e. $\mu_{\varepsilon_{B'}(\varphi)} \preceq \mu_{\varepsilon_B(\varphi)}$);*
- *anti-extensivity if B is reflexive: if $\forall \omega \in \Omega, B(\omega, \omega) = 1$, then $\forall \varphi \in \Phi, \vdash \varepsilon_B(\varphi) \rightarrow \varphi$ (i.e. $\mu_{\varepsilon_B(\varphi)} \preceq \mu_\varphi$);*
- *erosion does not commute with the supremum and only an inclusion holds: $\forall B \in \mathcal{B}, \forall(\varphi, \psi) \in \Phi^2, \vdash \varepsilon_B(\varphi) \vee \varepsilon_B(\psi) \rightarrow \varepsilon_B(\varphi \vee \psi)$ (i.e. $\mu_{\varepsilon_B(\varphi)} \vee \mu_{\varepsilon_B(\psi)} \preceq \mu_{\varepsilon_B(\varphi \vee \psi)}$);*
- *iterativity property: $\forall(B, B') \in \mathcal{B}^2, \forall \varphi \in \Phi, \vdash \varepsilon_B(\varepsilon'_B(\varphi)) \leftrightarrow \varepsilon_{\delta_B(B')}(\varphi)$, where $\mu_{\delta_B(B')}(\omega, \omega') = \bigvee_{\omega'' \in \Omega} t(B'(\omega, \omega''), B(\omega'', \omega'))$ (t is a t-norm).*

3.2. Examples

Let us consider the second example in Section 2.4. We consider structuring elements built from the Hamming distance d_H between worlds, as in [5, 20, 21] ($d_H(\omega, \omega')$ is equal to the number of symbols instantiated differently in ω and ω'). As a first example, we consider crisp structuring elements defined as the balls of this distance. A structuring element of size 1 is then B such that $\mu_B(\omega, \omega') = 1$ if $d_H(\omega, \omega') \leq 1$ and $\mu_B(\omega, \omega') = 0$ otherwise. Denoting by ε^n the erosion of size n, i.e. by a structuring element of size n, we have $\mu_{\varepsilon^n(\varphi)}(\omega) = \wedge_{\omega' | d_H(\omega, \omega') \leq n} \mu_\varphi(\omega')$, and the iterativity property then simply writes: $\vdash \varepsilon^n(\varepsilon^{n'}(\varphi)) \leftrightarrow \varepsilon^{n+n'}(\varphi)$. Note that B is reflexive and the erosion is thus anti-extensive. An example of erosion by B is illustrated in Figure 2 (left). Note that since only the minimum operator is involved in the computation of the membership values, this computation can be done qualitatively and only an ordering of the membership values (here colors) are needed. Let us however detail this example with numerical membership values. The initial formula φ has the following fuzzy set of models: $\mu_\varphi(abc) = 1$, $\mu_\varphi(\neg abc) = \mu_\varphi(ab\neg c) = \mu_\varphi(\neg ab\neg c) = 0.8$, $\mu_\varphi(a\neg bc) = \mu_\varphi(\neg a\neg bc) = \mu_\varphi(a\neg b\neg c) = 0.4$, $\mu_\varphi(\neg a\neg b\neg c) = 0$. The erosion by B has the following fuzzy set of models: $\mu_{\varepsilon_B(\varphi)}(abc) = \mu_{\varepsilon_B(\varphi)}(\neg abc) = \mu_{\varepsilon_B(\varphi)}(ab\neg c) = \mu_{\varepsilon_B(\varphi)}(a\neg bc) = 0.4$, $\mu_{\varepsilon_B(\varphi)}(\neg ab\neg c) = \mu_{\varepsilon_B(\varphi)}(\neg a\neg bc) = \mu_{\varepsilon_B(\varphi)}(a\neg b\neg c) = \mu_{\varepsilon_B(\varphi)}(\neg a\neg b\neg c) = 0$.

Let us now consider a fuzzy structuring element B', with $\mu_{B'}(\omega, \omega') = 1$ if $d_H(\omega, \omega') = 0$, $\mu_{B'}(\omega, \omega') = 0.5$ if $d_H(\omega, \omega') = 1$, and $\mu_{B'}(\omega, \omega') = 0$ otherwise. The result of the fuzzy erosion for the same φ as in Figure 1 (right) is displayed in Figure 2 (right). A larger result is obtained, since $\mu_{B'} \preceq \mu_B$, according to the decreasingness property of the erosion with respect to the structuring element.

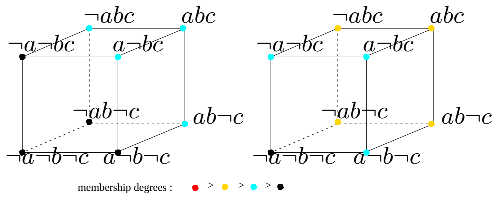

Figure 2: Left: Erosion with a crisp structuring element B (ball of radius 1 of the Hamming distance) of the example in Figure 1 (right). Right: erosion with a fuzzy structuring element B'.

3.3. Partial ordering on Ω

A natural partial ordering on Ω, with respect to a formula φ (or a theory Σ as in the next section) and a structuring element B can be defined from successive erosions. This relies on the fact that, assuming that Ω is connected by B[1], successive erosions (if not equivalent to the identity mapping) lead at some point to inconsistent formulas (with empty set of models). Let us denote by B_0 the trivial structuring element, such that $\forall \omega \in \Omega, \mu_{B_0}(\omega, \omega) = 1$ and $\forall(\omega, \omega') \in \Omega^2 \mid \omega \neq \omega', \mu_B(\omega, \omega') = 0$.

Proposition 3 *Let φ be a formula such that $\exists \omega_0 \in \Omega, \mu_\varphi(\omega_0) = 0$, and B a structuring element such that $B \neq B_0$. Then*

$$\forall \omega \in \Omega, \exists n \in \mathbb{N} \mid \mu_{\varepsilon^n(\varphi)}(\omega) = 0,$$

where $\varepsilon^n(\varphi)$ denotes the erosion of size n of φ by B (i.e. n iterations of the erosion by B), and $\varepsilon^0(\varphi)$ is the identity mapping.

Definition 4 *Let us consider a formula φ such that $\exists \omega_0 \in \Omega, \mu_\varphi(\omega_0) = 0$, and a structuring element B such that $B \neq B_0$. A* rank function *$r_{\varphi,B}$ is defined on Ω as:*

$$\forall \omega \in \Omega, r_{\varphi,B}(\omega) = \min\{n \in \mathbb{N} \mid \mu_{\varepsilon^n(\varphi)}(\omega) = 0\}.$$

[1] i.e. $\forall(\omega, \omega') \in \Omega, \exists(\omega_i)_{i=0\dots n} \mid \omega_0 = \omega, \omega_n = \omega', \forall i < n, \mu_B(\omega_i, \omega_{i+1}) > 0$

This rank function defines a stratification of Ω. Note that $r_{\varphi,B}(\omega_0) = 0$. In order to take membership values into account, a further ordering at each level of the stratification can be provided by the values of $\mu_{\varepsilon^k(\varphi)}(\omega)$, with $k = r_{\varphi,B}(\omega) - 1$, i.e. the last non zero value of ω during the successive erosions.

Let us consider again the example in Figure 1 (right). Erosions of size 2 and 3 are illustrated in Figure 3, and the corresponding stratification is provided in Table 2. Note that using this binary structuring element B, the maximum rank is at most equal to $|PS|$. In this example, at level 1 of the stratification, we can distinguish between $\neg ab\neg c$ which has a higher membership value to $[\![\varepsilon^0(\varphi)]\!] = [\![\varphi]\!]$ than $a\neg b\neg c$ and $\neg a\neg bc$, which refines the ordering.

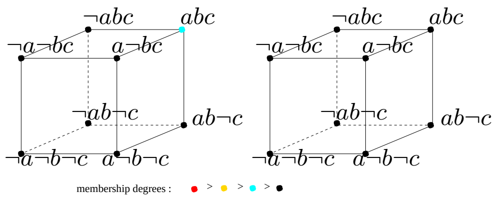

Figure 3: Successive erosions for the example in Figure 1 (right). $\varepsilon^1(\varphi)$ is shown in Figure 2 (left). Left: $\varepsilon^2(\varphi)$. Right: $\varepsilon^3(\varphi)$.

$r_{\varphi,B}$	ω
0	$\neg a\neg b\neg c$
1	$\neg ab\neg c$
	$a\neg b\neg c$, $\neg a\neg bc$
2	$ab\neg c$, $\neg abc$, $a\neg bc$
3	abc

Table 2: Stratification of Ω for the example in Figures 2 (left) and 3.

4. Two explanatory relations

Relying on the morphological operations described above, we now define explanatory relations and formulas γ such that $\Sigma \cup \{\gamma\} \vdash \varphi$, where Σ is a background theory. As in [5, 6], we propose to exploit erosions to define the "most central part" of a formula. This is performed using successive erosions, until a minimality criterion is reached. Two explanatory relations are then derived:

- $\varphi \rhd^{\ell ne} \gamma$: γ is a formula entailing the most central part of $\Sigma \wedge \varphi$;
- $\varphi \rhd^{\ell c} \gamma$: a sequence converging towards the most central part of Σ is defined by successive erosions, and γ is a formula entailing the conjunction of φ with the closest element of the sequence which is consistent with φ.

Note that from a general abduction perspective, the second approach matches the idea that the theory should be modified as least as possible [1, 22].

In the crisp case, the first approach amounts to erode $\Sigma \wedge \varphi$ until it becomes inconsistent, and the second one to erode Σ until it becomes inconsistent with φ. Then the last erosion before these inconsistencies occur defines the set of preferred explanations. In the fuzzy case, inconsistency may be too strong and may lead to last erosions with very low membership values (although non zero). Therefore we suggest to replace the strict inconsistency by a minimality criterion depending on a threshold value α on the membership values in the following definitions. From now on, we assume that erosions are anti-extensive, i.e. performed with a reflexive B.

Definition 5 *The* explanatory relation $\rhd^{\ell ne}$ *is defined as follows: given a threshold value $\alpha \in [0,1]$, for each formula φ, γ is a* preferred explanation *of φ, denoted by $\varphi \rhd^{\ell ne} \gamma$, if $\mu_\gamma \preceq \mu_{\varepsilon^{l_\alpha}(\Sigma\wedge\varphi)}$ (or equivalently in syntactic form: $\vdash_\Sigma \gamma \rightarrow \varepsilon^{l_\alpha}(\Sigma \wedge \varphi)$) and $\exists \omega \in \Omega \mid \alpha < \mu_\gamma(\omega) (\leq \mu_{\varepsilon^{l_\alpha}(\Sigma\wedge\varphi)}(\omega))$, with $l_\alpha = \max\{n \in \mathbb{N} \mid \exists \omega \in \Omega, \mu_{\varepsilon^n(\Sigma\wedge\varphi)}(\omega) > \alpha\}$.*

Note that the strict consistency criterion is obtained for $\alpha = 0$.

An example is displayed in Figure 4. Let us assume that colors correspond to membership degrees 1, 0.8, 0.4 and 0. The last erosion satisfying the minimality criterion for any $\alpha < 0.4$ has a support restricted to abc, with a membership value 0.4, and is obtained after two erosions. The preferred explanations γ are such that $\alpha < \mu_\gamma(abc) \leq 0.4$ and $\forall \omega \neq abc, \mu_\gamma(\omega) = 0$. If a larger value of α is required (e.g. $0.4 \leq \alpha < 0.8$), then the last erosion satisfying the minimality criterion is obtained for an erosion of size 1 and $[\![\gamma]\!]$ should contain at least abc with a degree in $(\alpha, 0.8]$. For $\alpha \geq 0.8$, then the last erosion is obtained for a size 0 (i.e. identity) and $[\![\gamma]\!]$ should contain at least abc with a degree in $(\alpha, 1]$.

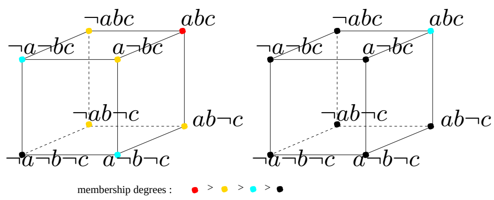

Figure 4: Left: models of $\Sigma \wedge \varphi$. Right: last erosion with a crisp structuring element B (ball of radius 1 of the Hamming distance) for α small enough (see text).

Definition 6 *The* explanatory relation $\rhd^{\ell c}$ *is defined as follows: given a threshold value $\alpha \in [0,1]$, for each formula φ, γ is a* preferred explanation *of φ, denoted $\varphi \rhd^{\ell c} \gamma$, if $\mu_\gamma \preceq \mu_{\varepsilon^{l_\alpha}(\Sigma)\wedge\varphi}$ (or equivalently in syntactic form: $\vdash_\Sigma \gamma \rightarrow \varepsilon^{l_\alpha}(\Sigma) \wedge \varphi$) and $\exists \omega \in \Omega \mid \alpha < \mu_\gamma(\omega)$, with $l_\alpha = \max\{n \in \mathbb{N} \mid \exists \omega \in \Omega, \mu_{\varepsilon^n(\Sigma)\wedge\varphi}(\omega) > \alpha\}$.*

Again the strict consistency criterion is obtained for $\alpha = 0$.

Let us illustrate this definition on the example in Figure 5, where the models of Σ and its successive erosions are shown.

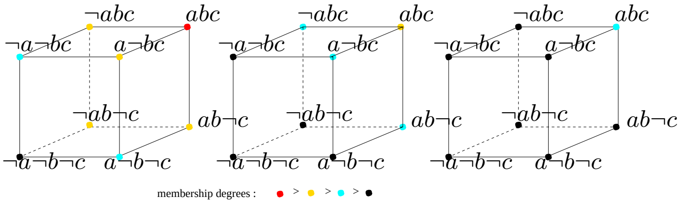

Figure 5: Models of Σ and of its successive erosions with a crisp structuring element B (ball of radius 1 of the Hamming distance).

Table 3 details the models of these erosions and the conjunction with φ. The last consistent erosion satisfying the minimality criterion for $\alpha < 0.4$ is obtained for an erosion of size 1, for $0.4 \leq \alpha < 0.8$ for an erosion of size 0, and the minimality criterion cannot be satisfied for $\alpha \geq 0.8$. For $\alpha < 0.4$, γ should satisfy $\alpha < \mu_\gamma(a\neg bc) \leq 0.4$ or $\alpha < \mu_\gamma(ab\neg c) \leq 0.4$ and $\forall \omega \in \Omega \setminus \{a\neg bc, ab\neg c\}, \mu_\gamma(\omega) = 0$.

Let us now consider an example with non crisply instantiated symbols, as in Section 2.4. Figure 6 illustrates the successive erosions of the formula $a \to b$ with a binary structuring element such that $B(\omega, \omega') = 1$ if $d_H(\omega, \omega') \leq 0.5$ and $B(\omega, \omega') = 0$ otherwise. Let us consider that this formula is $\Sigma \wedge \varphi$. Let us denote by ψ the formula that has a set of models reduced to $(0, 1)$, with membership value 0.5 (yellow dot on the last figure). Then the last erosion satisfying the minimality criterion with $\alpha < 0.5$ is ψ. Hence $\varphi \rhd^{\ell ne} \gamma$ is obtained for γ such that $\alpha < \mu_\gamma(0, 1) \leq 0.5$ and $\mu_\gamma(\omega) = 0$ elsewhere. Now let us consider that $a \to b$ represents Σ, and let us take $\varphi = (b = 1)$. The last consistent erosion satisfying the minimality criterion to a degree $\alpha < 0.5$ is again ψ, and $\psi \wedge \varphi = \psi$. Hence $\varphi \rhd^{\ell c} \gamma$ is obtained for γ such that $\alpha < \mu_\gamma(0, 1) \leq 0.5$ and $\mu_\gamma(\omega) = 0$ elsewhere.

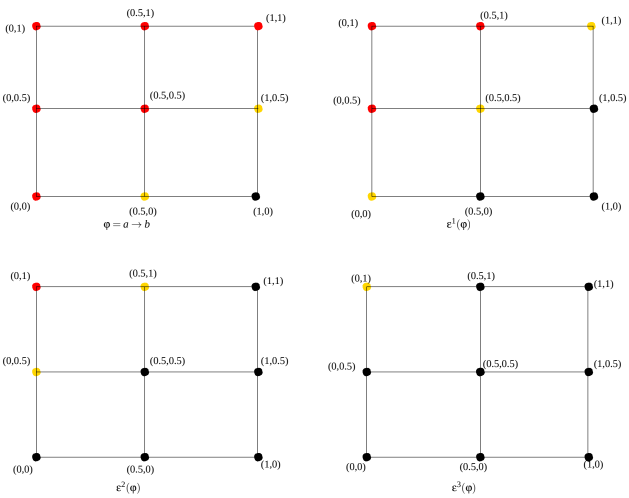

Figure 6: Successive erosions of $a \to b$ in the three-valued logic BL. Membership values are represented by colors (red = 1, yellow = 0.5, black =0).

Proposition 4 *Definitions 5 and 6 are equivalent to the ones proposed in the crisp case in [5] if $[\![\varphi]\!]$ is crisp, and the erosions are performed with a crisp structuring element.*

Proposition 5 *Considering the partial ordering introduced in Definition 4, the explanations according to $\rhd^{\ell c}$ are obtained for the smallest rank such that the minimality criterion depending on α is satisfied.*

The following notations will be used next: $\varphi \rhd \gamma \Leftrightarrow [\![\gamma]\!] \preceq [\![Expl(\varphi)]\!] \Leftrightarrow \mu_\gamma \preceq \mu_{Expl(\varphi)}$ where $Expl(\varphi)$ denotes the preferred explanations of φ (i.e. $Expl(\varphi) = \varepsilon^{l_\alpha}(\Sigma \wedge \varphi)$ or $Expl(\varphi) = \varepsilon^{l_\alpha}(\Sigma) \wedge \varphi$).

5. Rationality postulates

In this section, we consider the rationality postulates introduced by Pino-Perez and Uzcategui in [4]. It has been shown in [5] that all of them hold in the crisp case for $\rhd^{\ell c}$, while for $\rhd^{\ell ne}$ most of them hold and for a few of them only weaker forms are satisfied.

In the present context, these rationality postulates are expressed in Table 4. Both syntactic and semantic expressions are provided.

The intended meaning and motivation for these postulates can be found in [4].

Proposition 6 *The explanatory relation $\rhd^{\ell c}$ derived from fuzzy erosions with any structuring element B satisfies all rationality postulates of Table 4.*

Proposition 7 *The explanatory relation $\rhd^{\ell ne}$ derived from fuzzy erosions with any structuring element B satisfies* LLE$_\Sigma$, RLE$_\Sigma$, E-Reflexivity, E-Con$_\Sigma$, ROR, RS. *It does not satisfy* E-CM, E-C-Cut, E-R-Cut, LOR, E-DR.

Let us provide a counter-example for E-CM. We consider as before a simple example with three propositional symbols, and a binary structuring element such that $B(\omega, \omega') = 1 \Leftrightarrow d_H(\omega, \omega') \leq 1$. In Table 5, the membership functions for each $\omega \in \Omega$ to the fuzzy sets of models of formulas and their erosions are provided. The last erosion is $\varepsilon^{l_\alpha}(\Sigma \wedge \varphi) = \varepsilon^2(\Sigma \wedge \varphi)$ for $\alpha < 0.5$. The preferred explanations of φ are γ such that $\alpha < \mu_\gamma(\neg abc) \leq 0.5$ and $\forall \omega \in \Omega \setminus \{\neg abc\}, \mu_\gamma(\omega) = 0$. The last erosion of $\Sigma \wedge \varphi \wedge \varphi'$ is $\varepsilon^{l_\alpha}(\Sigma \wedge \varphi \wedge \varphi') = \varepsilon^1(\Sigma \wedge \varphi \wedge \varphi')$ for $\alpha < 0.5$. We have $\gamma \vdash_\Sigma \varphi'$, but γ is not an explanation of $\varphi \wedge \varphi'$ since $[\![\gamma]\!] \wedge [\![\varepsilon^{l_\alpha}(\Sigma \wedge \varphi \wedge \varphi')]\!] = \emptyset$.

Let us now provide a counter-example for E-C-Cut for $\rhd^{\ell ne}$. The details are in Table 6. The last erosions for $\Sigma \wedge \varphi$ and $\Sigma \wedge \varphi'$ are obtained for a size 2, for $\alpha < 0.5$. The one for $\Sigma \wedge \varphi \wedge \varphi'$ is obtained for a size 1. Let $\varphi \wedge \varphi' \rhd^{\ell ne} \gamma$ with $\alpha < \mu_\gamma(a\neg bc) \leq 0.5$, $\alpha < \mu_\gamma(ab\neg c) \leq 0.5$ and $\mu_\gamma(\omega) = 0$ for all other ω. All preferred explanations of φ verify $\alpha < \mu_\delta(abc) \leq 0.5$ and $\forall \omega \neq abc, \mu_\delta(\omega) = 0$, and $\delta \vdash_\Sigma \varphi'$. But γ is not a preferred explanation of φ.

As suggested in [5] for the crisp case, let us introduce weaker versions of E-CM and E-C-Cut. Their

	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
Σ	1	0.8	0.8	0.8	0.4	0.8	0.4	0
$\varepsilon^1(\Sigma)$	0.8	0.4	0.4	0.4	0	0	0	0
$\varepsilon^2(\Sigma)$	0.4	0	0	0	0	0	0	0
φ	0	0	0.8	1	0	0	1	0
$\Sigma \wedge \varphi$	0	0	0.8	0.8	0	0	0.4	0
$\varepsilon^1(\Sigma) \wedge \varphi$	0	0	0.4	0.4	0	0	0	0
$\varepsilon^2(\Sigma) \wedge \varphi$	0	0	0	0	0	0	0	0

Table 3: Illustration of the computation of $\rhd^{\ell c}$.

LLE$_\Sigma$:	$\dfrac{\vdash_\Sigma \varphi \leftrightarrow \varphi' \;;\; \varphi \rhd \gamma}{\varphi' \rhd \gamma}$	$\dfrac{[\![\varphi]\!] = [\![\varphi']\!] \;;\; \mu_\gamma \preceq \mu_{Expl(\varphi)}}{\mu_\gamma \preceq \mu_{Expl(\varphi')}}$
RLE$_\Sigma$:	$\dfrac{\vdash_\Sigma \gamma \leftrightarrow \gamma' \;;\; \varphi \rhd \gamma}{\varphi \rhd \gamma'}$	$\dfrac{[\![\gamma]\!] = [\![\gamma']\!] \;;\; \mu_\gamma \preceq \mu_{Expl(\varphi)}}{\mu_{\gamma'} \preceq \mu_{Expl(\varphi)}}$
E-Reflexivity:	$\dfrac{\varphi \rhd \gamma}{\gamma \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)}}{\mu_\gamma \preceq \mu_{Expl(\gamma)}}$
E-Con$_\Sigma$:	$\nvdash_\Sigma \neg\varphi$ iff there is γ such that $\varphi \rhd \gamma$	$[\![\varphi]\!] \neq \emptyset$ iff there is γ such that $\mu_\gamma \preceq \mu_{Expl(\varphi)}$
E-CM:	$\dfrac{\varphi \rhd \gamma \;;\; \gamma \vdash_\Sigma \varphi'}{(\varphi \wedge \varphi') \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)} \;;\; \mu_\gamma \preceq \mu_{\varphi'}}{\mu_\gamma \preceq \mu_{Expl(\varphi\wedge\varphi')}}$
E-C-Cut:	$\dfrac{(\varphi \wedge \varphi') \rhd \gamma \;;\; \forall\delta\ [\text{if } \varphi \rhd \delta \text{ then } \delta \vdash_\Sigma \varphi']}{\varphi \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi\wedge\varphi')} \;;\; \forall\delta\ [\text{if } \mu_\delta \preceq \mu_{Expl(\varphi)} \text{ then } \mu_\delta \preceq \mu'_\varphi]}{\mu_\gamma \preceq \mu_{Expl(\varphi)}}$
E-R-Cut:	$\dfrac{(\varphi \wedge \varphi') \rhd \gamma \;;\; \exists\delta\ [\varphi \rhd \delta \text{ and } \delta \vdash_\Sigma \varphi']}{\varphi \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi\wedge\varphi')} \;;\; \exists\delta\ [\mu_\delta \preceq \mu_{Expl(\varphi)} \text{ and } \mu_\delta \preceq \mu'_\varphi]}{\mu_\gamma \preceq \mu_{Expl(\varphi)}}$
RS:	$\dfrac{\varphi \rhd \gamma \;;\; \gamma' \vdash_\Sigma \gamma \;;\; \gamma' \nvdash_\Sigma \bot}{\varphi \rhd \gamma'}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)} \;;\; \mu_{\gamma'} \preceq \mu_\gamma \;;\; [\![\gamma']\!] \neq \emptyset}{\mu_{\gamma'} \preceq \mu_{Expl(\varphi)})}$
ROR:	$\dfrac{\varphi \rhd \gamma \;;\; \varphi \rhd \gamma'}{\varphi \rhd (\gamma \vee \gamma')}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)} \;;\; \mu_{\gamma'} \preceq \mu_{Expl(\varphi)}}{\mu_{\gamma\vee\gamma'} \preceq \mu_{Expl(\varphi)}}$
LOR:	$\dfrac{\varphi \rhd \gamma \;;\; \varphi' \rhd \gamma}{(\varphi \vee \varphi') \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)} \;;\; \mu_\gamma \preceq \mu_{Expl(\varphi')}}{\mu_\gamma \preceq \mu_{Expl(\varphi\vee\varphi')}}$
E-DR:	$\dfrac{\varphi \rhd \gamma \;;\; \varphi' \rhd \gamma'}{(\varphi \vee \varphi') \rhd \gamma \text{ or } (\varphi \vee \varphi') \rhd \gamma'}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)} \;;\; \mu_{\gamma'} \preceq \mu_{Expl(\varphi')}}{\mu_\gamma \preceq \mu_{Expl(\varphi\vee\varphi')} \text{ or } \mu_{\gamma'} \preceq \mu_{Expl(\varphi\vee\varphi')}}$

Table 4: Rationality postulates expressed in syntactic and semantic forms.

	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
$\Sigma \wedge \varphi$	1	1	0.8	0.8	0.8	0.8	0	0.5
$\varepsilon^1(\Sigma \wedge \varphi)$	0.8	0.8	0	0	0.5	0.5	0	0
$\varepsilon^2(\Sigma \wedge \varphi)$	0	0.5	0	0	0	0	0	0
φ'	0	1	1	1	1	1	1	1
$\Sigma \wedge \varphi \wedge \varphi'$	0	1	0.8	0.8	0.8	0.8	0	0.5
$\varepsilon^1(\Sigma \wedge \varphi \wedge \varphi')$	0	0	0	0	0.5	0.5	0	0

Table 5: Counter-example illustrating that $\rhd^{\ell ne}$ does not satisfy E-CM.

	abc	$\neg abc$	$a\neg bc$	$ab\neg c$	$\neg a\neg bc$	$\neg ab\neg c$	$a\neg b\neg c$	$\neg a\neg b\neg c$
$\Sigma \wedge \varphi$	1	1	0.8	0.8	0.8	0.8	0.5	0
$\varepsilon^1(\Sigma \wedge \varphi)$	0.8	0.8	0.5	0.5	0	0	0	0
$\varepsilon^2(\Sigma \wedge \varphi)$	0.5	0	0	0	0	0	0	0
$\Sigma \wedge \varphi'$	0.5	0	0.8	0.8	0.5	0.5	1	0.8
$\varepsilon^1(\Sigma \wedge \varphi')$	0	0	0.5	0.5	0	0	0.8	0.5
$\varepsilon^2(\Sigma \wedge \varphi')$	0	0	0	0	0	0	0.5	0
$\Sigma \wedge \varphi \wedge \varphi'$	0.5	0	0.8	0.8	0.5	0.5	0.5	0
$\varepsilon^1(\Sigma \wedge \varphi \wedge \varphi')$	0	0	0.5	0.5	0	0	0	0

Table 6: Counter-example illustrating that $\rhd^{\ell ne}$ does not satisfy E-C-Cut.

E-W-CM:	$\dfrac{\varphi \rhd \gamma \;;\; \varphi' \rhd \gamma}{(\varphi \wedge \varphi') \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi)} \;;\; \mu_\gamma \preceq \mu_{Expl(\varphi')}}{\mu_\gamma \preceq \mu_{Expl(\varphi \wedge \varphi')}}$
E-W-C-Cut:	$\dfrac{(\varphi \wedge \varphi') \rhd \gamma \;;\; \forall \delta \text{ [if } \varphi \rhd \delta \text{ then } \varphi' \rhd \delta \text{]}}{\varphi \rhd \gamma}$	$\dfrac{\mu_\gamma \preceq \mu_{Expl(\varphi \wedge \varphi')} \;;\; \forall \delta \text{ [if } \mu_\delta \preceq \mu_{Expl(\varphi)} \text{ then } \mu_\delta \preceq \mu_{Expl(\varphi')} \text{]}}{\mu_\gamma \preceq \mu_{Expl(\varphi)}}$

Table 7: Weak forms of some rationality postulates, expressed in syntactic and semantic forms.

syntactic and semantic expressions are given in Table 7. A weak version of E-R-Cut can be defined in a similar way.

Proposition 8 *The explanatory relation $\rhd^{\ell ne}$ derived from fuzzy erosions with any structuring element B satisfies* E-W-CM *and* E-W-C-Cut.

6. Conclusion

New explanatory relations have been proposed for knowledge representation based on logics with fuzzy sets of models, thus accounting with the approximate nature of abductive reasoning. The algebraic properties of the involved mathematical morphology operators lead to good properties of the proposed relations in terms of rationality properties. Future work aims at further developing examples, at investigating the potential role of α for balancing specialization and generalization of the solution, and at extending the formalism to other types of fuzzy logics.

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