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Spatial reasoning under imprecision using fuzzy set theory, formal logics and mathematical morphology

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Abstract

In spatial reasoning, in particular for applications in image understanding, structure recognition and computer vision, a lot of attention has to be paid to spatial relationships and to the imprecision attached to information and knowledge to be handled. Two main components are knowledge representation and reasoning. We show in this paper that the fuzzy set framework associated to the formalism provided by mathematical morphology and formal logics allows us to derive appropriate representations and reasoning tools.

Keywords: Spatial reasoning; Imprecision; Spatial relationships; Fuzzy sets; Fuzzy mathematical morphology; Morpho-logics; Fusion; Decision making; Image interpretation; Model-based recognition

1. Introduction

In this paper, we consider spatial reasoning from the point of view of mathematical morphology. Spatial reasoning can be defined as the domain of spatial knowledge representation, in particular spatial relations between spatial entities, and of reasoning on these entities and relations. This field has been largely developed in artificial intelligence, in

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particular using qualitative representations based on logical formalisms. In image interpretation and computer vision it is much less developed and is mainly based on quantitative representations. A typical example in this domain concerns model-based structure recognition in images, where the model represents spatial entities and relationships between them. This paper extends the work presented in [1], where examples of model-based structure recognition in images are shown.

Two main components of this domain are spatial knowledge representation and reasoning. In particular spatial relationships constitute an important part of the knowledge we have to handle. Imprecision is often attached to spatial reasoning in images, and can occur at different levels, from knowledge to the type of question we want to answer. The reasoning component includes fusion of heterogeneous spatial knowledge, decision making, inference, recognition. Two types of questions are raised when dealing with spatial relationships:

(1) given two objects (possibly fuzzy), assess the degree to which a relation is satisfied;
(2) given one reference object, define the area of space in which a relation to this reference is satisfied (to some degree).

In order to answer these questions and address both representation and reasoning issues, we rely on three different frameworks and their combination:

- mathematical morphology, which is an algebraic theory that has extensions to fuzzy sets and to logical formulas, and can elegantly unify the representation of several types of relationships;
- fuzzy set theory, which has powerful features to represent imprecision at different levels, to combine heterogeneous information and to make decisions;
- formal logics and the attached reasoning and inference power.

The association of these three frameworks for spatial reasoning is an original contribution of this paper. It allows us to match two important requirements: expressiveness and completeness with respect to the types of spatial information we want to represent [2]. Complexity issues are not addressed in this paper.

In Section 2, we illustrate the importance of modeling imprecision by looking at some other domains where spatial relationships play an important role, such as linguistics or cognition. In Section 3, we address the question of spatial knowledge representation and propose different types of representations in quantitative, semi-quantitative (fuzzy) and qualitative settings, all based on mathematical morphology. In Section 4, we address the question of reasoning on these representations.

2. Spatial relationships and imprecision

Spatial reasoning (in particular in images) has to deal with imprecision. Imprecision is often inherent to images, and its causes can be found at several levels: observed phenomenon (imprecise limits between structures or objects), acquisition process (limited resolution, numerical reconstruction methods), image processing steps (imprecision induced by a filtering for instance). This may induce imprecision on the objects to be recognized (due to the absence of strong contours or to a rough segmentation). But imprecision can be found also in semantics of some relationships (such as ''left of'', ''quite far'', etc.), or in the type of knowledge available about the structures (for instance anatomical textbooks describe the caudate nucleus as ''an internal brain structure which is very close to the lateral ventricles'') or even in the type of question we would like to answer (in mobile robotics for instance, we may want a robot ''go towards some object while remaining at some secure distance of it''). These examples show that even for relationships that are well defined in a mathematical sense, such as distances, we may have to deal with them in an imprecise way. This also becomes apparent when we are looking at other domains where spatial relationships are involved [3]. These domains constitute an important source of inspiration for developing spatial reasoning models in computer science and image interpretation.

2.1. Linguistics

Natural languages usually offer a rich variety of lexical terms for describing spatial location of entities. These terms are not only numerous, they also concern all lexical categories, such as nouns, verbs, adjectives, adverbs, prepositions [4], with significant differences between languages [5].

The domain of linguistics is a source of inspiration of many works on qualitative spatial information representation and qualitative spatial reasoning [6]. Modeling qualitative spatial relations strongly relies on the way these relations are expressed verbally. Several properties are exhibited, such as the asymmetry of some expressions, the non-bijective relation between language and spatial concepts (in particular for prepositions [4,7]), the interaction between distances and orientation, etc., [7,8].

A remarkable feature is that representation and communication are often achieved without using numbers [4]. Conversely, apparently precise statements (for instance containing crisp numbers) should not always be understood as really precise, but rather as order of magnitudes. Let us consider for instance the sentence *Paris and Toulouse are at a distance of 700 km*. The number 700 should not be considered as an exact value. It gives an idea of the distance, and its interpretation is subject to some considerations such as the areas of Paris and of Toulouse that are really concerned, the way to travel from one city to the other, etc.

Too precise statements can even become inefficient if they make the message too complex. This appears typically in the problem of route description for helping navigation and path finding. The example of giving directions in Venice is particularly eloquent [9].

Moreover, the way to describe spatial situations, the vision and the representation of space are not fixed and are likely to be modified depending on perceptual data and on discourse situation [4]. In linguistic statements about space and distance, the geometrical terms of the language that are involved in these statements are usually not sufficient to get a clear meaning. The statement context is also of prime importance, as well as functional properties of the considered physical entities.

2.2. Human perception

Let us consider for instance distances. A number of factors influence the perception of distance, leading to different measures [6]:

- purely spatial measures, in a geometric sense, give rise to "metric distances", and are related to intrinsic properties of the objects (involving not only purely geometrical distances, but also topological, size, shape properties of the objects);
- temporal measures lead to distances expressed as travel time, and can be considered of extrinsic type, as opposed to the previous class; this advocates for treating space and time together;
- economic measures, in terms of costs to be invested (also of extrinsic type);
- perceptual measures lead to distance of deictic type; they are related to an external point of view, which can be concrete or just a mental representation, which can be influenced by environmental features, by subjective considerations, leading to distances that are not necessarily symmetrical; the discourse situation also plays a role at this level, as mentioned above.

As mentioned in [10,11], the perception of distance between objects also depends on the presence or absence of other objects in the environment. If there are no other objects, the perception and human reasoning are mainly of geometrical type and distances are absolute. On the contrary when there are other objects, the perception of distance becomes relative. The size of the area and the frame of reference also play a crucial role in the perception of distances [6], in particular by defining the scale and the upper bound of the perceived distances. Perception is therefore not scale-independent [12], while language is to a large extent scale-independent [8]. Finally, attractiveness of the objects strongly affects the perception of proximity [11].

2.3. Cognition

The cognitive understanding of a spatial environment, in particular in large-scale spaces, is issued from two types of processes [6,13]:

- route knowledge acquisition, which consists in learning from sensori-motor experience (i.e. actual navigation) and implies an order information between visited landmarks;
- survey knowledge acquisition, from symbolic sources such as maps, leading to a global view ("from above") including global features and relationships, which is independent of the order of landmarks.

As for the internal representation of space in the brain, a distinction is usually made between egocentric and allocentric representations [6,14]. Although the notion of "map in the head" has recognized limitations as a cognitive theory, it is still quite popular, and corresponds to the allocentric representations. It is important to note that the psychological space does not need to mirror the physical space.

Cognitive studies report that distance and direction are quite dissociated. On the contrary, as mentioned for the perception, from a cognitive point of view, time and space cannot be easily separated.

The importance of the frame of reference, highlighted in all domains, has also a cognitive flavor: cognitive studies have shown that multiple frames of reference are usually used and appear as necessary for understanding and navigating in a spatial environment [6,15].

These cognitive concepts have been intensively used in several works in the modeling and conception of geographic information systems (GIS), where spatial information is the core [16,17]. Another field where cognitive aspects about space inspire the development of frameworks and systems is the domain of mobile robotics. The work by Kuipers is fundamental in this respect [15,18]. His spatial semantic hierarchy is a model of knowledge of large-scale space including both qualitative and quantitative representations, and is strongly inspired by the properties of the human cognitive map. It aims at providing methods for robot exploration and map building. The hierarchy consists of sensory, control, causal, topological and metrical levels. We are concerned in this paper mainly by the last two levels. Finally, it is worth mentioning the new approach proposed in [19], called Conceptual Spaces. These spaces can be considered as a representation of cognitive systems, intermediate between the high level symbolic representations and the subconceptual connectionist representations. They emphasize orders and measures, and a key notion is distances between concepts, leading to geometrical representations, but using quality dimensions.

3. Knowledge representation using mathematical morphology

In this section, we show that mathematical morphology constitutes an unifying framework for spatial knowledge representation for several reasons: (i) the spatial imprecision attached to objects can be conveniently modeled by morphological operations; (ii) several relationships can be expressed based on morphological operators (in particular dilation); (iii) the algebraic nature of this theory allows to convert these relations into algebraic terms (including relations such as distances), which leads to an easy translation to the fuzzy case (through fuzzy mathematical morphology) and to the qualitative case (through morpho-logics).

3.1. Mathematical morphology in quantitative, semi-quantitative and qualitative settings

3.1.1. Classical morphology

Let us first recall the definitions of dilation and erosion of a set X by a structuring element B in a space $\mathscr{S}$ (e.g. $\mathbb{R}^n$, or $\mathbb{Z}^n$ for discrete spaces like images), denoted respectively by $D_B(X)$ and $E_B(X)$ [20]:

$$D_B(X) = \{x \in \mathscr{S} | B_x \cap X \neq \emptyset\}, \tag{1}$$

$$E_B(X) = \{x \in \mathscr{S} | B_x \subseteq X\}, \tag{2}$$

where B_x denotes the translation of B at point x. In these equations, B defines a neighborhood that is considered at each point. It can also be seen as a relationship between points. From these two fundamental operations, a lot of others can be built [20].

3.1.2. Fuzzy mathematical morphology

Several definitions of mathematical morphology on fuzzy sets with fuzzy structuring elements have been proposed in the literature (see e.g. [21–23]). Here we use the approach using t-norms and t-conorms as fuzzy intersection and fuzzy union. However, what follows applies as well if other definitions are used. Erosion and dilation of a fuzzy set μ by a fuzzy structuring element ν, both defined in a space $\mathscr{S}$, are respectively defined as

$$E_\nu(\mu)(x) = \inf_{y\in\mathcal{S}} \perp [c(\nu(y-x)), \mu(y)], \tag{3}$$

$$D_\nu(\mu)(x) = \sup_{y\in\mathcal{S}} \top[\nu(y-x), \mu(y)], \tag{4}$$

where $\top$ is a t-norm, c a strong fuzzy negator (fuzzy complementation), and $\perp$ is the t-conorm associated to $\top$ with respect to c. These definitions guarantee that most properties of morphological operators are preserved [21,24]. Note that in all these morphological definitions, we use the conventions of [20], which lead to dilations and erosions which are dual from each other with respect to the complementation. Another approach, which is more rigorous from an algebraic point of view, relies on the notion of adjunction [25], leading to different conventions in cases where the structuring element is not symmetrical with respect to the origin. This approach has been generalized to the case of fuzzy sets in [26], and leads to strong algebraic properties of the derived operators. Here we mostly consider symmetrical structuring elements.

3.1.3. Morpho-logics

Now, we express morphological operations in a symbolic framework, using logical formulas. Let us consider a language generated by a finite set of propositional symbols and the usual connectives. Kripke's semantics is used. The set of all worlds is denoted by Ω. The set of worlds where a formula φ is satisfied is $\mathrm{Mod}(\varphi) = \{\omega \in \Omega \mid \omega \models \varphi\}$. The underlying idea for constructing morphological operations on logical formulas is to consider set interpretations of formulas and worlds. Since in classical propositional logics, the set of formulas is isomorphic to the quotient space $2^\Omega/\equiv$ ($\equiv$ denoting the logical equivalence), we can identify φ with $\mathrm{Mod}(\varphi)$, and then apply set-theoretic morphological operations. We recall that $\mathrm{Mod}(\varphi \vee \psi) = \mathrm{Mod}(\varphi) \cup \mathrm{Mod}(\psi)$, $\mathrm{Mod}(\varphi \wedge \psi) = \mathrm{Mod}(\varphi) \cap \mathrm{Mod}(\psi)$, and $\mathrm{Mod}(\varphi) \subseteq \mathrm{Mod}(\psi)$ iff $\varphi \models \psi$.

Using these equivalences, dilation and erosion of a formula φ are defined as [27]:

$$\mathrm{Mod}(D_B(\varphi)) = \{\omega \in \Omega | B(\omega) \cap \mathrm{Mod}(\varphi) \neq \emptyset\}, \tag{5}$$

$$\mathrm{Mod}(E_B(\varphi)) = \{\omega \in \Omega | B(\omega) \models \varphi\}, \tag{6}$$

where $B(\omega) \models \varphi$ means $\forall \omega' \in B(\omega)$, $\omega' \models \varphi$. The structuring element B represents a relationship between worlds and defines a "neighborhood" of worlds. It can be for instance defined as a ball of a distance between worlds [28]. The condition for dilation expresses that the set of worlds in relation to ω should be consistent with φ, i.e. $\exists \omega' \in B(\omega)$, $\omega' \models \varphi$. The condition for erosion is stronger and expresses that φ should be satisfied in all worlds in relation to ω.

Now we consider the framework of normal modal logics [29] and use an accessibility relation as relation between worlds. We define an accessibility relation from any structuring element B (or the converse) as: $R(\omega, \omega')$ iff $\omega' \in B(\omega)$. Let us now consider the two modal operators $\Box$ and $\Diamond$ defined from the accessibility relation as [29]

$$\mathcal{M}, \omega \models \Box\varphi \quad \text{iff } \forall \omega' \in \Omega,\ R(\omega, \omega') \ \Rightarrow \ \mathcal{M}, \omega' \models \varphi, \tag{7}$$

$$\mathcal{M}, \omega \models \Diamond\varphi \quad \text{iff } \exists \omega' \in \Omega,\ R(\omega, \omega') \ \text{and} \ \mathcal{M}, \omega' \models \varphi, \tag{8}$$

where $\mathcal{M}$ denotes a standard model related to R (it will be skipped in the following). Eq. (7) can be rewritten as

$$\omega \models \Box\varphi \iff B(\omega) \models \varphi, \tag{9}$$

which exactly corresponds to the definition of erosion of a formula, and Eq. (8) can be rewritten as

$$\omega \models \Diamond\varphi \iff B(\omega) \cap \mathrm{Mod}(\varphi) \neq \emptyset, \tag{10}$$

which exactly corresponds to a dilation. This shows that we can define modal operators derived from an accessibility relation as erosion and dilation with a structuring element:

$$\Box\varphi \equiv E_B(\varphi), \tag{11}$$

$$\Diamond\varphi \equiv D_B(\varphi). \tag{12}$$

The modal logic constructed from erosion and dilation has a number of theorems and rules of inference, detailed in [30], which increase its reasoning power.

All these definitions and properties extend to the fuzzy case, if we consider fuzzy formulas, for which $\mathrm{Mod}(\varphi)$ is a fuzzy set of Ω. A fuzzy structuring element can be interpreted as a fuzzy relation between worlds. Its usefulness will appear for expressing intrinsically vague spatial relationships such as directional relative position.

3.2. Spatial objects

We consider the general case of a 3D space $\mathcal{S}$, where objects can have any shape and any topology. In the quantitative framework, an object is simply a subset of $\mathcal{S}$.

If the objects are imprecise, as is often the case if they are extracted from images, then the semi-quantitative framework of fuzzy sets proved to be useful for their representation, as spatial fuzzy sets (i.e. fuzzy sets defined in the space $\mathcal{S}$). The use of fuzzy sets may represent different types of imprecision, either on the boundary of the objects (due for instance to partial volume effect, to the spatial resolution, or to a rough detection), or on the variability of these structures, etc. Fuzzy mathematical morphology can be used to make spatial imprecision explicit. For instance if an object is detected in an image (as a set or a fuzzy set), imprecision on its limits can be introduced by computing the fuzzy erosion and dilation of the object by a fuzzy structuring element modeling this imprecision. The erosion–dilation pair can be interpreted as a necessity–possibility pair, as often used in the fuzzy set community, as a rough set [31], or as a belief–plausibility pair [32].

In qualitative spatial reasoning based on logics, interpretations can represent spatial entities, like regions of space. Formulas then represent combinations of such entities, and define regions, objects, etc., which may be not connected. For instance, if a formula φ is a symbolic representation of a region X of space, it can be interpreted for instance as "the object we are looking at is in X". In an epistemic interpretation, it could represent the belief of an agent that the object is in X. The interest of such representations is also to deal with any kind of spatial entities, without referring to points, as highlighted also in the domain of mereotopology (see e.g. [33]). If φ represents some knowledge or belief about a region X of space, then $\Box\varphi$ represents a restriction of X. If we are looking at an object in X, then $\Box\varphi$ is a necessary region for this object. Similarly, $\Diamond\varphi$ represents an extension of X, and a possible region for the object.

3.3. Spatial relations

In this section we consider the problem of defining and computing spatial relationships. We consider both topological and metric relationships [18,34]. We distinguish also between relationships that are mathematically well defined (such as set relationships, adjacency, distances) and relationships that are intrinsically vague, like relative directional position, for which fuzzy definitions are appropriate. If the objects are imprecise, both types of relations have then to be extended to the fuzzy case. Results can also be semi-quantitative, and provided in the form of intervals or fuzzy numbers. Symbolic representations in the context of modal logics will be introduced as well. A synthesis of the main fuzzy spatial relations can be found in [35].

3.3.1. Quantitative and semi-quantitative settings

We first address question (1) raised in the introduction, i.e. given two objects or two fuzzy objects, assess the relations between them (or the degree to which some relation is satisfied).

Computing *set relationships*, like inclusion, intersection, etc., if the objects are precisely defined does not call for specific developments. If the objects are imprecise, stating whether they intersect or not, or whether one is included in the other, becomes a matter of degree. A degree of inclusion can be defined as the infimum of the membership values of the union (defined as a t-conorm) of one set and the complement of the other (as for erosion). A degree of intersection μ_{int} can be defined using the supremum of the membership values of the intersection (defined as a t-norm) between both fuzzy sets (as for fuzzy dilation) or using the fuzzy volume of the t-norm in order to take more spatial information into account. The degree of non-intersection is then simply defined by $\mu_{\neg\text{int}} = 1 - \mu_{\text{int}}$. The interpretations in terms of erosion and dilation allow us to include set relationships in the same mathematical morphology framework as the other relations.

Adjacency has a large interest in spatial reasoning, since it denotes an important relation between objects or regions. For any two subsets X and Y in the digital space $\mathbb{Z}^n$, the adjacency of X and Y can be expressed in terms of morphological dilation, as

$$X \cap Y = \emptyset \quad \text{and} \quad D_B(X) \cap Y \neq \emptyset, \quad D_B(Y) \cap X \neq \emptyset, \tag{13}$$

where B denotes the elementary structuring element associated to the chosen digital connectivity. This structuring element is usually symmetrical, which means that the two conditions $D_B(X) \cap Y \neq \emptyset$ and $D_B(Y) \cap X \neq \emptyset$ are equivalent, so only one needs to be checked. Adjacency between fuzzy sets can be defined by translating this expression into fuzzy terms, by using fuzzy dilation [36]. The binary concept becomes then a degree of adjacency between fuzzy sets μ and ν:

$$\mu_{\text{adj}}(\mu, \nu) = \top[\mu_{\neg\text{int}}(\mu, \nu), \mu_{\text{int}}[D_B(\mu), \nu], \mu_{\text{int}}[D_B(\nu), \mu]]. \tag{14}$$

This definition represents a conjunctive combination (through a t-norm $\top$) of a degree of non-intersection $\mu_{\neg\text{int}}$ between μ and ν and degrees of intersection μ_{int} between each fuzzy set and the dilation of the other. This definition is symmetrical, reduces to the binary definition if μ, ν and B are binary, and is invariant under geometrical transformations.

The importance of *distances* in spatial reasoning is well established. Mathematical morphology allows us to define distances between fuzzy sets that combine spatial information and membership comparison [3,37]. In the binary case, there exist strong links between mathematical morphology (in particular dilation) and distances (from a point to a set, and several distances between two sets), and this can also be exploited in the fuzzy case. The advantage is that distances are then expressed in set theoretical terms, and are therefore easier to extend with nice properties than usual analytical expressions. Here we only present the case of Hausdorff distance, which has the advantage of being a true distance in the crisp case. The binary equation defining the Hausdorff distance:

$$d_{\mathrm{H}}(X, Y) = \max\left[\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\right] \tag{15}$$

can be expressed in morphological terms as

$$d_{\mathrm{H}}(X, Y) = \inf\{n, X \subseteq D^n(Y) \text{ and } Y \subseteq D^n(X)\}. \tag{16}$$

A distance distribution, expressing the degree to which the distance between μ and μ' is less than n, is obtained by translating this equation into fuzzy terms:

$$\Delta_{\mathrm{H}}(\mu, \mu')(n) = \top\left[\inf_{x\in\mathcal{S}} \perp [D_\nu^n(\mu)(x), c(\mu'(x))], \inf_{x\in\mathcal{S}} \perp [D_\nu^n(\mu')(x), c(\mu(x))]\right], \tag{17}$$

where c is a complementation, $\top$ a t-norm and $\perp$ a t-conorm.

A distance density, expressing the degree to which the distance is equal to n, can be derived implicitly from this distance distribution. A direct definition of a distance density can be obtained from $d_{\mathrm{H}}(X, Y) = 0 \iff X = Y$, and for $n > 0$:

$$\begin{aligned} d_{\mathrm{H}}(X, Y) = n \quad \iff \quad & X \subseteq D^n(Y) \text{ and } Y \subseteq D^n(X) \\ & \text{and } \left(X \not\subseteq D^{n-1}(Y) \text{ or } Y \not\subseteq D^{n-1}(X)\right). \end{aligned} \tag{18}$$

Translating these equations leads to a definition of the Hausdorff distance between two fuzzy sets μ and μ' as a fuzzy number:

$$\delta_{\mathrm{H}}(\mu, \mu')(0) = \top\left[\inf_{x\in\mathcal{S}} \perp [\mu(x), c(\mu'(x))], \inf_{x\in\mathcal{S}} \perp [\mu'(x), c(\mu(x))]\right], \tag{19}$$

$$\begin{aligned} \delta_{\mathrm{H}}(\mu, \mu')(n) = \top\Bigg[&\inf_{x\in\mathcal{S}} \perp [D_\nu^n(\mu)(x), c(\mu'(x))], \inf_{x\in\mathcal{S}} \perp [D_\nu^n(\mu')(x), c(\mu(x))], \\ & \perp\left(\sup_{x\in\mathcal{S}} \top[\mu(x), c(D_\nu^{n-1}(\mu')(x))], \sup_{x\in\mathcal{S}} \top[\mu'(x), c(D_\nu^{n-1}(\mu)(x))]\right)\Bigg]. \end{aligned} \tag{20}$$

The obtained distance is positive (the support of this fuzzy number is included in $\mathbb{R}^+$). It is symmetrical with respect to μ and μ'. The separability property (i.e. $d(\mu, \nu) = 0 \iff \mu = \nu$) is not always satisfied. However, we have $\delta_{\mathrm{H}}(\mu, \mu')(0) = 1$ implies $\mu = \mu'$ for $\perp$ being the bounded sum ($\perp(a, b) = \min(1, a + b)$), while it implies μ and μ' crisp and equal for $\perp = \max$. The triangular inequality is not satisfied in general.

Relationships between objects can be partly described in terms of *directional relative position*, like "to the left of". Because of the inherent vagueness of such expressions, they may find a better understanding in the framework of fuzzy sets, as fuzzy relationships, even for crisp objects. A few works propose fuzzy approaches for assessing the directional relative position between objects, which is an intrinsically vague relation (see [38] for a comparative review).

The approach used here relies on a fuzzy dilation that provides a map (or fuzzy landscape) where the membership value of each point represents the degree of the satisfaction of the relation to the reference object. This approach has interesting features: it works directly in the spatial domain, without reducing the objects to points or histograms, and it takes the object shape into account.

We consider a (possibly fuzzy) object R in the s of $\mathscr{S}$, and denote by $\mu_\alpha(R)$ the fuzzy subset of $\mathscr{S}$ such that points of areas which satisfy to a high degree the relation "to be in the direction $\vec{u}_\alpha$ with respect to object R" have high membership values, where $\vec{u}_\alpha$ is a vector making an angle α with respect to a reference axis. We express $\mu_\alpha(R)$ as the fuzzy dilation of μ_R by v, where v is a fuzzy structuring element depending on α: $\mu_\alpha(R) = D_v(\mu_R)$ where μ_R is the membership function of the reference object R. This definition applies both to crisp and fuzzy objects and behaves well even in case of objects with highly concave shape. In polar coordinates (but this extends to 3D as well), v is defined by:[1] $v(\rho, \theta) = f(\theta - \alpha)$ and $v(0, \theta) = 1$, where $\theta - \alpha$ is defined modulo π and f is a decreasing function, e.g. $f(\beta) = \max[0, \cos\beta]^2$ for $\beta \in [-\pi/2, \pi/2]$.

Once we have defined $\mu_\alpha(R)$, we define the degree to which a given object A is in direction $\vec{u}_\alpha$ with respect to R. Let us denote by μ_A the membership function of the object A. The evaluation of relative position of A with respect to R is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all x in $\mathscr{S}$. The histogram of $\mu_\alpha(R)$ conditionally to μ_A is such a function. A summary of the contained information could be more useful in practice, and an appropriate tool for this is the fuzzy pattern matching approach [39]: the matching between two possibility distributions is summarized by two numbers, a necessity degree N (a pessimistic evaluation) and a possibility degree Π (an optimistic evaluation), as often used in the fuzzy set community. The possibility corresponds to a degree of intersection between the fuzzy sets A and $\mu_\alpha(R)$, while the necessity corresponds to a degree of inclusion of A in $\mu_\alpha(R)$. These operations can also be interpreted in terms of fuzzy mathematical morphology, since Π corresponds to a dilation, while N corresponds to an erosion.

3.3.2. Spatial representations

Now we address question (2) raised in the introduction: given a reference object, we define a spatial fuzzy set that represents the area of space where some relationship to this reference object is satisfied (to some degree). The advantage of these representations is that they map all types of spatial knowledge in the same space, which allows for their fusion and for spatial reasoning. This constitutes a new way to represent spatial knowledge in the spatial domain [40].

For each piece of knowledge, we consider its "natural expression", i.e. the usual form in which it is given or available, and translate it into a spatial fuzzy set in $\mathscr{S}$ having different semantics depending on the type of information (on objects, spatial imprecision, relationships to other objects, etc.). The numerical representation of membership values assumes that we can assign numbers that represent degrees of satisfaction of a relationship for instance. These numbers can be derived from prior knowledge or learned from examples, but usually there remain some quite arbitrary choices. However, we have to keep in mind that mostly the ranking is important, not the individual numerical values.

Set relationships specify whether areas where other objects can be localized are forbidden or possible. The corresponding region of interest is: $\mu_{\text{set}}(x) = \top[\mu_{O^{\text{in}}}(x), 1 - \mu_{O^{\text{out}}}(x)]$, where $\top$ is a t-norm, which expresses a conjunction between inclusion constraint in the objects O^{in} and exclusion constraint from the objects O^{out}. The properties of t-norms guarantee that good properties are satisfied.

[1] This definition of v is discontinuous at the origin. A continuous function could be obtained by modeling the fact that the direction of a point or of an object closed to the origin is imprecise.

Other *topological relations* (adjacency, etc.) can be treated in a similar way. For instance, an object that is a non tangential proper part of μ has to be searched in $E_v(\mu)$ where v is an elementary structuring element.

Morphological expressions of *distances*, directly lead to spatial representations of knowledge about distances. Let us assume that we want to determine B, subject to satisfy some distance relationship with an object A. According to the algebraic expressions of distances, dilation of A is an adequate tool for this. For example, if knowledge expresses that $d(A,B) \geqslant n$, then B should be looked for in $D^{n-1}(A)^C$. Or, if knowledge expresses that B should lie between a distance n_1 and a distance n_2 of A, i.e. the minimum distance should be greater than n_1 and the maximum distance should be less than n_2, then the possible domain for B is reduced to $D^{n_2}(A) \setminus D^{n_1-1}(A)$. In cases where imprecision has to be taken into account, fuzzy dilations are used, with the corresponding equivalences with fuzzy distances. The extension to approximate distances calls for fuzzy structuring elements. Let us consider the generalization to the fuzzy case of the previous example (minimum distance of at least n_1 and maximum distance of at most n_2 to a fuzzy set μ). Instead of defining an interval $[n_1,n_2]$, we consider a fuzzy interval, defined as a fuzzy set on $\mathbb{R}^+$ having a core equal to the interval $[n_1,n_2]$. The membership function μ_n is increasing between 0 and n_1 and decreasing after n_2 (this is but one example). Then we define two structuring elements, as

$$v_1(x) = \begin{cases} 1 - \mu_n(d_\mathrm{E}(x,0)), & \text{if } d_\mathrm{E}(x,0) \leqslant n_1, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

$$v_2(x) = \begin{cases} 1, & \text{if } d_\mathrm{E}(x,0) \leqslant n_2, \\ \mu_n(d_\mathrm{E}(x,0)), & \text{otherwise}. \end{cases} \tag{22}$$

where d_E is the Euclidean distance in $\mathcal{S}$ and 0 the origin. The spatial fuzzy set expressing the approximate relationship about distance to μ is then defined as

$$\mu_{\text{distance}} = \top[D_{v_2}(\mu), 1 - D_{v_1}(\mu)] \tag{23}$$

if $n_1 \neq 0$, and $\mu_{\text{distance}} = D_{v_2}(\mu)$ if $n_1 = 0$. The increasingness of fuzzy dilation with respect to both the set to be dilated and the structuring element [21] guarantees that these expressions do not lead to inconsistencies: we have $v_1 \subseteq v_2$, $v_1(0) = v_2(0) = 1$, and therefore $\mu \subseteq D_{v_1}(\mu) \subseteq D_{v_2}(\mu)$. In the case where $n_1 = 0$, we do not have $v_1(0) = 1$ any longer, but in this case, only the dilation by v_2 is considered. This case corresponds actually to a distance to μ less than "about n_2". These properties are indeed expected for representations of distance knowledge.

The definition of *directional position* between two sets described above relies on a spatial representation of the degree of satisfaction of the relation to the reference object. Therefore the first step of the proposed approach directly provides the desired representation as the fuzzy set $\mu_\alpha(R)$ in $\mathcal{S}$.

3.3.3. Symbolic representations

Now, we use the logical framework presented in Section 3.1. Let us first consider *topological relationships*, and two formulas φ and ψ representing two regions X and Y of space. Note that all what follows holds in both crisp and fuzzy cases. Simple topological relations such as inclusion, exclusion, intersection do not call for more operators than the standard ones of propositional logic. But other relations such that *X is a tangential part of Y* (one of the basic relations of Region Connection Calculus (RCC)) can benefit from the morphological modal operators. Such a relationship can be expressed as

$$\varphi \to \psi \text{ and } \Diamond\varphi \wedge \neg\psi \text{ consistent.} \tag{24}$$

Indeed, if X is a tangential part of Y, it is included in Y but its dilation is not. If we also want X to be a proper part, we have to add the condition:

$$\neg\varphi \wedge \psi \text{ consistent.} \tag{25}$$

Let us now consider *adjacency*. Saying that X is adjacent to Y means that they do not intersect and as soon as one region is dilated, it intersects the other. In symbolic terms, this relation can be expressed as

$$\varphi \wedge \phi \text{ inconsistent} \quad \text{and} \quad \Diamond\varphi \wedge \psi \text{ consistent} \quad \text{and} \quad \varphi \wedge \Diamond\psi \text{ consistent.} \tag{26}$$

Similarly, external connection in RCC (which implies some common boundary in contrary to our digital definition) can be expressed as

$$\varphi \wedge \phi \text{ consistent} \quad \text{and} \quad \Box\varphi \wedge \psi \text{ inconsistent} \quad \text{and} \quad \varphi \wedge \Box\psi \text{ inconsistent.} \tag{27}$$

It could be interesting to link these types of representations with the ones developed in the community of mereotopology and RCC, where such relations are defined respectively from parthood and connection predicates [33,41]. Interestingly enough, erosion is defined from inclusion (i.e. a parthood relationship) and dilation from intersection (i.e. a connection relationship). Some axioms of these domains could be expressed in terms of dilation. For instance from a parthood postulate $P(X, Y)$ between two spatial entities X and Y and from dilation, the tangential proper part could be defined as

$$\mathrm{TPP}(X,Y) = P(X,Y) \wedge \neg P(Y,X) \wedge \neg P(D(X),Y). \tag{28}$$

Let us consider again expressions of *distances* in terms of morphological dilations. The translation into a logical formalism is straightforward. Expressions like $d_{\mathrm{H}}(X, Y) = n$ translate into:

$$((\forall m, m < n), (\psi \wedge \neg\Diamond^m\varphi \text{ consistent or } \varphi \wedge \neg\Diamond^m\psi \text{ consistent})) \text{ and}$$
$$(\psi \to \Diamond^n\varphi \text{ and } \varphi \to \Diamond^n\psi). \tag{29}$$

The first condition corresponds to $d_{\mathrm{H}}(X, Y) \geqslant n$ and the second one to $d_{\mathrm{H}}(X, Y) \leqslant n$.

Let us consider an example of possible use of these representations for spatial reasoning. If we are looking at an object represented by ψ in an area which is at a distance in $[n_1, n_2]$ of a region represented by φ, this corresponds to a minimum distance greater than n_1 and to a Hausdorff distance less than n_2. Then we have to check the following relation:

$$\psi \to \neg\Diamond^{n_1}\varphi \wedge \Diamond^{n_2}\varphi. \tag{30}$$

This expresses in a symbolic way an imprecise knowledge about distances represented as an interval. If we consider a fuzzy interval, this extends directly using fuzzy dilation.

These expressions show how we can convert distance information, which is usually defined in an analytical way, into algebraic expressions through mathematical morphology, and then into logical ones through morphological expressions of modal operators.

For *directional relative position* we rely again on the approach where the reference object is dilated with a particular structuring element defined according to the direction of interest. Let us denote by D^d the dilation corresponding to a directional information in the direction d, and by $\Diamond^d$ the associated modal operator. Expressing that an object represented by ψ has to be in direction d with respect to a region represented by φ amounts to check the following relation: $\psi \to \Diamond^d\varphi$. In the fuzzy case, this relation can hold to some degree.

4. Reasoning on spatial relationships

In this section, we address the second important issue in spatial reasoning, namely reasoning. This includes fusion, since heterogeneous information has often to be combined in spatial reasoning, decision making and recognition (with a special focus of model-based recognition), and inference and logical reasoning.

4.1. Fusion

Spatial reasoning aiming for instance at recognizing structures in an image has to deal with the combination of knowledge and information represented and modeled as described in Section 3. Usually, to achieve recognition, several spatial relationships to one or several spatial entities have to be combined, as well as information extracted from the image itself. For this combination step, the advantages of fuzzy sets lie in the variety of combination operators, which may deal with heterogeneous information [42–44] expressed in a semi-quantitative framework. We proposed a classification of these operators with respect to their behavior (in terms of conjunctive, disjunctive, and compromise), the possible control of this behavior, their properties and their decisiveness, which proved to be useful for choosing an operator [45].

Operators such as t-norms, t-conorms and mean operators always behave respectively in a conjunctive, disjunctive and compromise way. Within each class, some operators are more severe or more indulgent, some are more or less discriminating, etc. Operators such as symmetrical sums behave differently depending on the values to be combined. Other operators depend on additional information such as conflict, reliability, context, and can be adapted to the situation at hand. Indeed, one often has to deal with situations where a piece of information is reliable only for some structures, or is not able to discriminate between two objects while another piece of information does. In this context, some operators are particularly powerful, like operators that behave differently depending on whether the values to be combined are of the same order of magnitude or not, whether they are small or high, and operators that depend on some global knowledge.

Let us give a few examples. If we have different constraints about an object (for instance concerning the relations it should have with respect to another object) which have all to be satisfied, these constraints can be combined using a t-norm (a conjunction). This is typically the case when an object is described using relations to several objects or several relations of different types to the same object. If one object has to satisfy one relation or another one then a disjunction represented by a t-conorm has to be used. This occurs for instance when two symmetrical structures with respect to the reference object can be found. Mean operators can be used to combine several estimations and try to find a compromise between them. Such operators have a compensation effect which is interesting in cases where both under-estimation and over-estimation occur. Operators with a variable behavior may also be of great interest. For instance associative symmetrical sums can be used for reinforcing the dynamics between high and low membership degrees, which has advantages for the decision step (since a better discrimination between different situations is achieved). Importance of a constraint or reliabilities can be easily introduced in adaptive operators. Several other examples can be found in different types of applications.

In the qualitative setting, logical tools for fusion have been proposed, in particular based on distances between worlds or between formulas [28,46,47]. Interestingly enough, several fusion rules can be expressed in terms of mathematical morphology [27]. For instance, merging two pieces of information represented by logical formulas can be performed by dilating both formulas until they become consistent. The conjunction of the results is a formula which is the closest one to both initial formulas. In possibilistic logic, a lot of work has been done for fusion of prioritized knowledge bases expressed as a set of (φ_i, a_i) where φ_i is a formula and a_i its degree of certainty or priority. Possibility distributions can be generated from such knowledge bases, and their fusion directly inherits the flexibility offered by fuzzy fusion operators [48]. If the possibility distribution is defined as a function of the distance of a world to a knowledge base, then the previous approach of distance-based fusion is recovered.

4.2. Decision making and recognition

Let us now consider the introduction of fusion in model-based recognition procedures. We summarize here two distinct approaches. Examples are illustrated in [49,50].

A first recognition approach, called global, uses the first type of question (1) raised in the introduction. The idea is to represent all available knowledge about the objects to be recognized. A typical example consists of graph-based representations. The model is then represented as a graph where nodes are objects and edges represent links between these objects. Both nodes and edges are attributed. Node attributes are characteristics of the objects, while edge attributes quantify spatial relationships between the objects. A data graph is then constructed from each image where the recognition has to be performed. Each region of the image (obtained after some processing) constitutes a node of this data graph, and edges represent spatial relationships between regions, as for the model graph. The comparison between representations is performed through the computation of similarities between model graph attributes and data graph attributes. The fusion takes mainly place at this level, in order to combine the similarity values for different relationships. The fusion results constitute an objective function to be optimized by a matching procedure. This approach can benefit from the huge literature on fuzzy comparison tools (see e.g. [51]) and from recent developments on fuzzy morphisms [52]. It has been used in facial feature recognition based on a rough model of a face [53] and brain structure recognition based on an anatomical atlas [54,55]. Mainly weighted average operators are used for the fusion. Such operators allow to weight differently node attributes and edge attributes, or to give more importance to some relationships than to other ones. This is particularly useful when characteristics of objects or of relations have not the same level of stability and variability. The similarity is located at an intermediate level, in the sense that it does not apply directly to the considered objects but to some global feature extracted from these objects. In order to cope with the summarization aspect of such a feature, it may be interesting to incorporate in the similarity measure a weight representing the quality of the relation. Typically a low confidence should be attached to a relation (like adjacency) between two objects that concerns only a few points. Such confidence values are easy to introduce in weighted operators. But other operators could be used as well, in order to exploit further the flexibility of the fuzzy set theory. Another aspect is that some relations are more sensitive than others to the definition of the regions to which they apply. For instance adjacency is very sensitive and may depend on only one point. It is then interesting to have a measure that is high only if both values are high. Indeed, the fact that two objects are adjacent like in the model is more relevant to recognition than the fact that they are not adjacent like in the model [36].

A second type of approach relies on the second type of question (2) raised in the introduction and is called here progressive. In such a progressive approach, objects are recognized sequentially and their recognition makes use of knowledge about their relations with respect to other objects. Relations with respect to previously obtained objects can be combined at two different levels of the procedure. First, fusion can occur in the spatial domain, using spatial fuzzy sets [49]. The result of this fusion allows to build a fuzzy region of interest in which the search of a new object will take place, in a process similar to focalization of attention. In a sequential procedure, the amount of available spatial relations increases with the number of processed objects. Therefore, the recognition of the most difficult structures, usually treated in the last steps, will be focused in a more restricted area. This approach has been used in medical imaging [49,56], as well as in mobile robotics to reason about the spatial position of the robot and the structure of its environment [57]. Another fusion level occurs during the final decision step, i.e. segmentation and recognition of a structure. For this purpose, it was suggested in [56] to introduce relations in the evolution scheme of a deformable model, in which they are fusioned with other types of numerical information, usually edge and regularity constraints.

4.3. Logical reasoning and inference

One of the advantages of logical representations is their inference and reasoning power. Rule-based systems can make use of the proposed representations in a quite straightforward way. But it is also interesting to note that several spatial logics contain ingredients that can be expressed equivalently in morphological terms. We show here some of these links but do not pretend to be exhaustive.

Some links with mereotopology and region connection calculus have already been mentioned in Section 3. They allow us to combine the expressive power of mathematical morphology and the reasoning power of RCC and mereotopology.

The "egg-yolk" structures, as developed e.g. in [58], can also lead to interpretations in terms of mathematical morphology. For instance in this model, establishing whether a yolk can be a mobile part (in translation) of its egg is based on the notion of congruence. This characterization can be expressed in a very simple way using morphological opening (erosion followed by a dilation): the opening of the egg by the yolk considered as the structuring element should be connected.

Let us now consider two examples of logics of distances. The first one defines a modality $A^{\leqslant a}$ by [59]:

$$(\omega \models A^{\leqslant a}\varphi) \quad \text{iff} \quad ((\forall u, d(\omega, u) \leqslant a), (u \models \varphi)), \tag{31}$$

where d is a distance between worlds. It is straightforward to show that $A^{\leqslant a}\varphi$ is equivalent to the erosion of φ by a ball of the distance d of radius a. The dual of $A^{\leqslant a}$ is equivalent to a dilation. Then we have direct correspondences between the axioms of this distance logics and the axioms of our modal morpho-logics as presented in [30]. Some theorems can be also directly deduced from properties of dilation or erosion. For instance, the following is proved to be a theorem:

$$(A^{\leqslant b}\varphi \rightarrow A^{\leqslant a}\varphi) \quad \text{for } a \leqslant b. \tag{32}$$

Using the morphological equivalence, this theorem is directly deduced from the decreasingness of erosion with respect to the size of the structuring element.

The second example concerns nearness logics [60], where the notion of "closer to" is modeled as

$$(x \models \langle N\rangle\varphi, \psi) \quad \text{iff } (\exists y, z), ((y \models \varphi) \wedge (z \models \psi) \wedge N(x, y, z)), \tag{33}$$

where $N(x, y, z)$ means that y is closer to x than z is. The meaning of this expression is that the nearest point distance of x to φ is less than the nearest point distance of x to ψ. An equivalent expression is therefore

$$x \models D^n(\psi) \rightarrow x \models D^n(\varphi), \tag{34}$$

which expresses that x is reached faster from φ than from ψ by dilations of these formulas.

Other links between linear logics or arrow logics and mathematical morphology exist, as already established in [60].

Finally, let us consider logics of convexity [60]:

$$(x \models C\varphi) \quad \text{iff } (\exists y, z), ((y \models \varphi) \wedge (z \models \varphi) \wedge (x \in [y, z])), \tag{35}$$

which expresses a linear closure, the iteration of which provides convexity. This iterative closure is clearly equivalent to morphological closing, where structuring elements are segments in all directions of infinite length (in practice, larger than the largest diameter of the considered spatial entities). All these examples show interesting links between different spatial logics which have not been exhibited before for most of them. They can be exploited in two ways: the properties of morphological operators can provide additional theorems to these logics; conversely spatial logics endow mathematical morphology with powerful inference and reasoning tools.

5. Conclusion

In this paper we considered spatial reasoning under the light of mathematical morphology. Due to its algebraic nature, it provides a unifying framework for representing different types of spatial relationships in quantitative, fuzzy or semi-quantitative, and symbolic or qualitative settings. Based on these knowledge representations, reasoning can be addressed in a numerical or in a logical way. In particular, we have exhibited some links between morphological representations and several spatial logics that can be of interest for reasoning. These links certainly deserve to be further developed.

Another interesting direction for further research is inference of relations from known ones, applying a sort of transitivity of relations. This has been addressed in a probabilistic framework in [61,62]. A lot remains to be done in an algebraic framework and for extended spatial entities.

Other relations could be modeled as well, such as "between" for instance, and introduced in a reasoning process. Modal logics of betweenness have been proposed [2] but less work has been done in quantitative and semi-qualitative frameworks [63].

Finally, complexity issues could be addressed.

References

[1] I. Bloch, Fuzzy spatial relationships for model-based pattern recognition in images and spatial reasoning under imprecision, in: WILF'03, Naples, Italy, 2003.

[2] M. Aiello, Spatial reasoning, theory and practice, Ph.D. thesis, University of Amsterdam, February 2002.

[3] I. Bloch, On fuzzy spatial distances, in: P. Hawkes (Ed.), Advances in Imaging and Electron Physics, vol. 128, Elsevier, Amsterdam, 2003, pp. 51–122.

[4] M. Aurnague, L. Vieu, A. Borillo, Représentation formelle des concepts spatiaux dans la langue, in: M. Denis (Ed.), Language et cognition spatiale, Masson, Paris, 1997, pp. 69–102.

[5] L. Talmy, Toward a Cognitive Semantics, MIT Press, Cambridge, MA, 2000.

[6] E. Clementini, P.D. Felice, D. Hernandez, Qualitative representation of positional information, Artificial Intelligence 95 (1997) 317–356.

[7] A. Herskovits, Language and spatial cognition, A Interdisciplinary Study of the Prepositions in English, Cambridge University Press, Cambridge, MA, 1986.

[8] L. Talmy, How language structures space, in: H.L. Pick, L.P. Acredolo (Eds.), Spatial Orientation: Theory, Research and Application, Plenum Press, New York, 1983.

[9] M. Denis, F. Pazzaglia, C. Cornoldi, L. Bertolo, Spatial discourse and navigation: an analysis of route directions in the city of venice, Applied Cognitive Psychology 13 (1999) 145–174.

[10] M. Gahegan, Proximity operators for qualitative spatial reasoning, in: A.U. Frank, W. Kuhn (Eds.), Spatial Information Theory: A Theoretical Basis for GIS, LNCS, vol. 988, Springer, 1995.

[11] H.W. Guesgen, J. Albrecht, Imprecise reasoning in geographic information systems, Fuzzy Sets and Systems 113 (2000) 121–131.

[12] D.R. Montello, Scale and multiple psychologies of space, in: International Conference COSIT'93, LNCS, vol. 716, Elba Island, Italy, 1993, pp. 312–321.

[13] R.A. Hart, G.T. Moore, The development of spatial cognition: a review, in: R.M. Downs, D. Stea (Eds.), Image and Environment: Cognitive Mapping and Spatial Behavior, Aldine, Chicago, 1973.

[14] L. Nadel, The psychobiology of spatial behavior: the hippocampal formation and spatial mapping, in: E. Alleva, H.-P. Lipp, L. Nadel, A. Fasolo, L. Ricceri (Eds.), Behavioral Brain Research in Naturalistic and Semi-Naturalistic Settings: Possibilities and Perspectives, Kluwer Press, 1995.

[15] B. Kuipers, The spatial semantic hierarchy, Artificial Intelligence 119 (2000) 191–233.

[16] D.J. Peuquet, Representations of geographical space: toward a conceptual synthesis, Annals of the Association of American Geographers 78 (3) (1988) 375–394.

[17] D.M. Mark, M.J. Egenhofer, Modeling spatial relations between lines and regions: combining formal mathematical models and human subjects testing, Cartography and Geographic Information Systems 21 (4) (1994) 195–212.

[18] B.J. Kuipers, T.S. Levitt, Navigation and mapping in large-scale space, AI Magazine 9 (2) (1988) 25–43.

[19] P. Gärdenfors, Conceptual Spaces: The Geometry of Thought, MIT Press, Cambridge, MA, 2000.

[20] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.

[21] I. Bloch, H. Maître, Fuzzy mathematical morphologies: a comparative study, Pattern Recognition 28 (9) (1995) 1341–1387.

[22] D. Sinha, E. Dougherty, Fuzzy mathematical morphology, Journal of Visual Communication and Image Representation 3 (3) (1992) 286–302.

[23] B. de Baets, Fuzzy morphology: a logical approach, in: B. Ayyub, M. Gupta (Eds.), Uncertainty in Engineering and Sciences: Fuzzy Logic, Statistics and Neural Network Approach, Kluwer Academic, 1997, pp. 53–67.

[24] M. Nachtegael, E.E. Kerre, Classical and fuzzy approaches towards mathematical morphology, in: E.E. Kerre, M. Nachtegael (Eds.), Fuzzy Techniques in Image Processing, Studies in Fuzziness and Soft Computing, Physica-Verlag, Springer, 2000, pp. 3–57 (Chapter 1).

[25] H.J.A.M. Heijmans, C. Ronse, The algebraic basis of mathematical morphology—Part I: dilations and erosions, Computer Vision, Graphics and Image Processing 50 (1990) 245–295.

[26] T.-Q. Deng, H. Heijmans, Grey-scale morphology based on fuzzy logic, Journal of Mathematical Imaging and Vision 16 (2002) 155–171.

[27] I. Bloch, J. Lang, Towards mathematical morpho-logics, in: 8th International Conference on Information Processing and Management of Uncertainty in Knowledge based Systems IPMU 2000, vol. III, Madrid, Spain, 2000, pp. 1405–1412.

[28] C. Lafage, J. Lang, Logical representation of preferences for group decision making, in: A.G. Cohn, F. Giunchiglia, B. Selman (Eds.), 7th International Conference on Principles of Knowledge Representation and Reasoning KR 2000, Morgan Kaufmann, San Francisco, CA, Breckenridge, CO, 2000, pp. 457–468.
[29] B. Chellas, Modal Logic, An Introduction, Cambridge University Press, Cambridge, 1980.
[30] I. Bloch, Modal logics based on mathematical morphology for spatial reasoning, Journal of Applied Non Classical Logics 12 (3–4) (2002) 399–424.
[31] I. Bloch, On links between mathematical morphology and rough sets, Pattern Recognition 33 (9) (2000) 1487–1496.
[32] I. Bloch, Using fuzzy mathematical morphology in the Dempster–Shafer framework for image fusion under imprecision, in: IFSA'97, Prague, 1997, pp. 209–214.
[33] N. Asher, L. Vieu, Toward a geometry of common sense: a semantics and a complete axiomatization of mereotopology, in: IJCAI'95, San Mateo, CA, 1995, pp. 846–852.
[34] S. Dutta, Approximate spatial reasoning: integrating qualitative and quantitative constraints, International Journal of Approximate Reasoning 5 (1991) 307–331.
[35] I. Bloch, Fuzzy spatial relationships for image processing and interpretation: a review, Image and Vision Computing 23 (2) (2005) 89–110.
[36] I. Bloch, H. Maître, M. Anvari, Fuzzy adjacency between image objects, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (6) (1997) 615–653.
[37] I. Bloch, On fuzzy distances and their use in image processing under imprecision, Pattern Recognition 32 (11) (1999) 1873–1895.
[38] I. Bloch, A. Ralescu, Directional relative position between objects in image processing: a comparison between fuzzy approaches, Pattern Recognition 36 (2003) 1563–1582.
[39] D. Dubois, H. Prade, C. Testemale, Weighted fuzzy pattern matching, Fuzzy Sets and Systems 28 (1988) 313–331.
[40] I. Bloch, Spatial representation of spatial relationships knowledge, in: A.G. Cohn, F. Giunchiglia, B. Selman (Eds.), 7th International Conference on Principles of Knowledge Representation and Reasoning KR 2000, Morgan Kaufmann, San Francisco, CA, Breckenridge, CO, 2000, pp. 247–258.
[41] D. Randell, Z. Cui, A. Cohn, A spatial logic based on regions and connection, in: B. Nebel, C. Rich, W. Swartout (Eds.), Principles of Knowledge Representation and Reasoning KR'92, Kaufmann, San Mateo, CA, 1992, pp. 165–176.
[42] D. Dubois, H. Prade, A review of fuzzy set aggregation connectives, Information Sciences 36 (1985) 85–121.
[43] R.R. Yager, Connectives and quantifiers in fuzzy sets, Fuzzy Sets and Systems 40 (1991) 39–75.
[44] D. Dubois, H. Prade, R. Yager, Merging fuzzy information, in: J. Bezdek, D. Dubois, H. Prade (Eds.), Handbook of Fuzzy Sets Series, Approximate Reasoning and Information Systems, Kluwer, 1999 (Chapter 6).
[45] I. Bloch, Information combination operators for data fusion: a comparative review with classification, IEEE Transactions on Systems, Man, and Cybernetics 26 (1) (1996) 52–67.
[46] S. Konieczny, R. Pino-Pérez, On the logic of merging, in: 6th International Conference on Principles of Knowledge Representation and Reasoning, Trento, Italy, 1998, pp. 488–498.
[47] S. Konieczny, R. Pino-Pérez, Merging with integrity constraints, in: A. Hunter, S. Parsons (Eds.), ECSQARU'99, LNCS, vol. 1638, Springer, London, 1999, pp. 232–244.
[48] S. Benferhat, D. Dubois, S. Kaci, H. Prade, Logique possibiliste et fusion d'informations, in: I. Bloch, L. Cholvy (Eds.), Fusion de données numériques et d'informations symboliques, Techniques et Sciences Informatiques, vol. 22, Hermès-Lavoisier, 2003, pp. 1035–1064.
[49] I. Bloch, T. Géraud, H. Maître, Representation and fusion of heterogeneous fuzzy information in the 3D space for model-based structural recognition—application to 3D brain imaging, Artificial Intelligence Journal 148 (2003) 141–175.
[50] I. Bloch, O. Colliot, O. Camara, T. Géraud, Fusion of spatial relationships for guiding recognition. Example of brain structure recognition in 3D MRI, Pattern Recognition Letters 26 (2005) 449–457.
[51] B. Bouchon-Meunier, M. Rifqi, S. Bothorel, Towards general measures of comparison of objects, Fuzzy Sets and Systems 84 (2) (1996) 143–153.
[52] A. Perchant, I. Bloch, Fuzzy morphisms between graphs, Fuzzy Sets and Systems 128 (2) (2002) 149–168.
[53] R. Cesar, E. Bengoetxea, I. Bloch, Inexact graph matching using stochastic optimization techniques for facial feature recognition, in: International Conference on Pattern Recognition ICPR 2002, vol. 2, Québec, 2002, pp. 465–468.

[54] A. Perchant, C. Boeres, I. Bloch, M. Roux, C. Ribeiro, Model-based scene recognition using graph fuzzy homomorphism solved by genetic algorithm, in: GbR'99 2nd International Workshop on Graph-Based Representations in Pattern Recognition, Castle of Haindorf, Austria, 1999, pp. 61–70.
[55] E. Bengoetxea, P. Larranaga, I. Bloch, A. Perchant, C. Boeres, Inexact graph matching by means of estimation of distribution algorithms, Pattern Recognition 35 (2002) 2867–2880.
[56] O. Colliot, O. Camara, R. Dewynter, I. Bloch, Description of brain internal structures by means of spatial relations for MR image segmentation, in: SPIE Medical Imaging, vol. 5370, San Diego, CA, USA, 2004, pp. 444–455.
[57] I. Bloch, A. Saffiotti, On the representation of fuzzy spatial relations in robot maps, in: IPMU 2002, vol. III, Annecy, France, 2002, pp. 1587–1594.
[58] M. Cristani, A.G. Cohn, B. Bennett, Spatial locations via morpho-mereology, in: A.G. Cohn, F. Giunchiglia, B. Selman (Eds.), 7th International Conference on Principles of Knowledge Representation and Reasoning KR 2000, Morgan Kaufmann, San Francisco, CA, Breckenridge, CO, 2000, pp. 15–25.
[59] O. Kutz, H. Sturm, N.-Y. Suzuki, F. Wolter, M. Zakharyaschev, Axiomatizing distance logics, Journal of Applied Non Classical Logics 12 (3–4) (2002) 425–440.
[60] M. Aiello, J. van Benthem, A modal walk through space, Journal of Applied Non Classical Logics 12 (3–4) (2002) 319–364.
[61] S. Dehak, I. Bloch, H. Maître, Inference of directional spatial relationship between points: a probabilistic approach, in: IEEE International Conference on Image Processing, ICIP 2001, vol. III, Thessaloniki, Greece, 2001, pp. 704–707.
[62] R. Dehak, I. Bloch, H. Maître, Spatial reasoning with incomplete information about relative positioning, IEEE Transactions on Pattern Analysis and Machine Intelligence 27 (9) (2005) 1473–1484.
[63] I. Bloch, O. Colliot, R.M. Cesar, Mathematical modeling of the relationship "between" based on morphological operators, in: ISMM 2005, Paris, France, 2005, pp. 299–308.