Non-Classical Logic via Mathematical Morphology

Logique non classique et
morphologie mathématique

Mitsuhiko Fujio[†] and Isabelle Bloch[‡]

Technical Report

[†]Kyushu Institute of Technology (KIT)
[‡]École Nationale Supérieure des Télécommunications (ENST)

Abstract

In this report, we apply the method of non-linear analysis using mathematical morphology to non-classical logics, i.e., modal logics, intuitionistic logic and linear logics. For modal logic, we extend the method of construction of modal logics based on standard models to those based on Kripke frames in order to avoid the assumption of surjectivity of valuation as well as to make it dependent purely on the mathematical morphology. We also show that a pair of modal operators defined from an adjunction constitutes a part of a quadruple of temporal operators. For intuitionistic (respectively linear) logic, we characterize subframes (resp. subquantales) in a frame (resp. quantale) in terms of interior (resp. closure) operators. Furthermore, we give a necessary and sufficient condition for a Kripke frame gives rise to an interior (resp. a closure) operator through an adjuntion. We also consider the case of those operators come from Galois connections.

Keywords Non-Classical Logics, Modal Logics, Temporal Logics, Intuitionistic Logic, Intuitionistic Linear Logics, Linear Logics, Mathematical Morphology, Adjunction, Erosion, Dilation, Galois Connection.

Résumé

Dans ce rapport, nous exploitons les méthodes non linéaires de la morphologie mathématique dans le cadre des logiques non-classiques : logiques modale, intuitionniste et linéaire. Nous étendons la méthode de construction des logiques modales à partir de modèles standard aux "frames" de Kripke afin de se passer de l'hypothèse de surjectivité de la fonction de valuation et de rendre la logique purement dépendante de la morphologie mathématique. Nous montrons aussi qu'une paire d'opérateurs modaux définis à partir d'une adjonction constitue une partie d'un 4-uple d'opérateurs temporels. Pour la logique intuitionniste (respectivement linéaire), nous caractérisons les sous-frames (resp. sous-quantales) dans une frame (resp. quantale) en termes d'opérateur d'intérieur (resp. fermeture). De plus, nous établissons une condition nécessaire et suffisante pour qu'une frame de Kripke donne un opérateur d'intérieur (resp. fermeture) via une adjonction. Nous considérons aussi le cas où ces opérateurs sont définis à partir d'une connexion de Galois.

Mot-Clés logiques non classiques, logiques modales, logiques temporels, logique intuitionniste, logiques linéaire intuitionniste, logiques linéaire, morphologie mathématique, adjonction, érosion, dilatation, connexion de Galois.

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Contents

Chapter 1

Introduction

Between mathematical morphology operators and modal logic operators, several similarities are observed in their behavior. One of the authors proposed a construction of modal operators $(\Box, \Diamond)$ from morphological erosion E_R and dilation D_R [7], [5], and extended it for adjunctions [6] on logical formulas in relevance to the generalized notions of erosion and dilation on lattices [27], [19], [18].

Modal logics based on standard models are considered in both approaches. Let $\mathscr{M}$ be a standard model composed of a set of possible worlds Ω, an accessability relation R on Ω and a valuation $V : \Psi \to \mathfrak{P}(\Omega)$. Here Ψ is the set of propositional symbols and $\mathfrak{P}(\Omega)$ is the powerset of Ω. The valuation is extended to the full set of modal formulas Φ_m in usual Kripke semantics. On the other hand, morphological erosion $E_R X$ and dilation $D_R X$ of a set $X \in \mathfrak{P}(\Omega)$ is defined based on the binary relation R on Ω, regarded as a structuring element [26].

In [5], erosion $E_R\phi$ and dilation $D_R\phi$ of a formula ϕ are defined in order

$$V(E_R\phi) = E_R(V(\phi)), \qquad V(D_R\phi) = D_R(V(\phi))$$

be satisfied. We note that surjectivity of $V : \Phi_m \to \mathfrak{P}(\Omega)$ is assumed to define operators on fomulas genuinely (modulo $\mathscr{M}$-equivalence $\equiv$). Then so defined morphological operators are related with modal operators by

$$\Box\phi \equiv E_R\phi, \qquad \Diamond\phi \equiv D_R\phi,$$

and it is shown that they give rise to a normal modal logic. Also, several properties about schemas of the derived modal logic are investigated in relevance to that of accessibility relations R. Surjectivity of V plays an important role again there. In fact, this assumption allows us to replace each instance of the form $V(\phi)$ $(\phi \in \Phi_m)$ in arguments with $X \in \mathfrak{P}(\Omega)$. This enables us to reduce any arguments over fomulas to those over sets.

The assumption of surjectivity is essential in the process of relating schemas with properties of accessibility relation for modal logics based on standard models. However, as we will discuss in section 3.1.2, this assumption seems to be rather strong. To avoid this assumption, we propose to consider modal logics based on Kripke frames. We can cover any subset X with an instance of the form $V(\phi)$ for some valuation V and formula ϕ by considering arbitrary standard model based on a fixed Kripke frame. Furthermore, this approach seems to be natural to say "defining modal logic from mathematical morphology" since the morphological operations on the set Ω are defined dependently on the Kripke frame structure only.

A pair of modal operator $(\Box, \Diamond')$ forming an adjunction is defined as an extension of idea of defining modal operators as morphological operators, and investigated in [6]. Because the operators in this case are not dual to each other in general as is already pointed there, one use a different notation from the usual one. Although the dual of an arbitrary operator always exists in Boolean lattices, the adjoint may not. As a consequence, the assumption $(\Box, \Diamond')$ to be an adjunction produces outgrowth. Namely, by considering the dual operators of $\Box$ and $\Diamond'$, we have a quadruple $(\Box, \Diamond, \Box', \Diamond')$ of modal operators. Then we can show that, under our assumption of adjunction, these are temporal operators and thus we have a temporal logic. We also show that to give an adjiont pair is equivalent to give a quadruple of temporal operators.

Adjunctions are also observed in other non-classical logics, such as intuitionistic logic and linear logic. In fact their algebraic models are defined to have some adjunctions. On the other hand, their model theories make use of a sort of interior/closure operators to produces a family of models. In this article, we try to apply morphological analysis to these logics through possible world semantics. We will describe adjunctions contained in these logics in terms of binary relations. We also investigate opening/closing operators associated with the accessiblity relations through adjunction and give a necessary and sufficient condition that the accessibilty relation gives rise to a suitable interior/closure operator which defines a model for the logic by the set of open/closed elements. We also consider the case of Galois connection (a skew variant of adjunction) as well as adjunction.

Chapter 2 is devoted to introduction of notations (§2.1) and logical and algebraic requisites (§§2.2-3). In the first section (§3.1) of chapter 3, we discuss about the assumption of surjectivity of valuation after we give a brief review of precedent works. Then we define the notions of modal logics based on models and Kripke frames and we show an equivalence between a schema and a property of accessibility relation in a general form. We consider adjunctions in our appraoch in §3.2. There we show a relation between adjunction schema and bidirectional model frames and then establish the equivalence of ajdunction schema and temporal logic. In chapter 4, we apply morphological analysis to intuitionistic logic (LJ). After we introduce the notion of frame as an algebraic model for LJ and we give necessary and sufficient conditions for a Kripke frame to give a frame as the set of open sets for an interior operator (§4.1). In §4.2 we investigate possible world semantics for LJ. Chapter 5 for linear logic (LL) is parallel to chapter 4 for IL, but arguments are dual. We first introduce the notion of quantale as an algebraic model for intuitionistic linear logic (ILL), then we give necessary and sufficient conditions for a Kripke frame to give a quantale as the set of closed sets for a closure operator (§5.1). Possible world semantics for ILL and IL are respectively given in §5.2 and §5.3. Sequent calculi for logics we are concerned with are found in Appendix A.1. In Appendix A.2, we give Kripke semantics for n-modal logic.

Chapter 2

Preliminaries

2.1 Notations

We often regard a binary relation $R \subseteq X \times A$ as a correspondence $R : X \to A$ by

$$R : X \ni x \quad \mapsto \quad R(x) = \{a \in A \mid (x, a) \in R\} \subseteq A$$

and vise versa. The *transposition* ${}^tR$ of R is given by

$${}^tR \quad = \quad \{(a, x) \mid a \in A,\ x \in X,\ (x, a) \in R\} \subseteq A \times X$$

or in terms of correspondence,

$${}^tR : A \ni a \quad \mapsto \quad \{x \in X \mid (x, a) \in R\} \subseteq X.$$

We will use two types of "image" of subset $Y \subseteq X$ under the correspondence R:

$$R(Y) = \bigcup_{y \in Y} R(y), \qquad\qquad R^*(Y) = \bigcap_{y \in Y} R(y).$$

The former is the usual set theoretical R-image of Y, and we refer to it as the *existential image* of Y. On the contrary, the latter plays an important role in the context of Galois connection. We call it the *universal image* of Y. The usual set theoretical inverse image of a subset $B \subseteq A$ can be expressed as $R^{-1}(B) = {}^tR(B)$ in our notation.

For a structured set $\mathscr{X} = (X, R)$ with a binary relation $R \subseteq X \times X$, we call $\mathscr{X}^{op} = (X, {}^tR)$ the *opposite* of (X, R).

2.2 Logical preliminaries

2.2.1 Syntax

Formulas

We denote the set of denumerable number of propositional symbols by

$$\Psi \quad = \quad \{p_0, p_1, p_2, \ldots\}.$$

The set of fomulas generated by Ψ with usual connectives such as

$$\top, \bot, \neg, \vee, \wedge, \to, \leftrightarrow$$

and a finite set of additional connectives Γ is denoted by $\Phi(\Psi, \Gamma)$. Also the set of fomulas generated by Ψ with connectives used in the context of linear logic such as

$$\mathbf{1}, \mathbf{0}, \top, \otimes, \oplus, \&, \multimap$$

and a finite set of additional connectives Γ is denoted by $\Phi^L(\Psi, \Gamma)$.

Example 2.2.1.

(1) The set of formulas of classical logic:

$$\Phi_c = \Phi(\Psi). \tag{2.2.1}$$

(2) The set of formulas of modal logic:

$$\Phi_m = \Phi(\Psi, \Box, \Diamond). \tag{2.2.2}$$

The set of formulas of n-modal logic:

$$\Phi_{n,m} = \Phi(\Psi, \Box_1, \Diamond_1, \ldots, \Box_n, \Diamond_n). \tag{2.2.3}$$

The set of formulas of temporal logic:

$$\Phi_t = \Phi(\Psi, G, F, H, P) \quad (\cong \Phi_{2,m}) \tag{2.2.4}$$

(3) The set of formulas of intuitionistic logic:

$$\Phi_i = \Phi(\Psi) \quad (= \Phi_c). \tag{2.2.5}$$

(4) The set of formulas of intuitionistic linear logic:

$$\Phi_{i\ell} = \Phi^L(\Psi) \tag{2.2.6}$$

The set of formulas of linear logic:

$$\Phi_\ell = \Phi^L(\Psi, \perp, ^\perp, \parr) \tag{2.2.7}$$

Note. For formulas of linear logic, we employ the Girard's notations [15], [16]. Some authors use other notations. Then Girard's notations should be read

$\otimes$ as $\cdot$, $\oplus$ as $\vee$, $\&$ as $\wedge$, $\mathbf{0}$ as $\perp$, $\perp$ as 0

in [28], [21], [2], *etc.* Furthermore, $\phi^\perp$ denotes the *linear negation* of ϕ and $\parr$ is the *par* operator dual to $\otimes$.

Sequent calculi

Definition. Let Φ be a set of formulas. We call each expression of the form

$$\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n \tag{2.2.8}$$

a *sequent* in Φ, where m and n are non-negative integers and ϕ's and ψ's are fomulas in Φ. As special cases, each of

$$\begin{array}{lrl} m = 0 & & \vdash \;\; \psi_1, \ldots, \psi_n \\ n = 0 & \phi_1, \ldots, \phi_m & \vdash \\ m = n = 0 & & \vdash \end{array}$$

is a sequent.

A system of sequent calculus consists of a set of *axioms* and *inference rules.* Axioms and inference rules for classical logic, modal logic, intuitionistic logic, linear logic and intuitionistic logic are given in Appendix A.1

A sequent is called *provable* iff it is derivable from axioms by using inference rules.

Note. The meanings of a sequent $\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n$ is given according to the context as follows:

- In the context of non-linear logic :

  $\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n$ means $\phi_1 \wedge \cdots \wedge \phi_m \to \psi_1 \vee \cdots \vee \psi_n$.

- In the contex of linear logic :

  $\phi_1, \ldots, \phi_m \vdash \psi_1, \ldots, \psi_n$ means $\phi_1 \otimes \ldots \otimes \phi_m \multimap \psi_1 \parr \ldots \parr \psi_n$.

2.2.2 Semantics

Standard models

Definition. Let $R \subseteq \Omega \times \Omega$ be a binary relation. A relation structure $\mathscr{F} = (\Omega, R)$ is called a *Kripke frame*. Each element $\omega \in \Omega$ is called a *possible world* and the binary relation R is called the *accessibity relation*. A mapping $V : \Psi \to \mathfrak{P}(\Omega)$ is called a *valuation*. A *standard model* based on a Kripke frame $\mathscr{F}$ is a pair $\mathscr{M} = (\mathscr{F}, V)$.

In particular for n-modal logic, as a generalization of Kripke frame defined as above, n accessibility relations $R_1, \ldots, R_n$ is used. n-relation structure $\mathscr{F} = (\Omega, R_1, \ldots, R_n)$ is called an *n-frame*.

Let Φ be a set of formulas and $\mathscr{M} = (\Omega, R, V)$ be a standard model. The truth value of a formula $\phi \in \Phi$ at a possible world $\omega \in \Omega$ in the standard model $\mathscr{M}$ is recursively defined by its constructions. We denote ϕ is true at ω in $\mathscr{M}$ by $\models_\omega^{\mathscr{M}} \phi$. Similarly, we denote ϕ is false at ω in $\mathscr{M}$ by $\not\models_\omega^{\mathscr{M}} \phi$. The definition of truth values for n-modal fomulas is given in Appendix A.2. After the truth value of each formula $\phi \in \Phi$ is defined, we consider the valuation V is extended to Φ by

$$V(\phi) = \left\{ \omega \in \Omega \;\middle|\; \models_\omega^{\mathscr{M}} \phi \right\}. \tag{2.2.9}$$

We call $V(\phi)$ the *true set* of ϕ.

Definition. A formula $\phi \in \Phi$ is said to be *true* in a model $\mathscr{M}$ and denoted by $\models^{\mathscr{M}} \phi$ iff

$$V(\phi) = \Omega; \tag{2.2.10}$$

ϕ is said to be *valid* in a Kripke frame $\mathscr{F}$ and denoted by $\models^{\mathscr{F}} \phi$ iff $\models^{\mathscr{M}} \phi$ for every model $\mathscr{M}$ based on $\mathscr{F}$; ϕ is said to be *valid* in a class $\mathfrak{F}$ of Kripke frames and denoted by $\models^{\mathfrak{F}} \phi$ iff $\models^{\mathscr{F}} \phi$ for every frame $\mathscr{F}$ in $\mathfrak{F}$.

Note. In semantics for linear logics, the notion of truth of each formula in a model and hence, that of validity in a frame or in a class of frames are slightly modified. For more precision, see section 5.2.2.

2.3 Algebraic preliminaries

Notions of dilation and erosion were first extended to complete lattices in [27] and general properties are investigated in [19] (see also [18]). For the sake of development of morphological analysis on formal systems of logics which are not complete in general, we generalize these notions to partially ordered sets. To treat adjunctions and Galois connection in a similar way, we introduce the notion of connection. Galois connections were considered for partially ordered sets from its early study [3].

2.3.1 Erosion, dilation and connection

For notions introduced here, we basically follow [18] but we consider a slightly general case of non-complete lattices or simply partially ordered sets.

Definition. Let X, A be partially ordered sets. A mapping $\varepsilon : X \to A$ is called an *erosion* iff for any family $x_\lambda \in X$ that has an infimum $\bigwedge_\lambda x_\lambda \in X$, the family $\varepsilon(x_\lambda)$ also has an infimum and

$$\bigwedge_\lambda \varepsilon(x_\lambda) = \varepsilon\left(\bigwedge_\lambda x_\lambda\right)$$

is satisfied. A mapping $\delta : X \to A$ is called a *dilation* iff for any family $x_\lambda \in X$ that has a supremum $\bigvee_\lambda x_\lambda \in X$, the family $\delta(x_\lambda)$ also has a supremum and

$$\bigvee_\lambda \delta(x_\lambda) = \delta\left(\bigvee_\lambda x_\lambda\right)$$

is satisfied. A mapping $\gamma : X \to A$ is called a *connection* iff for any family $x_\lambda \in X$ that has a supremum $\bigvee_\lambda x_\lambda \in X$, the family $\gamma(x_\lambda)$ has an infimum and

$$\bigwedge_\lambda \gamma(x_\lambda) = \gamma\left(\bigvee_\lambda x_\lambda\right)$$

is satisfied.

Proposition 2.1. Every erosion or dilation is monotonous. Every connection is anti-monotonous.

Example 2.3.1. Let $R \subseteq X \times A$ be a binary relation. We define the following set operators from $\mathfrak{P}(A)$ into $\mathfrak{P}(X)$ (note that the direction is opposed):

$$E_R(B) = \{x \in X \mid R(x) \subseteq B\}, \tag{2.3.1}$$
$$D_R(B) = \{x \in X \mid R(x) \cap B \neq \emptyset\} \ \left(= {}^tR(B)\right), \tag{2.3.2}$$
$$C_R(B) = \{x \in X \mid B \subseteq R(x)\} \ \left(= {}^tR^*(B)\right) \tag{2.3.3}$$

for $B \in \mathfrak{P}(A)$. Then

- $E_R : \mathfrak{P}(A) \to \mathfrak{P}(X)$ is an erosion, which we call the *morphological erosion defined by R.*

- $D_R : \mathfrak{P}(A) \to \mathfrak{P}(X)$ is a dilation, which we call the *morphological dilation defined by* R.

- $C_R : \mathfrak{P}(A) \to \mathfrak{P}(X)$ is a connection, which we call the *connection defined by* R.

By considering the transposition ${}^tR$ of R, we also obtain operators $E_{{}^tR} : \mathfrak{P}(X) \to \mathfrak{P}(A)$, $D_{{}^tR} : \mathfrak{P}(X) \to \mathfrak{P}(A)$, $C_{{}^tR} : \mathfrak{P}(X) \to \mathfrak{P}(A)$. We note that all of erosions, dilations and connections of set lattices are obtained in this way.

2.3.2 Adjunction and Galois connection

Definition. Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be mappings. The pair (f, g) is called an *adjunction* (from X into A) iff $\forall x \in X, \forall a \in A$

$$a \le f(x) \Leftrightarrow g(a) \le x \tag{2.3.4}$$

is satisfied. f is called the *left adjoint* of g and also g is called the *right adjoint* of f [18].

Note. It should be remarked that some authors, especially in the context of category theory (*cf.* [24]), use the words "right" and "left" in the opposed manner. Here we follow the manner in the context of mathematical morphology [18].

Proposition 2.2. Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be mappings. The following conditions are equivalent:

(1) (f, g) is an adjunction from X into A.

(2) f is monotonous and $\forall a \in A$, $g(a) = \min f^{-1}\{b \in A \mid a \le b\}$ is satisfied.

(3) g is monotonous and $\forall x \in X$, $f(x) = \max g^{-1}\{y \in X \mid y \le x\}$ is satisfied.

Proof. $\underline{(1) \Rightarrow (2)}$ First we note that for any $x \in X$,

$$f(x) = \max\{a \in A \mid g(a) \le x\}.$$

In fact, $f(x) \le f(x)$ implies $g(f(x)) \le x$ and hence

$$f(x) \in \{a \in A \mid g(a) \le x\} \quad \text{and} \quad a \le f(x)\,(\forall a \in \{a \in A \mid g(a) \le x\}).$$

For monotonicity of f, let $x \leq x' \in X$.

$$\begin{aligned} x \leq x' \quad &\Rightarrow \quad \{a \in A \mid g(a) \leq x\} \subseteq \{a \in A \mid g(a) \leq x'\} \\ &\Rightarrow \quad f(x) = \max\{a \in A \mid g(a) \leq x\} \\ &\qquad\qquad \leq \max\{a \in A \mid g(a) \leq x'\} = f(x'). \end{aligned}$$

On the other hand, for $a \in A$,

$$\begin{aligned} g(a) \leq g(a) \quad &\Rightarrow \quad a \leq f(g(a)) \\ &\Rightarrow \quad f(g(a)) \in \{b \in A \mid a \leq b\} \\ &\Rightarrow \quad g(a) \in f^{-1}\{b \in A \mid a \leq b\}. \end{aligned}$$

And

$$a' \in f^{-1}\{b \in A \mid a \leq b\} \quad \Rightarrow \quad a \leq f(a') \;\Leftrightarrow\; g(a) \leq a'.$$

Thus $g(a)$ is the minimum element of $f^{-1}\{b \in A \mid a \leq b\}$.

(2) $\Rightarrow$ (1) For any $x \in X$, $a \in A$,

$$\begin{aligned} a \leq f(x) \quad &\Leftrightarrow \quad f(x) \in \{b \in A \mid a \leq b\} \\ &\Leftrightarrow \quad x \in f^{-1}\{b \in A \mid a \leq b\} \\ &\Leftrightarrow \quad g(a) \leq x. \end{aligned}$$

Hence (f, g) is an adjunction.

Equivalence of (1) and (3) can be shown similarly.

q.e.d.

The following proposion gives another characterization for a pair of monotonous mappings to give rise to an adjunction.

Proposition 2.3. Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be monotonous mappings. For the pair (f, g) to be an adjunction it is necessary and sufficient that

$$g(f(x)) \leq x, \qquad\qquad a \leq f(g(a)) \tag{2.3.5}$$

are satisfied $\forall x \in X$, $\forall a \in A$.

Proof. Suppose that (f, g) is an adjunction. Let $x \in X$ and put $a = f(x)$. Then by (2.3.4),

$$a = f(x) \quad \Rightarrow \quad g(f(x)) \leq x.$$

Similarly, for $a \in A$, put $x = g(a)$. Then again by (2.3.4), we have

$$x = g(a) \quad \Rightarrow \quad a \le f(g(a)).$$

Conversely, suppose that (f, g) satisfies (2.3.5). Since f, g are monotonous, $\forall x \in X$, $\forall a \in A$,

$$\begin{aligned} a \le f(x) \quad &\Rightarrow \quad g(a) \le g(f(x)) \xrightarrow{(2.3.5)} g(a) \le x, \\ g(a) \le x \quad &\Rightarrow \quad f(g(a)) \le f(x) \xrightarrow{(2.3.5)} a \le f(x). \end{aligned}$$

<u>*q.e.d.*</u>

Proposition 2.4. Let X, A be partially ordered sets and (f, g) be an adjunction from X into A. Then

(1) f is an erosion,

(2) g is a dilation.

Proof.

(1) Let $x_\lambda \in X$ be a family that has an infimum $x = \bigwedge_\lambda x_\lambda \in X$. Since f is monotonous,

$$f(x) \quad \le \quad f(x_\lambda).$$

On the other hand, suppose that a is a lower bound of $\{f(x_\lambda)\}$: *i.e.*, $a \le f(x_\lambda)$. Then by virtue of (2.3.4), we have $g(a) \le x_\lambda$ and hence $g(a) \le x = \bigwedge_\lambda x_\lambda$. Again by (2.3.4), we conclude that

$$a \quad \le \quad f(x).$$

By combining this and the result above, we conclude that $f(x)$ is the infimum of $\{f(x_\lambda)\}$. Thus f is an erosion.

(2) Similar to (1).

<u>*q.e.d.*</u>

The converse of Proposition 2.4 holds under some conditions:

Proposition 2.5. Let X, A be partially ordered sets.

(1) When A is a complete $\bigwedge$-lattice, for a mapping $f : X \to A$ to be an erosion, it is necessary and sufficient that f is monotonous and the pair (f, g) is an adjunction for the mapping defined by

$$g(a) \quad = \quad \bigwedge f^{-1} \{b \in A \mid a \le b\}.$$

(2) When X is a complete $\bigvee$-lattice, for a mapping $g : A \to X$ to be a dilation, it is necessary and sufficient that g is monotonous and the pair (f, g) is an adjunction for the mapping defined by

$$f(x) \;=\; \bigvee g^{-1} \{y \in X \mid y \leq x\} .$$

Proof.

(1) Sufficiency is clear from Proposition 2.4. For necessity, we first note that, since A is $\bigwedge$-complete, $g : A \to X$ is well-defined. Now assume that $f : X \to A$ is an erosion. It follows from Proposition 2.1 that f is monotonous. When $a \leq f(x)$,

$$\begin{aligned} a \leq f(x) \;&\Rightarrow\; x \in f^{-1} \{b \in A \mid a \leq b\} \\ &\Rightarrow\; x \geq \bigwedge f^{-1} (\{b \in A \mid b \leq a\}) = g(a). \end{aligned}$$

And conversely, when $g(a) \leq x$, by virtue of monotonicity of f,

$$\begin{aligned} f(x) \;&\geq\; f(g(a)) \\ &=\; f\left(\bigwedge f^{-1} \{b \in A \mid b \leq a\}\right) \\ &=\; \bigwedge (f (f^{-1} \{b \in A \mid b \leq a\})) \\ &\geq\; \bigwedge (\{b \in A \mid b \leq a\}) \;=\; a. \end{aligned}$$

This shows that (f, g) is an adjunction.

(2) Similar to (1).

q.e.d.

Example 2.3.2 (Monoid action). Suppose that a monoid M acts on a set X:

$$M \times X \ni (m, x) \mapsto mx \in X.$$

For $N \in \mathfrak{P}(M)$, $Y \in \mathfrak{P}(X)$, we put

$$N \cdot Y = \bigcup_{n \in N} nY, \qquad\qquad N \multimap Y = \{x \in X \mid N \cdot \{x\} \subseteq Y\} .$$

Since

$$\begin{aligned} Y \subseteq (N \multimap Z) \;&\Leftrightarrow\; (y \in Y \Rightarrow N \cdot \{y\} \subseteq Z) \\ &\Leftrightarrow\; (n \in N \wedge y \in Y \Rightarrow ny \in Z) \\ &\Leftrightarrow\; N \cdot Y \subseteq Z \end{aligned}$$

we have that $(N \multimap, N\cdot)$ is an adjunction from $\mathfrak{P}(X)$ into $\mathfrak{P}(X)$.

- In particular, a monoid structure itself gives rise to an adjunction regarding as $X = M$.
- We note that when M is a group and the action is a group action we have

$$N \multimap Y = \bigcap_{n \in N} n^{-1} Y.$$

- This example gives a generalization of Minkowski addition and subtraction where $M = X = \mathbb{R}^n$.

Example 2.3.3 (Binary relation). (*cf.* Example 2.3.1) Let $R \subseteq X \times A$ be a binary relation. Then the pair $(E_{{}^tR}, D_R)$ is an adjunction from $\mathfrak{P}(X)$ into $\mathfrak{P}(A)$. In fact, $\forall Y \in \mathfrak{P}(X)$, $\forall B \in \mathfrak{P}(A)$,

$$\begin{aligned} B \subseteq E_{{}^tR}(Y) &\Leftrightarrow \forall b \in B \, ({}^tR(b) \subseteq Y) \\ &\Leftrightarrow \forall b \in B, \forall x \in X \, (x \in {}^tR(b) \Rightarrow x \in Y) \\ &\Leftrightarrow \forall x \in X, \forall b \in B \, (b \in R(x) \Rightarrow x \in Y) \\ &\Leftrightarrow \forall x \in X \, (R(x) \cap B \neq \emptyset \Rightarrow x \in Y) \\ &\Leftrightarrow D_R(B) \subseteq Y. \end{aligned}$$

Similarly, the pair $(E_R, D_{{}^tR})$ is an adjunction from $\mathfrak{P}(A)$ into $\mathfrak{P}(X)$.

Example 2.3.4 (Heyting algebra). Let H be a lattice. Consider the mapping

$$h\wedge : H \ni x \mapsto h \wedge x \in H$$

defined for $h \in H$. H is called a *Heyting algebra* iff the mapping $h\wedge$ has a left adjoint for each $h \in H$. The left adjoint of "$h\wedge$" is denoted by "$h \to$":

$$x \leq (h \to a) \quad \Leftrightarrow \quad h \wedge x \leq a.$$

The topology $\mathfrak{O} \subseteq \mathfrak{P}(X)$ of any topological space $(X, \mathfrak{O})$ is a Heyting algebra.

A Heyting algebra is an algebraic model for intuitionistic logic.

Example 2.3.5 (IL algebra). Let L be a lattice with bottom 0, equipped with a commutative monoid structure with the multiplication $\cdot$ and the unit element e. Consider the mapping

$$\ell\cdot : L \ni x \mapsto \ell \cdot x \in L$$

defined for $\ell \in L$. L is called an *IL algebra* iff the mapping $\ell\cdot$ has a left adjoint for each $\ell \in L$. The left adjoint of "$\ell\cdot$" is denoted by "$\ell \multimap$":

$$x \leq (\ell \multimap a) \quad \Leftrightarrow \quad \ell \cdot x \leq a.$$

The set lattice $\mathfrak{P}(M)$ of any commutative monoid M is an IL algebra (*cf.* Example 2.3.2).

An IL algebra is an algebraic model for intuitionistic linear logic.

Definition. Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be mappings. The pair (f, g) is called a *Galois connection* (between X and A) iff $\forall x \in X$, $\forall a \in A$

$$x \leq g(f(x)), \qquad\qquad a \leq f(g(a)) \tag{2.3.6}$$

are satisfied (*cf.* [3], [25], [18]).

Proposition 2.6. Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be mappings. For the pair (f, g) to be an adjunction from X into A, it necessary and sufficient that the pair (f, g) is a Galois connection between X^{op} and A.

Example 2.3.6 (Binary relation). (*cf.* Example 2.3.1) Let $R \subseteq X \times A$ be a binary relation. Then the pair of mappings

$$C_{{}^tR} : \mathfrak{P}(X) \ni Y \quad \mapsto \quad R^*(Y) \in \mathfrak{P}(A), \tag{2.3.7}$$
$$C_R : \mathfrak{P}(A) \ni B \quad \mapsto \quad {}^tR^*(B) \in \mathfrak{P}(X) \tag{2.3.8}$$

constitutes a Galois connection between $\mathfrak{P}(X)$ and $\mathfrak{P}(A)$. In fact, for $Y \in \mathfrak{P}(X)$, let $y \in Y$ and $a \in C_{{}^tR}(Y)$. By definition, $y{}^tRa$ or equivalently, aRy holds. Since a is arbitrary element in $C_{{}^tR}(Y)$, we have $y \in C_R(C_{{}^tR}(Y))$. Thus we have $Y \subseteq C_R(C_{{}^tR}(Y))$. Similarly, we have that $B \subseteq C_{{}^tR}(C_R(B))$ for any $B \in \mathfrak{P}(A)$.

The universal image of Y (respectively B) is called the *polar set* of Y (resp. B) and often denoted by Y^* (resp. B^*) if there is no confusion about the correspondences.

Conversely, when an arbitrary Galois connection (f, g) between $\mathfrak{P}(X)$ and $\mathfrak{P}(A)$ is given, if we define a binary relation $R \subseteq X \times A$ by

$$xRa \quad := \quad a \in f(\{x\}) \, (\Leftrightarrow x \in g(\{a\}))$$

we have

$$C_{{}^tR} = f, \qquad\qquad C_R = g$$

and thus we obtain the Galois connection (f, g) again.

Note. In the context of Formal Concept Analysis ([14], [9]), X and A represent sets of "objects" and "attributes", respectively. Instead of R, $\models$ is used. Then $x \models a$ means that "an object x posseses an attribute a".

Example 2.3.7 (Group action). Suppose that a group G acts on a set X. We denote the lattice of subgroups of G by $\mathfrak{G}$. We put for $H \in \mathfrak{G}$, $Y \in \mathfrak{P}(X)$,

$$\begin{aligned} Fix(H) &= \{x \in X \mid hx = x \, (h \in H)\} \quad \text{(the fixed point set for } H\text{-action)}, \\ Iso(Y) &= \{g \in G \mid gy = y \, (y \in Y)\} \quad \text{(the isotoropic subgroup for } Y). \end{aligned}$$

We have $\forall H \in \mathfrak{G}$, $\forall Y \in \mathfrak{P}(X)$,

$$H \subseteq Iso(Fix(H)), \qquad\qquad Y \subseteq Fix(Iso(Y))$$

and thus a Galois connection (Iso, Fix) between $\mathfrak{P}(X)$ and $\mathfrak{G}$.

Note. In original Galois theory, X is a field and G is its automorphism group.

2.3.3 Closure operator and Moore family

Definition. Let X, A be partially ordered sets. A mapping $f : X \to A$ is called a *filter* (of X into A) iff it is monotonous ($x \leq y \Rightarrow f(x) \leq f(y)$) and idempotent ($f^2(x) = f(x)$). A *closure operator* $\varphi : X \to A$ is a filter that has the extensivity property ($x \leq \varphi x$). Similarly, a filter $\alpha : X \to A$ is called an *interior operator* iff it has the anti-extensivity property ($\alpha x \leq x$) (*cf.* [3], [25], [18]).

Example 2.3.8. Let $R \subseteq X \times A$ be a binary relation.

- $E_R \circ D_{{}^tR}$, $C_R \circ C_{{}^tR}$ are closure operators from $\mathfrak{P}(X)$ into $\mathfrak{P}(A)$ and $E_{{}^tR} \circ D_R$, $C_{{}^tR} \circ C_R$ are closure operators from $\mathfrak{P}(A)$ into $\mathfrak{P}(X)$.

- For $x \in X$, $Y \subseteq X$,

$$x \in E_R \circ D_{{}^tR}(Y) \Leftrightarrow R(x) \subseteq R(Y) \Leftrightarrow \forall a \in A(x \in {}^tR(a) \Rightarrow Y \cap {}^tR(a) \neq \emptyset)$$
$$x \in C_R \circ C_{{}^tR}(Y) \Leftrightarrow R^*(Y) \subseteq R(x) \Leftrightarrow \forall a \in A(x \in {}^tR(a)^c \Rightarrow Y \cap {}^tR(a)^c \neq \emptyset)$$

- $D_R \circ E_{{}^tR}$, $\overline{C_R} \circ \overline{C_{{}^tR}}$ are interior operators from $\mathfrak{P}(X)$ into $\mathfrak{P}(A)$ and $D_{{}^tR} \circ E_R$, $\overline{C_{{}^tR}} \circ \overline{C_R}$ are interior operators from $\mathfrak{P}(A)$ into $\mathfrak{P}(X)$, where $\overline{f}$ denotes the *dual operator* of f which is defined by

$$\overline{f}(Y) \;\; = \;\; (f(Y^c))^c.$$

- For $x \in X$, $Y \subseteq X$,

$$x \in D_R \circ E_{{}^tR}(Y) \Leftrightarrow R(x) \cap E_{{}^tR}(Y) \neq \emptyset \Leftrightarrow \exists a \in A\,(x \in {}^tR(a) \subseteq Y)$$
$$x \in \overline{C_R} \circ \overline{C_{{}^tR}}(Y) \Leftrightarrow R^c(x) \cap C_{{}^tR}(Y^c) \neq \emptyset \Leftrightarrow \exists a \in A\,(x \in {}^tR(a)^c \subseteq Y)$$

- We call $\varphi_R = E_R \circ D_{{}^tR}$ the *mophological closing defined by* R and $\alpha_R = D_R \circ E_{{}^tR}$ the *morphological opening defined by* R.

- Also, we call $\gamma_R = C_R \circ C_{{}^tR}$ the *Galois closing defined by* R and $\overline{\gamma_R} = \overline{C_R} \circ \overline{C_{{}^tR}}$ the *Galois opening defined by* R.

Definition. Let X be a partially ordered set. A subset $M \subseteq X$ is called a *Moore family* [3] if it satisfies that for any $S \subseteq M$, if the infimum $\bigwedge S$ exists in X then $\bigwedge S \in M$. Dually, we call M a *dual Moore family*, if it satisfies that for any $S \subseteq M$, if the supremum $\bigvee S$ exists in X then $\bigvee S \in M$.

Note. By definition, if X has a maximal element 1, any Moore family contains it. Similarly, if X has a minimal element 0, any dual Moore family contains it.

Proposition 2.7. Let X be a partially ordered set.

(1) For any closure operator $\varphi : X \to X$, the set of all φ-closed elements

$$\mathfrak{F}_\varphi \;\; = \;\; \{x \in X \mid \varphi x = x\}$$

is a Moore family.

(2) When X is complete, for any Moore family $M \subseteq X$, there exists a closure operator φ such that $\mathfrak{F}_\varphi = M$.

Proof.

(1) Let $S \subseteq \mathfrak{F}_\varphi$ such that there exists an infimum $\bigwedge S$ in X. For any $s \in S$, since $\bigwedge S \leq s$ and φ is monotonous, $\varphi(\bigwedge S) \leq \varphi(s) = s$. That is, $\varphi(\bigwedge S)$ is a lower bound of S. Since $\bigwedge S$ is the infimum of S, $\varphi(\bigwedge S) \leq \bigwedge S$. On the other hand, since φ is extensive, $\bigwedge S \leq \varphi(\bigwedge S)$. This implies that $\varphi(\bigwedge S) = \bigwedge S$, *i.e.*, $\bigwedge S \in M$.

(2) Suppose that a Moore family $M \subseteq X$ is given. We define $\varphi : X \to M$ by

$$\varphi a \quad = \quad \bigwedge \{m \in M \mid a \leq m\}$$

and show that φ is an operator that has the desired properties. It is immediate from the definition of Moore family that φ is a well-defined mapping from X into M. Monotonicity and extensivity are obvious. For idempotency, we first note that

$$\varphi m \quad = \quad m \quad (\forall m \in M).$$

Thus, for any $a \in X$, since $\varphi a \in M$, we have $\varphi(\varphi a) = \varphi a$. We have also shown that $M \subseteq \mathfrak{F}_\varphi$. On the other hand, for $a \in \mathfrak{F}_\varphi$,

$$a \quad = \quad \varphi a \quad = \quad \bigwedge \{m \in M \mid a \leq m\} \in M.$$

q.e.d.

Proposition 2.8. Let X be a partially ordered set.

(1) For any interior operator $\alpha : X \to X$, the set of all α-open elements

$$\mathfrak{O}_\alpha \quad = \quad \{x \in X \mid \alpha x = x\}$$

is a dual Moore family.

(2) When X is complete, for any dual Moore family $M \subseteq X$, there exists an interior operator α such that $\mathfrak{O}_\alpha = M$.

2.3.4 Involutions for erosion/dilation

Duality, transposition and adjunction

For erosions and dilations of Boolean lattices, there are three sorts of involutive transformations of operators, namely, duality, transposition and adjunction ([27], [18]).

Definition. Let X, A be Boolean lattices and $\varepsilon : X \to A$ be an erosion. Its *dual* $\overline{\varepsilon} : X \to A$, *transpose* ${}^t\varepsilon : A \to X$ and *adjoint* $\varepsilon^* : A \to X$ are repectively defined by

$$\begin{array}{rcll} \overline{\varepsilon}(x) & = & \neg\,(\varepsilon(\neg x)) & (x \in X), \\ {}^t\varepsilon(a) \vee x = 1 & \Leftrightarrow & a \vee \varepsilon(x) = 1 & (x \in X,\, a \in A), \\ \varepsilon^*(a) \leq x & \Leftrightarrow & a \leq \varepsilon(x) & (x \in X,\, a \in A). \end{array}$$

Although transpose and adjoint are implicitely defined, they are uniquely determined if they exist for the given erosion.

Similarly, for a dilation $\delta : X \to A$, its *dual* $\overline{\delta} : X \to A$, *transpose* ${}^t\delta : A \to X$ and *adjoint* $\delta^* : A \to X$ are respectively defined by

$$\begin{array}{rcll} \overline{\delta}(x) & = & \neg\,(\delta(\neg x)) & (x \in X), \\ {}^t\delta(a) \wedge x = 0 & \Leftrightarrow & a \wedge \delta(x) = 0 & (x \in X,\, a \in A), \\ x \leq \delta^*(a) & \Leftrightarrow & \delta(x) \leq a & (x \in X,\, a \in A). \end{array}$$

Note. We note that the complementation of Boolean lattice satisfies

$$x \wedge y^c = 0 \;\Leftrightarrow\; x \leq y \;\Leftrightarrow\; x^c \vee y = 1\,.$$

By using this, the conditions for transpose are rewritten as for erosion,

$$x^c \leq {}^t\varepsilon(a) \quad\Leftrightarrow\quad a^c \leq \varepsilon(x) \qquad (x \in X,\, a \in A),$$

and for dilation,

$${}^t\delta(a) \leq y^c \quad\Leftrightarrow\quad \delta(x) \leq a^c \qquad (x \in X,\, a \in A).$$

Proposition 2.9. Let X, A be Boolean lattices.

(1) For an erosion $\varepsilon : X \to A$ to have an adjoint ε^*, it is necessary and sufficient that it has a transpose ${}^t\varepsilon$.

(2) For a dilation $\delta : X \to A$ to have an adjoint δ^*, it is necessary and sufficient that it has a transpose ${}^t\delta$.

Proof. It is easily verified from the note above that the adjoint and the transpose of an operator are dual to each other. Thus the existence of one of them implies that of the other.

q.e.d.

Proposition 2.10. Let X, A be Boolean lattices.

(1) For an erosion $\varepsilon : X \to A$, the dual $\overline{\varepsilon}$ and the adjoint ε^* are dilations and the transpose ${}^t\varepsilon$ is an erosion.

(2) For a dilation $\delta : X \to A$, the dual $\overline{\delta}$ and the adjoint δ^* are erosions and the transpose ${}^t\delta$ is a dilation.

Properties

All of the transformations defined above are involutive. That is, let μ be a morphological operator and τ be one of these three transformations then

$$\tau(\tau(\mu)) \quad = \quad \mu.$$

On the other hand, consequent applications of several operators are independent of order. Futhermore, we have the following relations:

$$\begin{array}{lll} \overline{({}^t\varepsilon)} = {}^t(\overline{\varepsilon}) = \varepsilon^* & \overline{(\varepsilon^*)} = (\overline{\varepsilon})^* = {}^t\varepsilon & {}^t(\varepsilon^*) = ({}^t\varepsilon)^* = \overline{\varepsilon} \\ \overline{({}^t\delta)} = {}^t\left(\overline{\delta}\right) = \delta^* & \overline{(\delta^*)} = \left(\overline{\delta}\right)^* = {}^t\delta & {}^t(\delta^*) = ({}^t\delta)^* = \overline{\delta} \end{array}$$

In case $\varepsilon = E_R$ and $\delta = D_R$, we have more explicit relations:

$$\begin{array}{lll} \overline{E_R} = D_R, & {}^tE_R = E_{{}^tR}, & (E_R)^* = D_{{}^tR}. \\ \overline{D_R} = E_R, & {}^tD_R = D_{{}^tR}, & (D_R)^* = E_{{}^tR}, \end{array}$$

By virtue of these equalities, we only have to employ 4 operators among them, for example D_R, E_R, $D_{{}^tR}$ and $E_{{}^tR}$. The relations are diagrammatically represented as follows:

$$\begin{tikzcd} E_R \arrow[r, leftrightarrow, "-"] \arrow[d, leftrightarrow, "t"'] \arrow[dr, leftrightarrow] & D_R \arrow[d, leftrightarrow, "t"] \arrow[dl, leftrightarrow, "*"'] \\ E_{{}^tR} \arrow[r, leftrightarrow, "-"'] & D_{{}^tR} \end{tikzcd}$$

In this case, the diagonal pairs $(E_R, D_{{}^tR})$ and $(E_{{}^tR}, D_R)$ are adjunctions.

Chapter 3

Modal logic via mathematical morphology

3.1 Modal logics based on mathematical morphology

3.1.1 Modal operators from morphological erosion/dilation

In [7], modal operators are defined for $\Phi_m = \Phi(\Psi, \Box, \Diamond)$ through a standard model as follows. Let $\mathscr{M}$ be a standard model based on a Kripke frame $\mathscr{F} = (\Omega, R)$ with a valuation $V : \Psi \to \mathfrak{P}(\Omega)$. We consider the usual extension of V to the formula set Φ_m (*cf.* Appendix A.2). In particular, the interpretations of modal operators are defined by

$$V(\Box\phi) = \{\omega \in \Omega \mid \forall\varpi \in \Omega\, (\varpi \in R(\omega) \Rightarrow \varpi \in V(\phi))\}, \quad (3.1.1)$$
$$V(\Diamond\phi) = \{\omega \in \Omega \mid \exists\varpi \in \Omega\, (\varpi \in R(\omega) \text{ and } \varpi \in V(\phi))\}. \quad (3.1.2)$$

Also, we assume that $V : \Phi_m \to \mathfrak{P}(\Omega)$ is surjective. Under this assumption, the quotient space $\Phi_m/\equiv$ is isomorphic to the complete Boolean lattice $\mathfrak{P}(\Omega)$, where $\equiv$ is an equivalence relation called *$\mathscr{M}$-equivalence* and defined by

$$\phi \equiv \psi \quad \Leftrightarrow \quad V(\phi) = V(\psi).$$

Morphological operators for the set lattice $\mathfrak{P}(\Omega)$ equipped with a binary relation R are defined as (2.3.2) and (2.3.1) in Example 2.3.1 for $X \in \mathfrak{P}(\Omega)$,

$$E_R(X) = \{\omega \in \Omega \mid R(\omega) \subseteq X\},$$
$$D_R(X) = \{\omega \in \Omega \mid R(\omega) \cap X \neq \emptyset\}.$$

Then erosion and dilation on (the quotient lattice of) the formula set $\mathit{\Phi}_m$ are defined by

$$\begin{aligned} E_R\phi &= V^{-1}\left(E_R(V(\phi))\right), \\ D_R\phi &= V^{-1}\left(D_R(V(\phi))\right) \end{aligned}$$

for $\phi \in \mathit{\Phi}_m$. By comparing with the interpretations of modal operators, we have

$$V(E_R\phi) = E_R(V(\phi)) = V(\Box\phi), \tag{3.1.3}$$

$$V(D_R\phi) = D_R(V(\phi)) = V(\Diamond\phi), \tag{3.1.4}$$

or equivalently,

$$\Box\phi \equiv E_R\phi, \tag{3.1.5}$$

$$\Diamond\phi \equiv D_R\phi. \tag{3.1.6}$$

Then it is shown [7] that $\Diamond$ and $\Box$ satisfy

Df $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$,

K $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,

RN $\dfrac{\phi}{\Box\phi}$.

Thus we obtain a normal system modal logic [8].

3.1.2 Observation

We consider the assumption of surjectivity of $V : \mathit{\Phi} \to \mathfrak{P}(\mathit{\Omega})$.

Surjectivity allows us to replace each instance of the form $V(\phi)$ ($\phi \in \mathit{\Phi}_m$) in arguments by $X \in \mathfrak{P}(\mathit{\Omega})$. This enables us to reduce any arguments over formulas to that of sets. For example, a proof of the equivalece of schema **D** and seriality of R is give by showing the following diagramme:

$$\begin{array}{lcl}
\textbf{D}: & & \textit{serial}: \\
\quad \Box\phi \to \Diamond\phi \text{ is valid} & & \quad X \neq \emptyset \;\Rightarrow\; R(X) \neq \emptyset \\
\qquad\qquad (\forall\phi \in \mathit{\Phi}) & & \qquad\qquad (\forall X \in \mathfrak{P}(\mathit{\Omega})) \\
\qquad\quad \Updownarrow & & \qquad\quad \Updownarrow \\
E_R(V(\phi)) \subseteq D_R(V(\phi)) & \Longleftrightarrow & E_R(X) \subseteq D_R(X) \\
\qquad\qquad (\forall\phi \in \mathit{\Phi}) & & \qquad\qquad (\forall X \in \mathfrak{P}(\mathit{\Omega}))
\end{array}$$

In the horizontal equivalence, although the leftward direction $\Longleftarrow$ is automatically satisfied, the opposite needs surjectivity of the valuation V.

With similar arugument about the modal operators derived from morphological erosion/dilation, we can show the followings (we omit dual expressions):

D $\Box\phi \to \Diamond\phi$ $(\forall\varphi \in \Phi)$ iff R is serial

$$E_R X \subseteq D_R X \ (\forall X \in \mathfrak{P}(\Omega)) \quad \Leftrightarrow \quad (X \neq \emptyset \Rightarrow R(X) \neq \emptyset) \ (\forall X \in \mathfrak{P}(\Omega))$$

T $\Box\phi \to \phi$ $(\forall\varphi \in \Phi)$ iff R is reflexive

$$E_R X \subseteq X \ (\forall X \in \mathfrak{P}(\Omega)) \quad \Leftrightarrow \quad X \subseteq R(X) \ (\forall X \in \mathfrak{P}(\Omega))$$

B $\phi \to \Box\Diamond\phi$ $(\forall\varphi \in \Phi)$ iff R is symmetric

$$X \subseteq E_R D_R X \ (\forall X \in \mathfrak{P}(\Omega)) \quad \Leftrightarrow \quad R(X) \subseteq {}^tR(X) \ (\forall X \in \mathfrak{P}(\Omega))$$

4 $\Box\phi \to \Box\Box\phi$ $(\forall\varphi \in \Phi)$ iff R is transitive

$$E_R X \subseteq E_R E_R X \ (\forall X \in \mathfrak{P}(\Omega)) \quad \Leftrightarrow \quad R^2(X) \subseteq R(X) \ (\forall X, \forall Y \in \mathfrak{P}(\Omega))$$

5 $\Diamond\phi \to \Box\Diamond\phi$ $(\forall\varphi \in \Phi)$ iff R is Euclidean

$$D_R X \subseteq E_R D_R X \ (\forall X \in \mathfrak{P}(\Omega)) \quad \Leftrightarrow \quad {}^tR(X) \cap {}^tR(Y) \neq \emptyset \Rightarrow X \subseteq R(Y) \ (\forall X, \forall Y \in \mathfrak{P}(\Omega))$$

To define modal operators on formulas so as to satisfy (3.1.3) and (3.1.4), surjectivity of V is necessary not onto $\mathfrak{P}(\Omega)$ but onto a Boolean sublattice of $\mathfrak{P}(\Omega)$ which is invariant under the operation of the morphological dilation D_R (or equivalently, invariant under that of E_R). In fact, the later condition is also sufficient to define modal operators. But, as we obserbed above, to relate shematic properties with those of accessibility relation for modal logics based on standard models, the assumption of surjectivity of V onto $\mathfrak{P}(\Omega)$ is essential.

In the precedent articles, to actualize surjectivity of valuation, the number of propositional symbols are assumed to be finite. In general, this is not sufficient as we will see in the following. For this, we denote the number (or cardinality) of a set X by $|X|$ and we use $\aleph_0$ for $|\mathbb{N}|$. Then for a valuation $V : \Phi(\Psi, \Gamma) \to \mathfrak{P}(\Omega)$ to be surjective, it is necessary that

$$|\Phi(\Psi, \Gamma)| \geq |\mathfrak{P}(\Omega)| .$$

Under the usual assumption that $|\Psi| \leq \aleph_0$ and $|\Gamma|$ is finite, we have $|\Phi(\Psi,\Gamma)| \leq \aleph_0$. On the other hand, general cardinality analysis tells us that $|\mathfrak{P}(\Omega)| = 2^{|\Omega|}$. Thus, if we require the surjectivity of V, the set of possible world Ω can not be infinite. Furthermore, even under the assumption that Ψ is countably many and Ω is finite, V may not be surjective when some singletons are not covered with V. The following example gives a case breaking surjectivity under a countably many Ψ and a finite Ω.

Example 3.1.1. We consider the case $\Psi = \{p_0, p_1, \dots\}$ with the standard model given by

$$\Omega = \{\omega_0, \omega_1, \omega_2\},$$

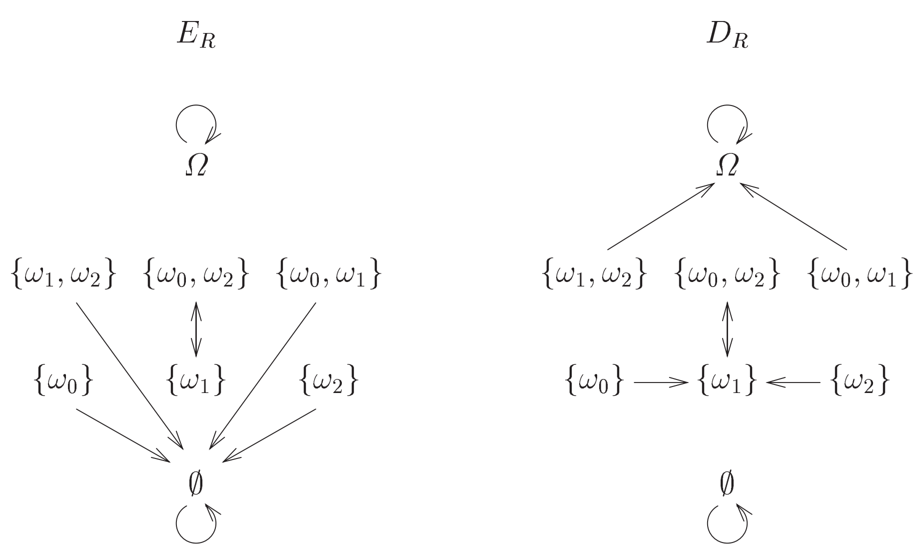
$$R(\omega) = \begin{cases} \{\omega_1\} & (\omega = \omega_0) \\ \{\omega_0, \omega_2\} & (\omega = \omega_1) \\ \{\omega_1\} & (\omega = \omega_2) \end{cases},$$

$$V(p_i) = \begin{cases} \{\omega_0, \omega_2\} & (i : \mathit{even}) \\ \{\omega_1\} & (i : \mathit{odd}) \end{cases}.$$

Then for a pure classical formula, *i.e.*, containing no modal operators, we obtain as true sets

$$\emptyset, \{\omega_1\}, \{\omega_0, \omega_2\}, \Omega.$$

On the other hand, the effects of morphological operators are as follows:



	$\emptyset$	$\{\omega_0\}$	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_1, \omega_2\}$	$\{\omega_0, \omega_2\}$	$\{\omega_0, \omega_1\}$	Ω
E_R	$\emptyset$	$\emptyset$	$\{\omega_0, \omega_2\}$	$\emptyset$	$\emptyset$	$\{\omega_1\}$	$\emptyset$	Ω
D_R	$\emptyset$	$\{\omega_1\}$	$\{\omega_0, \omega_2\}$	$\{\omega_1\}$	Ω	$\{\omega_1\}$	Ω	Ω

Thus the proper subalgebra

$$\{\emptyset, \{\omega_1\}, \{\omega_0, \omega_2\}, \Omega\}$$

is invariant under the operation of E_R and D_R. This means that V is not surjective. We note that the accessibilty relation R is serial and symmetric and hence modal operators satisfy **D** and **B**.

For a solution of eliminating of the surjectivity assumption, we can make use of the independency of operators of the effect of the valuation. More precisely, in our definition, we may omit the valuation "V" in the equalities

$$V(\Box\phi) = E_R(V(\phi)), \qquad V(\Diamond\phi) = D_R(V(\phi)),$$

in a sense. In fact, if we take another standard model $\mathscr{M}' = (\mathscr{F}, V')$ based on the same Kripke frame $\mathscr{F}$, we also have

$$V'(\Box\phi) = E_R(V'(\phi)), \qquad V'(\Diamond\phi) = D_R(V'(\phi)).$$

In other words, operators $\Diamond, \Box : \Phi_m \to \Phi_m$, D_R, $E_R : \mathfrak{P}(\Omega) \to \mathfrak{P}(\Omega)$ are defined independently of a choice of valuation $V : \Phi_m \to \mathfrak{P}(\Omega)$, so the following diagrammes are commutative:

$$\begin{tikzcd} \Phi_m \arrow[r, "\Diamond"] \arrow[d, "V"'] & \Phi_m \arrow[d, "V"] \\ \mathfrak{P}(\Omega) \arrow[r, "D_R"'] & \mathfrak{P}(\Omega) \end{tikzcd} \qquad \begin{tikzcd} \Phi_m \arrow[r, "\Box"] \arrow[d, "V"'] & \Phi_m \arrow[d, "V"] \\ \mathfrak{P}(\Omega) \arrow[r, "E_R"'] & \mathfrak{P}(\Omega) \end{tikzcd}$$

This means that the modal operators based on mathematical morphology depends not on each model but on the choice of Kripke frame. This is also clear from the fact that giving a pair of morphological erosion/dilation on a given set lattice is the same as giving a binary relation, or equivalently, giving a Kripke frame structure.

3.1.3 Systems of modal logic

A set of formulas $\Sigma \subseteq \Phi_m = \Phi(\Psi, \Diamond, \Box)$ is called a *system of modal logic* if it is closed under

RPL $\dfrac{\phi_1, \ldots, \phi_n}{\phi} \quad (n \geq 0)$

where ϕ is a tautological consequence of ϕ_1, ..., ϕ_n. Furthermore, if Σ contains the schema

Df$\Diamond$ $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$

and is closed under

RK $\dfrac{\phi_1, \ldots, \phi_n \Rightarrow \phi}{\Box\phi_1, \ldots, \Box\phi_n \Rightarrow \Box\phi}$

it is called *normal*.

Note. In the context of modal logic of sequent calculas, the term "schema" is used for that right hand side for a sequent in axioms. For example, requirement of containing the schema **Df**$\Diamond$ is equivalent to the set of axioms has all sequents of the form

$$\Rightarrow \Diamond\phi \leftrightarrow \neg\Box\neg\phi.$$

Lemma 3.1 ([8]). Let $\Sigma_i \subseteq \Phi_m$ be any family of subsets of modal laguage and put $\Sigma = \bigcap_i \Sigma_i$.

(1) If each of Σ_i is a system of modal logic, then so is Σ.

(2) If each of Σ_i is a normal system of modal logic, then so is Σ.

3.1.4 Modal logics based on standard models

The relation "a formula ϕ is true at a possible world $\omega \in \Omega$ in a model $\mathscr{M}$" can be considered as a correspondence $\models^{\mathscr{M}} : \Omega \to \Phi_m$. To each $\omega \in \Omega$, the existential image

$$\Lambda_{\mathscr{M},\omega} \;=\; \left\{\phi \in \Phi_m \;\middle|\; \models^{\mathscr{M}}_{\omega} \phi\right\} \subseteq \Phi_m$$

is a system of modal logic.

Definition. The *modal logic based on a standard model* $\mathscr{M}$ is defined by

$$\Lambda_{\mathscr{M}} \;=\; \bigcap_{\omega\in\Omega} \Lambda_{\mathscr{M},\omega}.$$

Namely, $\Lambda_{\mathscr{M}}$ consists of all formulas which are true in the model $\mathscr{M}$.

Proposition 3.2 ([8]). The modal logic $\Lambda_{\mathscr{M}}$ based on a standard model $\mathscr{M}$ is a normal system of modal logic.

Note. Any normal modal logic can be obtained by this way. In fact, for an arbitrary normal system of modal logic Σ, consider a canonical standard model $\mathscr{C} = (\Omega, R, V)$ for Σ. Then it is verified that

$$\phi \in \Lambda_{\mathscr{C}} \quad \Leftrightarrow \quad \phi \in \Sigma.$$

Thus Σ is the modal logic based on $\mathscr{C}$. For more details on the canonical model, see [8], [4].

3.1.5 Modal logics based on Kripke frames

The modal logic based on a standard model $\mathscr{M} = (\mathscr{F}, V)$ depends on the valuation V as well as the underlying Kripke frame $\mathscr{F}$. To eliminate the dependency of valuations, we put

$$\Lambda_{\mathscr{F}} \quad = \quad \bigcap_{\mathscr{M}/\mathscr{F}} \Lambda_{\mathscr{M}}.$$

and call it the *modal logic based on the Kripke frame* $\mathscr{F}$.

Theorem 3.3. The modal logic $\Lambda_{\mathscr{F}}$ based on a Kripke frame $\mathscr{F}$ is a normal system of modal logic. Especially, $\Lambda_{\mathscr{F}}$ is invariant under the operation of the modal operators $\Diamond$ and $\Box$.

3.1.6 Properties of modal logics based on mathematical morphology

For the sake of systematic treatment of the properties of the schemas which are satisfied by modal logics based on standard models or Kripke frames in terms of the accesability relation, we consider schema $\mathbf{G}^{k,\ell,m,n}$ and the notion of "k, ℓ, m, n-incestuality" as well as [8].

Definition. We call the schema

$$\Diamond^k \Box^\ell \phi \to \Box^m \Diamond^n \phi$$

as $\mathbf{G}^{k,\ell,m,n}$. On the other hand, a binary relation R of Ω is called k, ℓ, m, n-*incestual* iff for $\forall\omega$, $\forall\varpi \in \Omega$,

$$^tR^k(\omega) \cap {}^tR^m(\varpi) \neq \emptyset \quad \Rightarrow \quad R^\ell(\omega) \cap R^n(\varpi) \neq \emptyset. \tag{3.1.7}$$

To see the equivalence of $\mathbf{G}^{\boldsymbol{k,\ell,m,n}}$ and k,ℓ,m,n-incestuality, we show some equivalents of k,ℓ,m,n-incestuality:

Lemma 3.4. For a binary relation R of Ω, the following conditions are equivalent[1]:

(1) R is k,ℓ,m,n-incestual.

(2) R satisfies

$$D^m_{{}^tR}\left(D^k_R(X)\right) \subseteq D^n_R\left(D^\ell_{{}^tR}(X)\right) \quad (\forall X \in \mathfrak{P}(\Omega)). \tag{3.1.8}$$

(3) R satisfies

$$D^k_R\left(E^\ell_R(X)\right) \subseteq E^m_R\left(D^n_R(X)\right) \quad (\forall X \in \mathfrak{P}(\Omega)). \tag{3.1.9}$$

(4) R satisfies

$${}^tR^k(X) \cap {}^tR^m(Y) \neq \emptyset \Rightarrow R^\ell(X) \cap R^n(Y) \neq \emptyset \quad (\forall X, \forall Y \in \mathfrak{P}(\Omega)). \tag{3.1.10}$$

Proof. $\underline{(1) \Rightarrow (2).}$ Since $D^k_R(X) = {}^tR^k(X)$, $D^\ell_{{}^tR}(X) = R^\ell(X)$,

$$\begin{aligned}
\varpi \in D^m_{{}^tR}\left(D^k_R(X)\right) &\Leftrightarrow {}^tR^m(\varpi) \cap {}^tR^k(X) \neq \emptyset \\
&\Leftrightarrow \exists \omega \in X\left({}^tR^m(\varpi) \cap {}^tR^k(\omega) \neq \emptyset\right) \\
&\overset{(2)}{\Longrightarrow} \exists \omega \in X\left(R^n(\varpi) \cap R^\ell(\omega) \neq \emptyset\right) \\
&\Leftrightarrow R^n(\varpi) \cap R^\ell(X) \neq \emptyset \\
&\Leftrightarrow \varpi \in D^n_R\left(D^\ell_{{}^tR}(X)\right).
\end{aligned}$$

Thus we have (3.1.8).

$\underline{(2) \Rightarrow (3).}$ Let $X \in \mathfrak{P}(\Omega)$. Relation (3.1.8) applied for $E^\ell_R(X)$ becomes

$$D^m_{{}^tR}\left(D^k_R\left(E^\ell_R(X)\right)\right) \subseteq D^n_R\left(D^\ell_{{}^tR}\left(E^\ell_R(X)\right)\right).$$

[1]We note that for the powers of a binary relation and operators

$$D^n_R = D_{R^n}, \qquad\qquad E^n_R = E_{R^n}$$

hold. These relations come from the more general ones for composions of binary relations and operators:

$$D_{RR'} = D_{R'}D_R, \qquad\qquad E_{RR'} = E_{R'}E_R.$$

On the other hand, since $D^\ell_{{}^tR}E^\ell_R$ is an opening and D^n_R is monotonous,

$$D^n_R\left(D^\ell_{{}^tR}\left(E^\ell_R(X)\right)\right) \subseteq D^n_R(X).$$

Thus

$$D^m_{{}^tR}\left(D^k_R\left(E^\ell_R(X)\right)\right) \subseteq D^n_R(X).$$

Then by using the fact that $(E^m_R, D^m_{{}^tR})$ is an adjunction, we have (3.1.9).

<u>(3) ⇒ (4).</u> Suppose that $R^\ell(X) \cap R^n(Y) = \emptyset$. Then

$$\begin{aligned}
R^\ell(X) \cap D^n_{{}^tR}(Y) = \emptyset &\Leftrightarrow R^\ell(X) \subseteq \left(D^n_{{}^tR}(Y)\right)^c = E^n_{{}^tR}(Y^c) \\
&\Leftrightarrow X \subseteq E^\ell_R\left(E^n_{{}^tR}(Y^c)\right) \\
&\Rightarrow D^k_R(X) \subseteq D^k_R\left(E^\ell_R\left(E^n_{{}^tR}(Y^c)\right)\right) \\
&\overset{(3)}{\Rightarrow} D^k_R(X) \subseteq E^m_R\left(D^n_R\left(E^n_{{}^tR}(Y^c)\right)\right) \subseteq E^m_R(Y^c) \\
&\Leftrightarrow {}^tR^k(X) \subseteq E^m_R(Y^c) = \left(D^m_R(Y)\right)^c \\
&\Leftrightarrow {}^tR^k(X) \cap D^m_R(Y) = \emptyset
\end{aligned}$$

Thus we can conclude that ${}^tR^k(X) \cap {}^tR^m(Y) = \emptyset$.

<u>(4) ⇒ (1).</u> By taking as $X = \{\omega\}$, $Y = \{\varpi\}$ in (3.1.10), we have (3.1.7).

q.e.d.

Theorem 3.5. Let $\mathscr{F} = (\Omega, R)$ be a Kripke frame and $\mathscr{M} = (\mathscr{F}, V)$ be a standard model based on $\mathscr{F}$.

(1) For the modal logic $\Lambda_{\mathscr{M}}$ based on the standard model $\mathscr{M}$ to satisfy $\mathbf{G}^{k,\ell,m,n}$, it is sufficient that R is k, ℓ, m, n-incestual. When $V : \Phi_m \to \mathfrak{P}(\Omega)$ is surjective, the converse is true.

(2) For the modal logic $\Lambda_{\mathscr{F}}$ based on the Kripke frame $\mathscr{F}$ to satisfy $\mathbf{G}^{k,\ell,m,n}$, it is necessary and sufficient that R is k, ℓ, m, n-incestual.

Proof.

(1) For the modal logic $\Lambda_{\mathscr{M}}$ based on the standard model $\mathscr{M}$, the schema $\mathbf{G}^{k,\ell,m,n}$ is equivalently rewitten in terms of the valuation and morphological operators as:

$$\mathbf{G}^{k,\ell,m,n} \quad \Leftrightarrow \quad \forall\phi \in \Phi_m\left(D^k_R(E^\ell_R(V(\phi))) \subseteq E^m_R(D^n_R(V(\phi)))\right).$$

Thus, by Lemma 3.4, k, ℓ, m, n-incestuality of R is sufficient for $\mathbf{G}^{k,\ell,m,n}$. When the valuation V is surjective, since any subset $X \in \mathfrak{P}(\Omega)$ can be a true set $V(\phi)$ for some $\phi \in \Phi_m$, k, ℓ, m, n-incestuality is necessary.

(2) For the modal logic $\Lambda_{\mathscr{F}}$ based on the Kripke frame $\mathscr{F}$ the schema $\mathbf{G}^{k,\ell,m,n}$ is equivalently rewitten in terms of models based on $\mathscr{F}$ and morphological operators as:

$$\mathbf{G}^{k,\ell,m,n} \quad \Leftrightarrow \quad \forall \mathscr{M}/\mathscr{F},\ \forall \phi \in \Phi_m \, (D_R^k(E_R^\ell(V(\phi))) \subseteq E_R^m(D_R^n(V(\phi)))).$$

By (1), for any standard model $\mathscr{M}$ based on $\mathscr{F}$, k, ℓ, m, n-incestuality of R is sufficient for $\mathbf{G}^{k,\ell,m,n}$. Thus, also for $\Lambda_{\mathscr{F}}$, it is sufficient. Conversely, suppose that $\Lambda_{\mathscr{F}}$ satisfies $\mathbf{G}^{k,\ell,m,n}$. Then for any $X \in \mathfrak{P}(\Omega)$, by taking a standard model $\mathscr{M} = (\mathscr{F}, V)$ based on $\mathscr{F}$ and a formula $\phi \in \Phi_m$ such that $V(\phi) = X$, we have

$$D_R^k(E_R^\ell(X)) \subseteq E_R^m(D_R^n(X)).$$

By virtue of Lemma 3.4, this is equivalent to k, ℓ, m, n-incestuality of R.

q.e.d.

Other schemas **D**, **T**, **B**, **4**, **5** are obtained from $\mathbf{G}^{\boldsymbol{k,\ell,m,n}}$ with particular values of k, ℓ, m, n:

(k,ℓ,m,n)	$(0,1,0,1)$	$(0,1,0,0)$	$(0,0,1,1)$	$(0,1,2,0)$	$(1,0,1,1)$
$\mathbf{G}^{\boldsymbol{k,\ell,m,n}}$	**D**	**T**	**B**	**4**	**5**

Corollary 3.6. Let $\Lambda_{\mathscr{F}}$ be the modal logic based on a Kripke frame $\mathscr{F} = (\Omega, R)$.

(1) $\Lambda_{\mathscr{F}}$ satisfies $\mathbf{D} : \Box\phi \to \Diamond\phi$ iff R is serial : $R(\omega) \neq \emptyset \, (\forall \omega \in \Omega)$

(2) $\Lambda_{\mathscr{F}}$ satisfies $\mathbf{T} : \Box\phi \to \phi$ iff R is reflexive : $\omega \in R(\omega) \, (\forall \omega \in \Omega)$

(3) $\Lambda_{\mathscr{F}}$ satisfies $\mathbf{B} : \phi \to \Box\Diamond\phi$ iff R is symmetric : $\omega \in {}^tR(\varpi) \Rightarrow \omega \in R(\varpi)$ $(\forall \omega, \varpi \in \Omega)$

(4) $\Lambda_{\mathscr{F}}$ satisfies $\mathbf{4} : \Box\phi \to \Box\Box\phi$ iff R is transitive : $\omega \in {}^tR^2(\varpi) \Rightarrow \varpi \in R(\omega)$ $(\forall \omega, \varpi \in \Omega)$

(5) $\Lambda_{\mathscr{F}}$ satisfies $\mathbf{5} : \Diamond\phi \to \Box\Diamond\phi$ iff R is Euclidean : ${}^tR(\omega) \cap {}^tR(\varpi) \neq \emptyset$ $\Rightarrow \omega \in R(\varpi) \, (\forall \omega, \varpi \in \Omega)$

3.2 Modal logics based on adjunctions

3.2.1 Modal operators from algebraic erosion/dilation

In [6], modal operators are defined by an adjunction on the formula sets as follows. As seen in the previous chpater, the notion of adjunction can be defined for partially ordered sets, and their components are erosion and dilation. To introduce the notion of adjunction on the set of formulas, all we need is a partial order on formulas. For this we regard the connective "→" as a partial order. In fact, it can be regarded as a partial order in the quotient space by $\equiv$. Then a pair $(\Box, \Diamond')$ of modal operators is called adjunction iff

$$\textbf{Ad}. \qquad \frac{\phi \to \Box\psi}{\Diamond'\phi \to \psi} \qquad \text{and} \qquad \frac{\Diamond'\phi \to \psi}{\phi \to \Box\psi}$$

is satisfied for all formulas ϕ, ψ. Since $\Diamond$ and $\Box'$ are not necessarily dual to each other, we use different notations from the usual ones. Moreover, we should consider the formula set $\Phi_{2,m} = \Phi(\Psi, \Box, \Diamond, \Box', \Diamond')$ of 2-modal logic rather than $\Phi_{m'} = \Phi(\Psi, \Box, \Diamond')$, where $\Diamond$ and $\Box'$ are the dual to $\Box$ and $\Diamond'$ respectively. In this case we should modify $\equiv$ according to a standard model $\mathscr{M} = (\Omega, R, R', V)$ for 2-modal logic (see Appendix A.2). In particular,

$$\begin{aligned}
V(\Box\phi) &= \{\omega \in \Omega \mid \forall\varpi \in \Omega\,(\varpi \in R(\omega) \Rightarrow \varpi \in V(\phi))\}, && (3.2.1)\\
V(\Diamond\phi) &= \{\omega \in \Omega \mid \exists\varpi \in \Omega\,(\varpi \in R(\omega) \text{ and } \varpi \in V(\phi))\}, && (3.2.2)\\
V(\Box'\phi) &= \{\omega \in \Omega \mid \forall\varpi \in \Omega\,(\varpi \in R'(\omega) \Rightarrow \varpi \in V(\phi))\}, && (3.2.3)\\
V(\Diamond'\phi) &= \{\omega \in \Omega \mid \exists\varpi \in \Omega\,(\varpi \in R'(\omega) \text{ and } \varpi \in V(\phi))\}. && (3.2.4)
\end{aligned}$$

In formal, we define the *2-modal logic based on the standard model* $\mathscr{M}$ by

$$\Lambda_{\mathscr{M}} = \{\phi \in \Phi_{2,m} \mid V(\phi) = \Omega\}.$$

Then it is shown ([6]) that $\Lambda_{\mathscr{M}}$ satisfies

Df $\Diamond\phi \leftrightarrow \neg\Box\neg\phi$ and $\Diamond'\phi \leftrightarrow \neg\Box'\neg\phi$,

K $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ and $\Box'(\phi \to \psi) \to (\Box'\phi \to \Box'\psi)$,

RN $\dfrac{\phi}{\Box\phi}$ and $\dfrac{\phi}{\Box'\phi}$.

Thus we obtain a normal system of 2-modal logic.

Proposition 3.7. The 2-modal logic $\Lambda_{\mathscr{M}}$ based on a standard model $\mathscr{M}$ is a normal system of 2-modal logic.

Similarly to the case of mathematical morphology, we define the 2-modal logic based on a 2-frame $\mathscr{F}$ by

$$\Lambda_{\mathscr{F}} \;=\; \bigcap_{\mathscr{M}/\mathscr{F}} \Lambda_{\mathscr{M}}.$$

Then we also have

Theorem 3.8. The 2-modal logic $\Lambda_{\mathscr{F}}$ based on a 2-frame $\mathscr{F}$ is a normal system of 2-modal logic. Especially, $\Lambda_{\mathscr{F}}$ is invariant under the operation of the modal operators $\Box$, $\Diamond$, $\Box'$ and $\Diamond'$.

3.2.2 Properties of modal logics based on adjunction

A 2-modal logic based on an arbitrary standard model or 2-frame does not a priori satisfies the schema **Ad**.

Theorem 3.9. Let $\mathscr{F} = (\Omega, R, R')$ be a 2-frame and $\mathscr{M} = (\mathscr{F}, V)$ be a standard model based on $\mathscr{F}$.

(1) For the modal logic $\Lambda_{\mathscr{M}}$ based on the standard model $\mathscr{M}$ to satisfy **Ad**, it is sufficient that $R' = {}^tR$. When $V : \Phi_{2,m} \to \mathfrak{P}(\Omega)$ is surjective, the converse is true.

(2) For the modal logic $\Lambda_{\mathscr{F}}$ based on the 2-frame $\mathscr{F}$ to satisfy **Ad**, it is necessary and sufficient that $R' = {}^tR$.

Proof.

(1) For the modal logic $\Lambda_{\mathscr{M}}$ based on the standard model $\mathscr{M}$, the schema **Ad** is equivalently rewritten in terms of the valuation and morphological operators as:

$$\mathbf{Ad} \;\Leftrightarrow\; \forall \phi,\, \psi \in \Phi_{2,m}\, (V(\phi) \subseteq E_R(V(\psi)) \;\Leftrightarrow\; D_{R'}(V(\phi)) \subseteq V(\psi)).$$

Thus, by Example 2.3.3 $R' = {}^tR$ is sufficient for **Ad**. When the valuation V is surjective, since any subset of $\mathfrak{P}(\Omega)$ can be a true set for some formula in $\Phi_{2,m}$, we have

$$\mathbf{Ad} \;\Leftrightarrow\; \forall X,\, Y \in \mathfrak{P}(\Omega)\; (X \subseteq E_R(Y) \;\Leftrightarrow\; D_{R'}(X) \subseteq Y).$$

Thus the pair $(E_R, D_{R'})$ is an adjunction of the set lattice $\mathfrak{P}(\Omega)$. Thus $R' = {}^tR$ is necessary.

(2) Argument is parallel to the proof for the part (2) of Theorem 3.5.

q.e.d.

A 2-frame $\mathscr{F} = (\Omega, R, {}^tR)$ is called a *bidirectional frame* ([4]). Also, a standard model based on a bidirectional frame is called a *bidirectional model*.

3.2.3 Temporal logic

It is known that the logic of bidirectional frames is the minimal temporal logic $\mathbf{K}_t$, which means that for the class of all bidirectional frames $\mathfrak{B}$,

$$\mathbf{K}_t \;=\; \left\{\phi \in \Phi_{2,m} \;\middle|\; \models^{\mathfrak{B}} \phi\right\}. \tag{3.2.5}$$

More precisely, a normal 2-modal logic satisfying the schema

$$\textbf{Tmp}. \qquad \phi \to \Box\Diamond'\phi \qquad \text{and} \qquad \phi \to \Box'\Diamond\phi$$

is called a *temporal logic* and $\mathbf{K}_t$ is defined as the minimal one. In Appendix A.1, we give a definition of the minimal temporal logic $\mathbf{K}_t$ as a deductive system. For more details and a proof of (3.2.5), see [4].

By combining this fact and Theorem 3.9, we can conclude that a 2-modal logic based on a 2-frame having an adjunction is nothing but a temporal logic. Also, a 2-modal logic based on a standard model having an adjunction is nothing but a temporal logic, provided that, as usual, the valuation is surjective.

From the point of view of schemas, we can grasp this more clearly.

Lemma 3.10. For a normal 2-modal logic Λ, the following conditions are equivalent:

(1) Λ satisfies the schema **Ad**.

(2) Λ satisfies the schema

$$\textbf{Ad'}. \qquad \frac{\phi \to \Box'\psi}{\Diamond\phi \to \psi} \qquad \text{and} \qquad \frac{\Diamond\phi \to \psi}{\phi \to \Box'\psi}.$$

(3) Λ satisfies the schema **Tmp**.

(4) Λ satisfies the schema

$$\textbf{Tmp'}. \qquad \phi \to \Box\Diamond'\phi \qquad \text{and} \qquad \Diamond'\Box\phi \to \phi.$$

Note. We note that every proof is parallel to that in the case of adjunction on partially ordered sets but $\Phi_{2,m}$ is not a partially ordered set. To introduce a partial order on Λ independently of models, we consider a provable equivalence:

$$\phi \sim \psi \;\overset{def}{\Longleftrightarrow}\; \phi \leftrightarrow \psi \in \Lambda \qquad (\textit{i.e.}, \text{ a theorem of } \Lambda).$$

Then, as before, we can regard the connective "$\to$" as a partial order in the quotient space $\Lambda/\sim$.

Proof. The equivalence between (1) and (2) comes from the fact that the pair $(\Box, \Diamond')$ is an adjunciton iff so is the dual pair $(\Box', \Diamond)$. The equivalence between (3) and (4) comes from

$$\phi \to \Box'\Diamond\phi \quad \Leftrightarrow \quad \Diamond'\Box\phi \to \phi,$$

but this is nothing but the duality. The equivalence between (1) and (4) follows from Proposition 2.3.

q.e.d.

Theorem 3.11. For a normal 2-modal logic to have a pair of modal operators forming an adjunction it is necessary and sufficient that it is a temporal logic.

Note. In usual notation, by putting

$$G = \Box, \qquad F = \Diamond, \qquad H = \Box', \qquad P = \Diamond', \tag{3.2.6}$$

we have the set of formulas of temporal logic $\varPhi_t = \varPhi(\varPsi, G, F, H, P)$ (*cf.* [4], [13]).

Chapter 4

Intuitionistic logic via mathematical morphology

4.1 Frames for intuitionistic logic

Here we intorduce another notion of "frame", which is studied in the field of pointless topology [22], [30]. Although the notion of frame and that of Kripke frame are quite different from each other, both of them are found in the context of semantics for intuitionistic logic. We will use the naked word "frame" only for the former.

Definition. A $\bigvee$-complete lattice F is called a *frame* iff the operator

$$a\wedge : F \ni x \quad \mapsto \quad a \wedge x \in F$$

is a dilation for each $a \in F$.

Proposition 4.1. A frame is a Heyting algebra. Conversely, a complete Heyting algebra is a frame.

Proof. Immediate from Proposition 2.5.

q.e.d.

Definition. Let H be a Heyting algebra. An operator $\alpha : H \to H$ is called an *interior operator of Heyting algebra* iff it is an interior operator of partially ordered set and satisfies

$$\alpha a \wedge \alpha b \quad \leq \quad \alpha(a \wedge b) \qquad (a, b \in H). \tag{4.1.1}$$

Lemma 4.2. Let H be a Heyting algebra and $\alpha : H \to H$ be an interior operator of Heyting algebra. Then $\forall a, b \in H$,

$$\alpha(\alpha a \wedge \alpha b) = \alpha(a \wedge b).$$

Proof. Since α is monotone, we have a priori,

$$\alpha(a \wedge b) \quad \leq \quad \alpha a \wedge \alpha b.$$

By monotonicity again and by idempotency, we have

$$\alpha(a \wedge b) \quad = \quad \alpha(\alpha(a \wedge b)) \;\leq\; \alpha(\alpha a \wedge \alpha b).$$

Conversly, by using monotonicity and idempotency, we have from (4.1.1) that

$$\alpha(\alpha a \wedge \alpha b) \quad \leq \quad \alpha(\alpha(a \wedge b)) \;=\; \alpha(a \wedge b).$$

<u>*q.e.d.*</u>

Proposition 4.3. Let α be an interior operator of a Heyting algebra H. Then the set of all α-open elements $\mathfrak{O}_\alpha = \{a \in H \mid \alpha a = a\}$ is a Heyting subalgebra.

Proof. By Proposition 2.8, $\mathfrak{O}_\alpha$ is a dual Moore set and hence it is closed under $\bigvee$ operation. To show that $\mathfrak{O}_\alpha$ is closed under $\bigwedge$ operation, we first note that for any $a, b \in H$,

$$\alpha(a \to b) \leq \alpha a \to \alpha b. \tag{4.1.2}$$

In fact,

$$\begin{array}{rcl} a \to b \leq a \to b & \stackrel{\text{adjunction}}{\Longleftrightarrow} & a \wedge (a \to b) \leq b \\ & \stackrel{\text{monotonicity}}{\Longrightarrow} & \alpha(a \wedge (a \to b)) \leq \alpha b, \\ & \stackrel{(4.1.1)}{\Longrightarrow} & \alpha a \wedge \alpha(a \to b) \leq \alpha b \\ & \stackrel{\text{adjunction}}{\Longleftrightarrow} & \alpha(a \to b) \leq \alpha a \to \alpha b. \end{array}$$

Now suppose that $a, b \in \mathfrak{O}_\alpha$, then

$$\begin{array}{rcl} a \wedge b \leq a \wedge b & \stackrel{\text{adjunction}}{\Longleftrightarrow} & b \leq a \to a \wedge b \\ & \stackrel{\text{monotonicity}}{\Longrightarrow} & \alpha b \leq \alpha(a \to a \wedge b) \\ & \stackrel{(4.1.2)}{\Longrightarrow} & \alpha b \leq \alpha a \to \alpha(a \wedge b) \\ & \stackrel{\text{adjunction}}{\Longleftrightarrow} & \alpha a \wedge \alpha b \leq \alpha(a \wedge b). \end{array}$$

But, since a, b are α-open, we have $a \wedge b \leq \alpha(a \wedge b)$. On the other hand, by anti-extensivity, $a \wedge b \leq \alpha(a \wedge b)$ holds. Thus we have $\alpha(a \wedge b) = a \wedge b$ for a, $b \in \mathfrak{O}_\alpha$ which means that $\mathfrak{O}_\alpha$ is closed under $\wedge$ operation. Finally, we show that for each $a \in \mathfrak{O}_\alpha$,

$$a\wedge : \mathfrak{O}_\alpha \ni x \quad \mapsto \quad a \wedge x \in \mathfrak{O}_\alpha$$

has the left adjoint

$$a \to_\alpha : \mathfrak{O}_\alpha \ni x \quad \mapsto \quad \alpha(a \to x) \in \mathfrak{O}_\alpha.$$

In fact, for any $y \in \mathfrak{O}_\alpha$,

$$\begin{aligned} y \leq a \to x \quad &\Rightarrow \quad y = \alpha(y) \leq \alpha(a \to x), \\ y \leq \alpha(a \to x) \quad &\Rightarrow \quad y \leq a \to x. \end{aligned}$$

Hence for any x, $y \in \mathfrak{O}_\alpha$,

$$y \leq \alpha(a \to x) \quad \Leftrightarrow \quad y \leq a \to x \;\Leftrightarrow\; a \wedge y \leq x.$$

q.e.d.

Proposition 4.4. Let F be a frame.

(1) For any interior operator of Heyting algebra $\alpha : F \to F$, the set of all α-open elements $\mathfrak{O}_\alpha$ is a subframe.

(2) For any subframe $E \subseteq F$, there exists an interior operator of Heyting algebra α such that $\mathfrak{O}_\alpha = E$.

Proof.

(1) By Proposition 4.3, $\mathfrak{O}_\alpha$ is a Heyting subalgebra. Thus, by virtue of Proposition 4.1, it is sufficient to show that $\mathfrak{O}_\alpha$ is complete. Let $S \subseteq \mathfrak{O}_\alpha$. Since F is complete, S has a supremum $\bigvee S$ in F. But, since $\mathfrak{O}_\alpha$ is a dual Moore family, $\bigvee S \in \mathfrak{O}_\alpha$. Thus $\mathfrak{O}_\alpha$ is complete.

(2) Since E is $\bigvee$-complete, it is a dual Moore family. Then by Proposition 2.8, there exists an interior operator α such that $\mathfrak{O}_\alpha = E$. Thus all we have to show is that α satisfies (4.1.1). We note that α is explicitly written as

$$\alpha a \quad = \quad \bigvee \{x \in E \mid x \leq a\} \qquad (a \in F).$$

Let $a, b \in F$. For any $x, y \in E$, since E is subframe, $x \wedge y \in E$. Thus

$$\begin{aligned} x \le a\,(x \in E) \text{ and } y \le b\,(y \in E) \quad &\Rightarrow \quad x \wedge y \le a \wedge b\,(x \wedge y \in E) \\ &\Rightarrow \quad x \wedge y \le \alpha(a \wedge b). \end{aligned}$$

Since x and y are arbitrary elements in E satisfying $x \le a$ and $y \le b$, we conclude (4.1.1).

q.e.d.

Example 4.1.1. (*cf.* Example 2.3.4) Let R be a binary relation on a set X. For the morphological opening $\alpha_R = D_R \circ E_{{}^tR}$ defined by R to be an interior operator of the Heyting algebra $\mathfrak{P}(X)$ iff for any $a, b \in X$,

$$\bigcup_{{}^tR(c) \subseteq {}^tR(a) \cap {}^tR(b)} {}^tR(c) \quad = \quad {}^tR(a) \cap {}^tR(b). \tag{4.1.3}$$

$(\because)$ For α_R, the condition (4.1.1) becomes

$$D_R E_{{}^tR}(A) \cap D_R E_{{}^tR}(B) \quad \subseteq \quad D_R E_{{}^tR}(A \cap B) \qquad (A, B \in \mathfrak{P}(X)).$$

By Example 2.3.8, this is equivalent to the condition that $\forall x \in X$,

$$\begin{aligned} &\exists a \in X\,(x \in {}^tR(a) \subseteq A) \text{ and } \exists b \in X\,(x \in {}^tR(b) \subseteq B) \\ &\qquad\qquad\qquad\qquad \Rightarrow \exists c \in X\,(x \in {}^tR(c) \subseteq A \cap B). \end{aligned} \tag{4.1.4}$$

It is easily verified that the equivalence between of (4.1.3) and (4.1.4) holds.

□

Any quasi-order (*i.e.*, reflexive and transitive relation) R satisfies (4.1.3). In fact, let R be a quasi-order $\le$. Then for $a \in X$,

$${}^tR(a) = \{x \in X \mid x \le a\},$$

and for any $a, b \in X$,

$$c \in {}^tR(a) \cap {}^tR(b) \quad \Rightarrow \quad c \in R(c) \subseteq {}^tR(a) \cap {}^tR(b).$$

(4.1.3) follows from this immediately.

Note. For a morphological opening α_R defined by a quasi-order R, a subset is α_R-open iff it is *anti-hereditary*:

$$\forall x, y \in X \,(x \in A \text{ and } y \leq x \Rightarrow y \in A).$$

Possible world semantics using hereditary (opposite of anti-hereditary) relations is called *Kripke semantics for intuitionistic logic*. It is known that the intuitionistic logic is sound and complete with respect to its Kripke semantics, and it has finite model property [29].

Example 4.1.2. (*cf.* 2.3.4) Let R be a binary relation on a set X. For the Galois opening $\overline{\gamma_R} = \overline{C_R} \circ \overline{C_{{}^tR}}$ defined by R to be an interior operator of the Heyting algebra $\mathfrak{P}(X)$ iff for any a, $b \in X$,

$$\bigcap_{{}^tR(a)\cup{}^tR(b)\subseteq{}^tR(c)} {}^tR(c) \;=\; {}^tR(a) \cup {}^tR(b). \tag{4.1.5}$$

($\because$) For $\overline{\gamma_R}$, the condition (4.1.1) becomes

$$\overline{C_R}\,\overline{C_{{}^tR}}(A) \cap \overline{C_R}\,\overline{C_{{}^tR}}(B) \;\subseteq\; \overline{C_R}\,\overline{C_{{}^tR}}(A \cap B) \qquad (A, B \in \mathfrak{P}(X)).$$

By Example 2.3.8, this is equivalent to the condition that $\forall x \in X$,

$$\begin{aligned} &\exists a \in X\,(x \in {}^tR(a)^c \subseteq A) \text{ and } \exists b \in X\,(x \in {}^tR(b)^c \subseteq B) \\ &\qquad\qquad\qquad\qquad \Rightarrow \exists c \in X\,(x \in {}^tR(c)^c \subseteq A \cap B). \end{aligned} \tag{4.1.6}$$

It is easily verified that the equivalence between of (4.1.5) and (4.1.6) holds.

$\square$

A total order R satisfies (4.1.6). In fact, let R be a total order $\leq$. Then for $a \in X$,

$${}^tR(a) = \{x \in X \mid x \leq a\},$$

and for any a, $b \in X$,

$${}^tR(a) \cup {}^tR(b) \;=\; {}^tR(\max\{a, b\}).$$

(4.1.5) follows from this immediately.

4.2 Semantics for intuitionistic logic

4.2.1 Morphological/Galois LJ frames

Definition. We call a Kripke frame $\mathscr{F} = (\Omega, R)$ a *morphological LJ frame* iff R satisfies (4.1.3). Similarly, we call $\mathscr{F}$ a *Galois LJ frame* iff R satisfies (4.1.5). The *interior operator associated* to an LJ frame $\mathscr{F}$ means α_R when $\mathscr{F}$ is morphological and $\overline{\gamma_R}$ when $\mathscr{F}$ is Galois.

Let $\mathscr{F} = (\Omega, R)$ be a morphological/Galois LJ frame and α be the interior operator associated to $\mathscr{F}$. By Proposition 4.4, the set of α-open sets $\mathfrak{O}_\alpha$ is a subframe of $\mathfrak{P}(\Omega)$. To each α-open set $G \in \mathfrak{O}_\alpha$,

$$E_G : \mathfrak{O}_\alpha \ni O \mapsto \alpha((G \cap O) \cup G^c) \in \mathfrak{O}_\alpha, \tag{4.2.1}$$

$$D_G : \mathfrak{O}_\alpha \ni O \mapsto G \cap O \in \mathfrak{O}_\alpha \tag{4.2.2}$$

defines an adjunction (E_G, D_G).

Note. The adjunction (E_G, D_G) comes from the identity relation restricted to G:

$$\boldsymbol{I}_G = \{(\omega, \omega) \mid \omega \in G\}.$$

We remark that every such adjunction should be distinguished from the one defined from the accessibility relation R of $\mathscr{F}$.

4.2.2 Morphological/Galois LJ models

Definition. A standard model $\mathscr{M}$ based on a morphological LJ frame $\mathscr{F}$ with a valuation valued in $\mathfrak{O}_{\alpha_R}$ is called a *morphological LJ model*. Similarly, a standard model $\mathscr{M}$ based on a Galois LJ frame $\mathscr{F}$ with a valuation valued in $\mathfrak{O}_{\overline{\gamma_R}}$ is called a *Galois LJ model*.

Let $\Phi_i = \Phi(\Psi)\,(= \Phi_c)$ be the set of formulas for intuitionisitic logic and $\mathscr{M}$ be an LJ model based on a LJ frame $\mathscr{F} = (\Omega, R)$ with a valuation $V : \Psi \to \mathfrak{O}_\alpha$, where α is the interior operator associated to $\mathscr{F}$. The valuation V can be extended to the formula set Φ_i by

(1) $V(p_i)$ for $p_i \in \Psi$.

(2) $V(\top) = \Omega$.

(3) $V(\bot) = \emptyset$.

(4) $V(\neg\phi) = \alpha(V(\phi)^c)$.

(5) $V(\phi \wedge \psi) = V(\phi) \cap V(\psi)$.

(6) $V(\phi \vee \psi) = V(\phi) \cup V(\psi)$.

(7) $V(\phi \to \psi) = \alpha(V(\phi)^c \cup V(\psi))$.

(8) $V(\phi \leftrightarrow \psi) = \alpha\left(V(\phi)^c \cup V(\psi)\right) \cap \left(V(\phi) \cup V(\psi)^c\right)$.

Then we have

$$\begin{aligned} V(\phi \wedge \psi) &= D_{V(\phi)} V(\psi), && (4.2.3) \\ V(\phi \to \psi) &= E_{V(\phi)} V(\psi). && (4.2.4) \end{aligned}$$

Note. Any Kripke semantics for intuitionistic logic is obtained as a morphological LJ model (*cf.* note just after Example 4.1.1).

Proposition 4.5.

(1) The intuitionistic logic is sound with respect to any morphological LJ frame.

(2) The intuitionistic logic is sound with respect to any Galois LJ frame.

Proof. For both cases, soundness is proved by a straightforwrd induction.

q.e.d.

Chapter 5

Linear logic via mathematical morphology

5.1 Quantales for intuitionistic linear logic

A quantale is an algebraic model for linear logics ([15], [20]).

Definition. A complete lattice Q is called a *quantale* iff it is equipped with a commutative monoid structure with the multiplication $\cdot$ and the unit element e such that

$$a\cdot : Q \ni x \quad \mapsto \quad a \cdot x \in Q$$

is a dilation for each $a \in Q$.

Proposition 5.1. A quantale is an IL algebra. Conversely, any complete IL algebra is a quantale.

Proof. Immediately from Proposition 2.5.

q.e.d.

Definition. Let L be a IL algebra. An operator $\varphi : L \to L$ is called a *closure operator of IL algebra* iff it is a closure operator of partially ordered set and satisfies

$$\varphi a \cdot \varphi b \quad \leq \quad \varphi(a \cdot b) \qquad (a, b \in L). \tag{5.1.1}$$

Lemma 5.2. Let L be an IL algebra and $\varphi : L \to L$ be a closure operator of IL algebra. Then for any family $a_\lambda \in L$, for which $\bigvee_\lambda a_\lambda$ and $\bigvee_\lambda \varphi(a_\lambda)$ exist,

(1) $\varphi\left(\bigvee_\lambda \varphi a_\lambda\right) = \varphi\left(\bigvee_\lambda a_\lambda\right)$

and for a, $b \in L$,

(2) $\varphi(\varphi a \cdot \varphi b) = \varphi(a \cdot b)$.

Proof.

(1) Since φ is monotonous, we have a priori,

$$\bigvee_{\lambda} \varphi a_{\mu} \;\; \leq \;\; \varphi\left(\bigvee_{\lambda} a_{\lambda}\right).$$

By monotonicity again and by idempotency, we have

$$\varphi\left(\bigvee_{\lambda} \varphi a_{\mu}\right) \;\; \leq \;\; \varphi\varphi\left(\bigvee_{\lambda} a_{\lambda}\right) \; = \; \varphi\left(\bigvee_{\lambda} a_{\lambda}\right).$$

Conversely, by extensivity and monotonicity, we have

$$a_{\lambda} \leq \varphi a_{\lambda} \;\; \Rightarrow \;\; \bigvee_{\lambda} a_{\lambda} \leq \bigvee_{\lambda} \varphi a_{\lambda} \; \Rightarrow \; \varphi\left(\bigvee_{\lambda} a_{\lambda}\right) \leq \varphi\left(\bigvee_{\lambda} \varphi a_{\lambda}\right)$$

(2) Since $a \leq \varphi a$, $b \leq \varphi b$,

$$a \cdot b \leq \varphi a \cdot \varphi b \;\; \Rightarrow \;\; \varphi(a \cdot b) \leq \varphi(\varphi a \cdot \varphi b).$$

Conversely, by using monotonicity and idempotency, we have from (5.1.1) that

$$\varphi(\varphi a \cdot \varphi b) \;\; \leq \;\; \varphi\varphi(a \cdot b) \; = \; \varphi(a \cdot b).$$

q.e.d.

Lemma 5.3. Let $\varphi : L \to L$ be a closure operator of IL algebra L and $\mathfrak{F}_{\varphi} = \{a \in \Omega \mid \varphi a = a\}$ be the set of all φ-closed elelments.

(1) $\mathfrak{F}_{\varphi} \ni a, b \Rightarrow a \wedge b \in \mathfrak{F}_{\varphi}$,

(2) $L \ni a$, $\mathfrak{F}_{\varphi} \ni b \Rightarrow a \multimap b \in \mathfrak{F}_{\varphi}$.

Proof.

(1) Let a, $b \in \mathfrak{F}_{\varphi}$. By monotonicity,

$$\varphi(a \wedge b) \;\; \leq \;\; \varphi a \wedge \varphi b \; = \; a \wedge b.$$

On the other hand, by extensivity,

$$a \wedge b \;\; \leq \;\; \varphi(a \wedge b).$$

Thus $a \wedge b$ is φ-closed.

(2) Let $a \in L$, $b \in \mathfrak{F}_\varphi$.

$$
\begin{aligned}
a \multimap b \leq a \multimap b \quad &\overset{\text{adjunction}}{\Longleftrightarrow} \quad a \cdot (a \multimap b) \leq b \\
&\overset{\text{monotonicity}}{\Longrightarrow} \quad \varphi(a \cdot (a \multimap b)) \leq \varphi b \\
&\overset{(5.1.1)}{\Longrightarrow} \quad \varphi a \cdot \varphi(a \multimap b) \leq \varphi b \\
&\overset{\text{adjunction}}{\Longleftrightarrow} \quad \varphi(a \multimap b) \leq \varphi a \multimap \varphi b = a \multimap b.
\end{aligned}
$$

On the other hand, by extensivity

$$a \multimap b \quad \leq \quad \varphi(a \multimap b).$$

Thus $a \multimap b$ is closed.

<u>*q.e.d.*</u>

Proposition 5.4. Let φ be a closure operator of an IL algebra L. Then the set of all φ-closed elements $\mathfrak{F}_\varphi$ is an IL subalgebra with

$$\bigvee_{\lambda}{}_{\varphi}\, a_\lambda = \varphi\left(\bigvee_\lambda a_\lambda\right), \qquad 0_\varphi = \varphi(0), \qquad a \cdot_\varphi b = \varphi(a \cdot b), \qquad e_\varphi = \varphi\, e\,.$$

Proof. Let $a_\lambda \in \mathfrak{F}_\varphi$. Then for $b \in \mathfrak{F}_\varphi$,

$$\varphi\left(\bigvee_\lambda a_\lambda\right) \leq b \quad \overset{\text{extensivity}}{\Longrightarrow} \quad \bigvee_\lambda a_\lambda \leq \varphi\left(\bigvee_\lambda a_\lambda\right) \leq b.$$

Conversely,

$$\bigvee_\lambda a_\lambda \leq b \quad \overset{\text{monotonicity}}{\Longrightarrow} \quad \varphi\left(\bigvee_\lambda a_\lambda\right) \leq \varphi b = b.$$

This implies that $\varphi\left(\bigvee_\lambda a_\lambda\right)$ is the supremum in $\mathfrak{F}_\varphi$ for the family a_λ, *i.e.*, $\bigvee_{\lambda}{}_{\varphi}\, a_\lambda$. On the other hand, since for any $c \in \mathfrak{F}_\varphi$,

$$0 \leq c \quad \Rightarrow \quad 0_\varphi \leq \varphi c = c,$$

0_φ is the minimum element of $\mathfrak{F}_\varphi$. Hence $(\mathfrak{F}_\varphi, \wedge, \vee_\varphi, 0_\varphi)$ is a lattice with the bottom 0_φ.

It is easily verified that associativity and commutativity of $\cdot_\varphi$ follows from those of $\cdot$. On the other hand, we have

$$a \cdot_\varphi e_\varphi \;=\; \varphi(a \cdot \varphi e) \;=\; \varphi(\varphi a \cdot \varphi e) \;=\; \varphi(a \cdot e) \;=\; \varphi a \;=\; a.$$

and similarly $e_\varphi \cdot_\varphi a = a$. Hence $(\mathfrak{F}_\varphi, \cdot_\varphi, e_\varphi)$ is a commutative monoid with the unit element e_φ.

Finally, we show the adjunction:

$$a \leq b \multimap c \iff a \cdot_\varphi b \leq c.$$

But, since $a \leq b \multimap c \Leftrightarrow a \cdot b \leq c$ in L, it is sufficient to show that

$$\varphi(a \cdot b) \leq c \iff a \cdot b \leq c.$$

$\Rightarrow$ follows from extensitivity and $\Leftarrow$ follows from monotonicity.

<u>*q.e.d.*</u>

Proposition 5.5. Let Q be a quantale.

(1) For any closure operator of IL algebra $\varphi : Q \to Q$, the set of all φ-closed elements $\mathfrak{F}_\varphi$ is a subquantale.

(2) For any subquantale $P \subseteq Q$, there exists a closure operator of IL algebra φ such that $\mathfrak{F}_\varphi = P$.

Proof.

(1) By Proposition 5.4, $\mathfrak{F}_\varphi$ is an IL subalgebra. Thus, by virtue of Proposition 5.1, it is sufficient to show that $\mathfrak{F}_\varphi$ is complete. Let $S \subseteq \mathfrak{F}_\varphi$. Since Q is complete, S has an infimum $\bigwedge S$ in Q. But, since $\mathfrak{F}_\varphi$ is a Moore family, $\bigwedge S \in \mathfrak{F}_\varphi$. Thus $\mathfrak{F}_\varphi$ is complete.

(2) Since P is $\bigwedge$-complete, it is a Moore family. Then by Proposition 2.7, there exists a closure operator φ such that $\mathfrak{F}_\varphi = P$. Thus all we have to show is that φ satisfies (5.1.1). We note that φ is explicitly written as

$$\varphi a \;=\; \bigwedge \{x \in P \mid a \leq x\} \qquad (a \in Q).$$

Let a, $b \in Q$. For any $x \in P$,

$$\begin{aligned}
a \cdot b \leq x \quad &\overset{\text{adjunction}}{\Longrightarrow} \quad a \leq b \multimap x \\
&\overset{\text{monotonicity}}{\Longrightarrow} \quad \varphi a \leq \varphi(b \multimap x) \overset{\text{Lemma 5.3}}{=\!=\!=} b \multimap x \\
&\overset{\text{adjunction}}{\Longrightarrow} \quad \varphi a \cdot b \leq x \\
&\overset{\text{adjunction}}{\Longrightarrow} \quad b \leq \varphi a \multimap x \\
&\overset{\text{monotonicity}}{\Longrightarrow} \quad \varphi b \leq \varphi(\varphi a \multimap x) \overset{\text{Lemma 5.3}}{=\!=\!=} \varphi a \multimap x \\
&\overset{\text{adjunction}}{\Longrightarrow} \quad \varphi a \cdot \varphi b \leq x.
\end{aligned}$$

Thus we have $\varphi a \cdot \varphi b \leq \varphi(a \cdot b)$.

q.e.d.

Example 5.1.1. (*cf.* Example 2.3.5) Let R be a binary relation on a commutative monoid M. For the morphological closing $\varphi_R = E_R \circ D_{{}^tR}$ defined by R to be a closure operator of the IL algbra $\mathfrak{P}(M)$ iff for a, $b \in M$ and A, $B \in \mathfrak{P}(M)$,

$$R(a) \subseteq R(A) \text{ and } R(b) \subseteq R(B) \quad \Rightarrow \quad R(a \cdot b) \subseteq R(A \cdot B). \tag{5.1.2}$$

($\because$) For φ_R, the condition (5.1.1) becomes

$$E_R D_{{}^tR}(A) \cdot E_R D_{{}^tR}(B) \quad \subseteq \quad E_R D_{{}^tR}(A \cdot B) \qquad (A, B \in \mathfrak{P}(M)).$$

But by Example 2.3.8, this is equivalent to (5.1.2).

□

A binary relation R on a commutative monoid M is said to be *compatible* (*with the multiplication*) iff

$$R(a) \cdot b \subseteq R(a \cdot b) \quad (a, b \in M).$$

Immediately from the defintion, for any compatible R, we have

$$R(a) \cdot R(b) \subseteq R(a \cdot b) \quad (a, b \in M), \tag{5.1.3}$$
$$R(A) \cdot R(B) \subseteq R(A \cdot B) \quad (A, B \in \mathfrak{P}(M)). \tag{5.1.4}$$

Any compatible quasi-order satisfies (5.1.2). In fact, let R be a compatible quasi-order $\leq$. Then for $a \in M$,

$$R(a) = \{m \in M \mid a \leq m\},$$

and for $a \in M$, $A \in \mathfrak{P}(M)$,

$$R(a) \subseteq R(A) \quad \Leftrightarrow \quad a \in R(A).$$

Therefore, if $R(a) \subseteq R(A)$ and $R(b) \subseteq R(B)$, then by (5.1.4),

$$a \cdot b \in R(A) \cdot R(B) \subseteq R(A \cdot B),$$

and hence we have $R(a \cdot b) \subseteq R(A \cdot B)$. This establishes (5.1.2).

Another example is a homogeneous relation. When, in the defintion of compatibility, equality holds instead of inclusion:

$$R(a) \cdot b = R(a \cdot b) \quad (a, b \in M),$$

we say that R is *homogeneous*. When M is a group, the compatibility coincides with the homogeneity:
($\because$) It follows from

$$\begin{aligned} R(a \cdot b) &= R(a \cdot b) \cdot (b^{-1} \cdot b) \\ &\subseteq R((a \cdot b) \cdot b^{-1}) \cdot b \;=\; R(a) \cdot b. \end{aligned}$$

$\square$

Any homogeneous relation R on a commutative monoid satisfies (5.1.2). In fact, if R is homogeneous, $R(a)$ for $a \in M$ can be written as

$$R(a) \quad = \quad R(e) \cdot a,$$

where e is the unit element. Furthermore, for $a \in M$ and $A \in \mathfrak{P}(M)$,

$$R(a) \subseteq R(A) \quad \Leftrightarrow \quad R(e) \cdot a \subseteq \bigcup_{x \in A} R(e) \cdot x.$$

Now suppose that $R(a) \subseteq R(A)$ and $R(b) \subseteq R(B)$. Then

$$\begin{aligned} R(a \cdot b) &= R(a) \cdot b \\ &\subseteq \left(\bigcup_{x \in A} R(e) \cdot x \right) \cdot b \;=\; \bigcup_{x \in A} (R(e) \cdot b) \cdot x \\ &\subseteq \bigcup_{x \in A} \left(\bigcup_{y \in B} R(e) \cdot y \right) \cdot x \;=\; \bigcup_{x \in A, y \in B} R(e) \cdot (x \cdot y) \\ &= R(A \cdot B). \end{aligned}$$

Thus (5.1.2) holds.

Example 5.1.2. (*cf.* Example 2.3.5) Let R be a binary relation on a commutative monoid M. For the Galois closing $\gamma_R = C_R \circ C_{{}^tR}$ defined by R to be a closure operator of the IL algebra $\mathfrak{P}(M)$ iff for any a, $b \in M$ and any A, $B \in \mathfrak{P}(M)$,

$$R^*(A) \subseteq R(a) \text{ and } R^*(B) \subseteq R(b) \quad \Rightarrow \quad R^*(A \cdot B) \subseteq R(a \cdot b). \quad (5.1.5)$$

($\because$) For γ_R, the condition (5.1.1) becomes

$$C_R C_{{}^tR}(A) \cdot C_R C_{{}^tR}(B) \;\; \subseteq \;\; C_R C_{{}^tR}(A \cdot B) \qquad (A, B \in \mathfrak{P}(M)).$$

But by Example 2.3.8, this is equivalent to (5.1.5).

$\square$

Similary to the previous example, on a commutative monoid M, a compatible quasi-order or a homogeneous relation satisfies (5.1.5).

If R is a compatible quasi-order $\leq$, then for $a \in M$, $A \in \mathfrak{P}(M)$,

$$R^*(A) \subseteq R(a) \quad \Leftrightarrow \quad A \subseteq R(a).$$

Therefore, if $R^*(A) \subseteq R(a)$ and $R^*(B) \subseteq R(b)$, then by (5.1.3),

$$A \cdot B \subseteq R(a) \cdot R(b) \subseteq R(a \cdot b),$$

and hence we have $R^*(A \cdot B) \subseteq R(a \cdot b)$. Thus we have shown (5.1.5).

On the other hand, if R is homegeneous, for $a \in M$ and $A \in \mathfrak{P}(M)$,

$$R^*(A) \subseteq R(a) \quad \Leftrightarrow \quad \bigcap_{x \in A} R(e) \cdot x \subseteq R(e) \cdot a.$$

Now suppose that $R^*(A) \subseteq R(a)$ and $R^*(B) \subseteq R(b)$. Then

$$\begin{aligned} R^*(A \cdot B) \;\; &= \;\; \bigcap_{x \in A, y \in B} R(x \cdot y) \;=\; \bigcap_{y \in B} \left(\bigcap_{x \in A} R(x) \right) \cdot y \;=\; \bigcap_{y \in B} R^*(A) \cdot y \\ &\subseteq \;\; \bigcap_{y \in B} R(a) \cdot y \;=\; \left(\bigcap_{y \in B} R(e) \cdot y \right) \cdot a \\ &\subseteq \;\; (R(e) \cdot b) \cdot a \;=\; R(a \cdot b). \end{aligned}$$

This means that (5.1.5).

Note. Consider the case where M is a commutative monoid of multisets of places of a Petri net and R is its reachability relation, *i.e.*, the reflexive and transitive closure of the firing relations, on M. Then R is a compatible quasi-order $\leq$. In this case, the result of the morphological closing φ_R defined by R for $A \in \mathfrak{P}(M)$ is nothing but taking upper closure:

$$\varphi_R(A) \;=\; \{x \in M \mid \exists a \in A\,(a \leq x)\}\,.$$

On the other hand, since operations of $C_{{}^tR}$ and C_R respectively coincide with that of taking upper bounds and lower bounds:

$$\begin{aligned} C_{{}^tR}(A) &= R^*(A) = \{x \in M \mid \forall a \in A\,(a \leq x)\} = \mathscr{U}(A), \\ C_R(A) &= {}^tR^*(A) = \{x \in M \mid \forall a \in A\,(x \leq a)\} = \mathscr{L}(A), \end{aligned}$$

the result of the Galois closing γ_R defined by R for $A \in \mathfrak{P}(M)$ can be rewritten as

$$\gamma_A(A) \;=\; \mathscr{L}(\mathscr{U}(A))\,.$$

This operator is known as *MacNeille completion* [17].

These closure operators are used in Petri net semantics for intuitionistic linear logic ([11], [10], [12], [20], [21]). It is known that the intuitionistic linear logic is sound with respect to Petri net semantics of both types, however, not complete with respect to the ones with upper closures, but complete with respect to the ones with MacNeille completions. For more details, see [20], [21].

5.2 Semantics for intuitionistic linear logic

5.2.1 Morphological/Galois ILL frames

Definition. We call a Kripke frame $\mathscr{F} = (\Omega, R)$ a *morphological ILL frame* iff Ω is a commutative monoid and R satisfies (5.1.2). Similarly, we call $\mathscr{F}$ a *Galois ILL frame* iff Ω is a commutative monoid and R satisfies (5.1.5). The *closure operator associated* to an ILL frame $\mathscr{F}$ means φ_R when $\mathscr{F}$ is morphological and γ_R when $\mathscr{F}$ is Galois.

Let $\mathscr{F} = (\Omega, R)$ be a morphological/Galois ILL frame and φ be the closure operator associated to $\mathscr{F}$. By Proposition 5.5, the set of φ-closed sets $\mathfrak{F}_\varphi$ is a subquantale of $\mathfrak{P}(\Omega)$. To each φ-closed set $F \in \mathfrak{F}_\varphi$,

$$E_F : \mathfrak{F}_\varphi \ni C \;\mapsto\; F \multimap C \in \mathfrak{F}_\varphi, \tag{5.2.1}$$

$$D_F : \mathfrak{F}_\varphi \ni C \;\mapsto\; \varphi(F \cdot C) \in \mathfrak{F}_\varphi \tag{5.2.2}$$

defines an adjunction (E_F, D_F).

Note. The adjunction (E_F, D_F) comes from the F-action:

$$\overset{F}{\rightarrow} \;=\; \{(\omega, \varpi) \mid \exists \tau \in F\,(\tau \cdot \omega = \varpi)\}\,.$$

We remark that every such adjunction should be distinguished from the one defined from the accessibility relation R of $\mathscr{F}$.

5.2.2 Morphological/Galois ILL models

Definition. A standard model $\mathscr{M}$ based on a morphological ILL frame $\mathscr{F}$ with a valuation valued in $\mathfrak{F}_{\varphi_R}$ is called a *morphological ILL model*. Simillarly, a standard model $\mathscr{M}$ based on a Galois ILL frame $\mathscr{F}$ with a valuation valued in $\mathfrak{F}_{\gamma_R}$ is called a *Galois ILL model*.

Let $\Phi_{i\ell} = \Phi^L(\Psi)$ be the set of formulas for intuitionistic linear logic and $\mathscr{M}$ be an ILL model based on an ILL frame $\mathscr{F} = (\Omega, R)$ with a valuation $V : \Psi \to \mathfrak{F}_\varphi$, where φ is the closure operator associated to $\mathscr{F}$. The valuation V can be extended to the formula set $\Phi_{i\ell}$ by

(1) $V\,(p_i)$ for $p_i \in \Psi$.

(2) $V\,(\mathbf{1}) = \varphi(\{e\}) \quad (= e_\varphi)$.

(3) $V\,(\mathbf{0}) = \varphi(\emptyset) \quad (= 0_\varphi)$.

(4) $V\,(\top) = \Omega$.

(5) $V\,(\phi \otimes \psi) = \varphi\,(V\,(\phi) \cdot V\,(\psi)) \quad (= V\,(\phi) \cdot_\varphi V\,(\psi))$.

(6) $V\,(\phi \oplus \psi) = \varphi\,(V\,(\phi) \cup V\,(\psi)) \quad (= V\,(\phi) \vee_\varphi V\,(\psi))$.

(7) $V\,(\phi \,\&\, \psi) = V\,(\varphi) \cap V\,(\psi)$.

(8) $V\,(\phi \multimap \psi) = V\,(\varphi) \rightarrow V\,(\psi)$.

Then we have

$$V\,(\phi \otimes \psi) \;=\; D_{V(\phi)} V\,(\psi)\,, \tag{5.2.3}$$
$$V\,(\phi \multimap \psi) \;=\; E_{V(\phi)} V\,(\psi)\,. \tag{5.2.4}$$

The notion of truth of each formula for intuitionistic linear logic is modified from the usual one as follows:

Definition. A formula $\phi \in \Phi_{i\ell}$ is said to be *true* in an ILL model $\mathscr{M}$ and denoted by $\models^{\mathscr{M}} \phi$ iff

$$e_{\varphi} \subseteq V(\phi)\,; \tag{5.2.5}$$

ϕ is said to be *valid* in an ILL frame $\mathscr{F}$ and denoted by $\models^{\mathscr{F}} \phi$ iff $\models^{\mathscr{M}} \phi$ for every model $\mathscr{M}$ based on $\mathscr{F}$; ϕ is said to be *valid* in a class $\mathfrak{F}$ of ILL frames and denoted by $\models^{\mathfrak{F}} \phi$ iff $\models^{\mathscr{F}} \phi$ for every frame $\mathscr{F}$ in $\mathfrak{F}$.

Note. Any Petri net semantics for intuitionistic linear logic with MacNeille completion is obtained as a Galois ILL model (*cf.* note just after Example 5.1.2).

Proposition 5.6.

(1) The intuitionistic linear logic is sound with respect to any morphological ILL frame.

(2) The intuitionistic linear logic is sound with respect to any Galois ILL frame.

Proof. For both cases, soundness is proved by a straightforwrd induction.

q.e.d.

5.3 Linear logic

5.3.1 Linear algebras

Definition. An IL algebra L is called an *L algebra* (or a linear algebra) iff there exists an element $\perp \in L$ satisfying

$$a \;=\; (a \multimap \perp) \multimap \perp \tag{5.3.1}$$

for $\forall a \in L$. Similarly, a quantale Q is called a *classical* iff there exists an element $\perp \in Q$ satisfying

$$a \;=\; (a \multimap \perp) \multimap \perp \tag{5.3.2}$$

for $\forall a \in Q$. In both cases, we call $\perp$ a *falsity element.*

Proposition 5.7. A classical quantale is an L algebra. Conversely, any complete L algebra is a classical quantale.

Proof. Immediately from Proposition 5.1.

q.e.d.

A general method to construct an L algebra from a given IL algebra L using an arbitrary chosen falsity element ν is known [15]. Here we show that such a construction can be understood in the context of Galois connection.

Theorem 5.8. Let L be an IL algebra and $\nu \in L$ be an arbitrary element. Then the duplicate pair (γ_ν, γ_ν) of

$$\gamma_\nu : L \ni x \quad \mapsto \quad x \multimap \nu \in L$$

is a Galois connection. Futhermore γ_ν^2 is a closure operator of IL algebra and $\mathfrak{F}_{\gamma_\nu^2}$ is an L algebra with the falsity element

$$\perp_{\gamma_\nu} = \nu.$$

Proof. First we show that γ_ν is anti-monotonous. In fact, let a, $b \in L$ and suppose that $a \leq b$. Then

$$\begin{aligned} b \multimap \nu \leq b \multimap \nu \quad & \overset{\text{adjunction}}{\Longleftrightarrow} \quad b \leq (b \multimap \nu) \multimap \nu \\ & \overset{\text{assumption}}{\Longrightarrow} \quad a \leq (b \multimap \nu) \multimap \nu \\ & \overset{\text{adjunction}}{\Longleftrightarrow} \quad a \cdot (b \multimap \nu) \leq \nu \\ & \overset{\text{adjunction}}{\Longleftrightarrow} \quad b \multimap \nu \leq a \multimap \nu. \end{aligned}$$

For any $a \in L$,

$$a \multimap \nu \leq a \multimap \nu \quad \Rightarrow \quad a \cdot (a \multimap \nu) \leq \nu \;\Rightarrow\; a \leq (a \multimap \nu) \multimap \nu = \gamma_\nu^2 a.$$

This show that (γ_ν, γ_ν) is a Galois connection. Since γ_ν^2 is a closure operator of lattice, for γ_ν^2 to be a closure operator of IL algebra, it is sufficient to show that

$$\gamma_\nu^2 a \cdot \gamma_\nu^2 b \quad \leq \quad \gamma_\nu^2 (a \cdot b).$$

To do this, we first note that

$$a \multimap (b \multimap c) \quad = \quad (a \cdot b) \multimap c$$

holds for $\forall a,\, b,\, c \in L$ by virtue of commutativity and associativity of multiplication. Then the equality $a \multimap (b \multimap \nu) = (a \cdot b) \multimap \nu$ can be written as

$$a \multimap \gamma_\nu b \;\;=\;\; \gamma_\nu(a \cdot b). \tag{5.3.3}$$

From this

$$\begin{array}{lcl}
\gamma_\nu(a \cdot b) \leq a \multimap \gamma_\nu b & \overset{\text{adjunction}}{\Longleftrightarrow} & \gamma_\nu(a \cdot b) \cdot a \leq \gamma_\nu b \\
& \overset{\text{anti-monotonicity}}{\Longrightarrow} & \gamma_\nu^2 b \leq \gamma_\nu(\gamma_\nu(a \cdot b) \cdot a) \overset{(5.3.3)}{=\!=} \gamma_\nu(a \cdot b) \multimap \gamma_\nu a \\
& \overset{\text{adjunction}}{\Longleftrightarrow} & \gamma_\nu^2 b \cdot \gamma_\nu(a \cdot b) \leq \gamma_\nu a \\
& \overset{\text{anti-monotonicity}}{\Longrightarrow} & \gamma_\nu^2 a \leq \gamma_\nu \left(\gamma_\nu^2 b \cdot \gamma_\nu(a \cdot b)\right) \overset{(5.3.3)}{=\!=} \gamma_\nu^2 b \multimap \gamma_\nu^2(a \cdot b) \\
& \overset{\text{adjunction}}{\Longleftrightarrow} & \gamma_\nu^2 a \cdot \gamma_\nu^2 b \leq \gamma_\nu^2(a \cdot b).
\end{array}$$

Hence by Proposition 5.4, the set of γ_ν^2-closed elements $\mathfrak{F}_{\gamma_\nu^2}$ is an IL algbra. Finally, since

$$(a \multimap \bot_{\gamma_\nu}) \multimap \bot_{\gamma_\nu} \;\;=\;\; \gamma_\nu^2 a \;=\; a$$

for any $a \in \gamma_\nu^2(L)$, we conclude that $\bot_{\gamma_\nu} = \nu$ is a falsity element.

q.e.d.

Note. When L is the power set lattice of some commutative monoid, the L algebra $(\mathfrak{F}_{\gamma_\nu^2}, \nu)$ derived as above is called a *phase space* and each element of $\mathfrak{F}_{\gamma_\nu^2}$ is referred to as a *fact* [15]. The notion of LL frame we will introduce in the next section is a generalization of phase space construction to an arbitrary IL subalgebra of a power set.

5.3.2 Semantics for linear logic

Definition. We call a pair of ILL frame $\mathscr{F} = (\Omega, R)$ and a subset $\bot \subseteq \mathfrak{F}_\varphi$ an *LL frame*, where φ is the closure operator associated to $\mathscr{F}$. Also, we call a pair of ILL model $\mathscr{M} = (\Omega, R, V)$ and a subset $\bot \subseteq \mathfrak{F}_\varphi$ an *LL model*. We call $\bot$ the *set of falsity*.

Let $\Phi_\ell = \Phi^L(\Psi, \bot, {}^\perp, \invamp)$ be the set of formulas for linear logic and $\mathscr{M}$ be an LL model based on an LL frame $\mathscr{F} = (\Omega, R, \bot)$ with a valuation $V : \Psi \to \mathfrak{F}_\varphi$, where φ is the closure operator associated to $\mathscr{F}$. The valuation V can be extended to the formula set Φ_ℓ as follows: For $X \in \mathfrak{F}_\varphi$, we use the notation $X^\perp$ for representing $X \multimap \bot$.

(1) $V\,(p_i)$ for $p_i \in \Psi$.

(2) $V(\mathbf{1}) = \varphi(\{e\})^{\perp\perp} \quad \left(= {e_\varphi}^{\perp\perp}\right)$.

(3) $V(\mathbf{0}) = \varphi(\emptyset)^{\perp\perp} \quad \left(= {0_\varphi}^{\perp\perp}\right)$.

(4) $V(\top) = \Omega$.

(5) $V(\perp) = \perp$.

(6) $V\left(\phi^{\perp}\right) = V(\phi)^{\perp} \quad (= X \multimap \perp)$.

(7) $V(\phi \otimes \psi) = \varphi\left(V(\phi) \cdot V(\psi)\right)^{\perp\perp} \quad \left(= \left(V(\phi) \cdot_\varphi V(\psi)\right)^{\perp\perp}\right)$.

(8) $V(\phi \parr \psi) = \varphi\left(V(\phi)^{\perp} \cdot V(\psi)^{\perp}\right)^{\perp} \quad \left(= \left(V(\phi)^{\perp} \cdot_\varphi V(\psi)^{\perp}\right)^{\perp}\right)$.

(9) $V(\phi \oplus \psi) = \varphi\left(V(\phi) \cup V(\psi)\right)^{\perp\perp} \quad \left(= \left(V(\phi) \vee_\varphi V(\psi)\right)^{\perp\perp}\right)$.

(10) $V(\phi \,\&\, \psi) = V(\varphi) \cap V(\psi)$.

(11) $V(\phi \multimap \psi) = V(\varphi) \multimap V(\psi)$.

Note. When the accessibiliy relation R is the identity relation $\boldsymbol{I}$:

$$\omega \boldsymbol{I} \varpi \quad \Leftrightarrow \quad \omega = \varpi,$$

morphological LL models gives rise to Phase space semantics for linear logic, and any Phase space semantics is obtained by a morphological LL model. Similarly, when the accessibiliy relation R is the complementary identity relation $\boldsymbol{I}^c$:

$$\omega \boldsymbol{I}^c \varpi \quad \Leftrightarrow \quad \omega \neq \varpi,$$

Galois LL models gives rise to Phase space semantics for linear logic, and any Phase space semantics is obtained by a Galois LL model. It is known that linear logic is complete and sound with respect to the class of Phase space semantics [15].

Proposition 5.9.

(1) The linear logic is sound with respect to any morphological LL frame.

(2) The linear logic is sound with respect to any Galois LL frame.

Proof. For both cases, soundness is proved by a straightforwrd induction.

q.e.d.

Chapter 6

Conclusion

This work is motivated by the question of what will happen when mathematical morphology is applied to logic. Through this work, we have got several solutions as well as further problems.

From the point of view of erosion and dilation, we can say that mathematical morphology defines modal logics. If we consider a pair of morphological operators (a dual pair), we obtain a normal modal logics. On the other hand, if we consider a pair of algebraic ones (an adjunction), we obtain a normal 2-dimensional modal logics. Most of knowledge about this was already obtained by one of the authors in the precedent works ([7], [5], [6]). Contribution of this work for this view is as follows:

To clarify the mechanism of how mathematical morphology defines modal logic, we proposed definitions of modal logics based on Kripke frames for both cases and then we showed equivalence existing between fundamental schemes and properties of accessibilty relations. By this approach, we can exclude the assumption of finiteness of propositional symbols and that of surjectivity of valuations of models. We also established the equivalence between a 2-dimensional modal logic based on an adjunction and a temporal logic.

From the point of view of opening and closing, mathematical morphology provides possible world semantics for non-classical logics (LJ, ILL and LL).

In the effort of systematic understanding of so called "closure consturuction of model spaces for linear logic" in the context of mathematical morphology (and Galois connection), we reached the notions of LJ, ILL and LL-frames. For each one of these classes, we gave a necessary and sufficient condition for a given Kripke frame to belong to it explicitly in terms of accessibility relations or equivalently, in terms of erosion and dilation, or connections. Each of LJ, ILL and LL is sound with respect to corresponding classes. For the completeness, we need further research.

For the future works, several topics can be offered:

Many-dimensional modal logic

As a direct extension, properties of temporal operators derived from an adjuntion should be described.

In relevance to temporal logic and modal logic, we can pose the Temporalization Problem. This can be explained as follows: for any normal modal logic, we can extend it to a temporal logic since every normal modal logic can be represented as a modal logic based on a Kripke model and each Kripke model is embeded into a bidirectional model. Of course we have an ambiguity of choice of embedding. Leaving it out of concern, our problem is that of deciding the number of different temporalizations of a given normal logic, and if possible, to parametrize it.

Many-dimensional modal logics apply to logics of spaces with several attributes represented by modal operators. Also, curved many-dimensional modal logics and its parallelizability would have a link with fusion problems.

More on morphological analysis

Mathematical morphology applied to non-classical logics in this report plays two different roles:

- erosion/dilation for modal logics, as analyzing tools;
- opening/closing for other non-classical (*i.e.*, intuitionistic/linear) logics, as construction tools;

What about exchanging the combination?

⟨Modal logics based on opening/closing⟩

As related topics about this interest, topological interpretation and neighbourhood semantics ([1]) are already mentioned in [6]. Although there closing/opening of a dual pair are considered, it seems profitable considering those of adjunction, because of idempotency, which is required for topological closures/interiors. We note that (4.1.3) shows a property of topological basis provided that R is serial ($\Leftrightarrow {}^tR(X) = X$). Thus for topological consideration, it seems better to take the intuitionistic (modal) logic.

Natural extension of this direction is the region of spatial interpretation in spatial logics, such as distance logic, mereotopology *etc.* Spatial interpretation accompanied with temporal logic would provide us with a method of spatio-temporal reasoning. We also remark that reasoning about dynamical systems are given by Kripke semantics (as LTS : labeled transition systems) and by Petri net semantics.

⟨Minimal models and further semantics⟩

We also note that neighbourhood semantics inspires us to consider "minimal models" of modal logic [8], which requires a correspondence $\Gamma : \Omega \to \mathfrak{P}(\Omega)$ instead of accessibility relation $R : \Omega \to \Omega$. As well as normal modal logics are obtained from standard models, *classical modal logics* are obtained from minimal models.

Taking into account minimal models is important to characterize modal operators. Because modal operators compatibe with taking finite infimum/supremum can be characterized as *regular modal logics* in the context of classical logic [8]. To know when this compatiblity with finite infimum/supremum turns into the one for infinite infimum/supremum is nothing else than to know when modal operators are derived from algebraic erosion/dilation.

Other directions of generalization of semantics are sheaf model (Kripke-Joyal semantics) and categorical model. The notion of sheaf presents local data, which are distributed locally and can be patched together, and is a natural extension of simple topology. Category theory includes generalized sheaves as special case of topoi, and there, adjunctions and Galois connections naturally occur [23].

⟨Morphological analysis on intuitionistic/linear logics⟩

In intuitionistic/linear case, a Kripke frame does not play a role of giving mathematical morphology yet.

But there are many symptoms of possibility of morphological analysis:

(1) In the first place, intuitionistic/linear logics are accompanied with a series of adjuntions by nature, $(\phi\wedge, \phi\to)$ in intuitionistic logic, $(\phi\otimes, \phi\multimap)$ in linear logic;

(2) For intuitionistic logic, $\mathfrak{O}_{\alpha_R}$ is closed under D_{t_R}, and E_R is a right inverse of D_{t_R} on $\mathfrak{O}_{\alpha_R}$ *etc.*, thus we have a filtration of spaces:

$$\begin{array}{ccccccccc}
D^0_{t_R}(\Omega) & \underset{\overset{\longleftarrow}{E_R}}{\overset{\overset{D_{t_R}}{\longrightarrow}}{\supseteq}} & D^1_{t_R}(\Omega) & \underset{\overset{\longleftarrow}{E_R}}{\overset{\overset{D_{t_R}}{\longrightarrow}}{\supseteq}} & D^2_{t_R}(\Omega) & \underset{\overset{\longleftarrow}{E_R}}{\overset{\overset{D_{t_R}}{\longrightarrow}}{\supseteq}} & D^3_{t_R}(\Omega) & \underset{\overset{\longleftarrow}{E_R}}{\overset{\overset{D_{t_R}}{\longrightarrow}}{\supseteq}} & \cdots \\
|| & & || & & || & & || & & \\
\Omega & & \mathfrak{O}_{\alpha_R} & & \mathfrak{O}_{\alpha_{R^2}} & & \mathfrak{O}_{\alpha_{R^3}} & &
\end{array}$$

Similarly, for intuitionistic linear logic, $\mathfrak{F}_{\varphi_R}$ is closed under E_{t_R}, and D_R is a right inverse of E_{t_R} on $\mathfrak{F}_{\varphi_R}$ *etc.*, and thus again we have a filtration of spaces:

$$\begin{array}{ccccccccc}
E^0_{t_R}(\Omega) & \underset{\xleftarrow[D_R]{}}{\overset{\xrightarrow{E_{t_R}}}{\supseteq}} & E^1_{t_R}(\Omega) & \underset{\xleftarrow[D_R]{}}{\overset{\xrightarrow{E_{t_R}}}{\supseteq}} & E^2_{t_R}(\Omega) & \underset{\xleftarrow[D_R]{}}{\overset{\xrightarrow{E_{t_R}}}{\supseteq}} & E^3_{t_R}(\Omega) & \underset{\xleftarrow[D_R]{}}{\overset{\xrightarrow{E_{t_R}}}{\supseteq}} & \cdots \\
\| & & \| & & \| & & \| & & \\
\Omega & & \mathfrak{F}_{\varphi_R} & & \mathfrak{F}_{\varphi_{R^2}} & & \mathfrak{F}_{\varphi_{R^3}} & &
\end{array}$$

(3) Operations appearing in (1) can be restricted to the model space $\mathfrak{O}_{\alpha_R}/\mathfrak{F}_{\varphi_R}$.

The filtration $D^n_{t_R}(\Omega)/E^n_{t_R}(\Omega)$ $(n = 0, 1, \dots)$ with a series of adjunctions $(\phi\wedge, \phi\rightarrow)$ $(\phi \in \Phi_i)$/ $(\phi\,\cdot, \phi\multimap)$ $(\phi \in \Phi_{i\ell})$ seems to provide a systematic analyzing tool for intuitionistic/linear logic. We hope such a analysis extends classical morphological methods such as granulometry, pattern spectrum analysis *etc.*

Completeness

As noted in this report, we have partial information about the completeness:

- the class of morphological LJ models contains all of Kripke semantics for intuitionistic logic, with respect to which LJ is complete;
- the class of morphological ILL models contains all of Petri net semantics for intuitionistic linear logic with upper closure, with respect to which ILL is not complete;
- the class of Galois ILL models contains all of Petri net semantics for intuitionistic linear logic with MacNeille completion, with respect to which ILL is complete;
- the class of morphological/Galois LL models contains all of Phase space semantics for linear logic, with respect to which LL is complete.

Related to this direction, we should clarify the difference between general frames and well known semantics or examples presented in this work: for example, it is interesting to develop examples of $\mathscr{F} = (\Omega, R)$ such that

- morphological LJ but R is not a quasi-order (*i.e.*, Kripke semantics for intuitionistic logic);
- Galois LJ but R is not a total order;
- morphological ILL but R is not a compatible quasiorder (thus not Petri net semantics with upper closure for intuitionistic linear logic) nor homogeneous;

- Galois ILL but R is not a compatible quasiorder (thus not Petri net semantics MacNeille completion for intuitionistic linear logic) nor homogeneous.

Appendix A

Logical supplements

A.1 Sequent calculi

In what follows, Γ, Δ *etc*, denote finite (may be zero) sequences of formulas. We use the notation $\Gamma \Leftrightarrow \Delta$ for the abbreviation of "$\Gamma \Rightarrow \Delta$ and $\Delta \Rightarrow \Gamma$".

A.1.1 Classical logic : LK

Formulas

$$\Phi_c = \Phi(\Psi). \tag{1.1.1}$$

Axioms

$$\phi \Rightarrow \phi \quad (\phi \in \Phi_c) \tag{1.1.2}$$

$$\Rightarrow \top \tag{1.1.3}$$

$$\bot \Rightarrow \tag{1.1.4}$$

Inference rules

Structural rules

(weakening left)

$$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}$$

(weakening right)

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}$$

(contraction left)

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}$$

(contraction right)

$$\frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}$$

(exchange left)

$$\frac{\Gamma, \phi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \phi, \Pi \Rightarrow \Delta}$$

(exchange right)

$$\frac{\Gamma \Rightarrow \Delta, \phi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \phi, \Lambda}$$

(cut)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical rules

($\wedge$ left 1)

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}$$

($\wedge$ left 2)

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}$$

($\wedge$ right)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

($\vee$ left)

$$\frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}$$

($\vee$ right 1)

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$$

($\vee$ right 2)

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$$

($\rightarrow$ left)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Pi \Rightarrow \Lambda}{\phi \rightarrow \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

($\rightarrow$ right)

$$\frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \psi}$$

($\neg$ left)

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\neg\phi, \Gamma \Rightarrow \Delta}$$

($\neg$ right)

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\phi}$$

A.1.2 Modal logic : K

Formulas

$$\Phi_m = \Phi(\Psi, \Diamond, \Box). \tag{1.1.5}$$

Axioms

All axioms of LK ((1.1.2), (1.1.3), (1.1.4)) +

$$\text{(Df)} \qquad \Box\phi \Leftrightarrow \neg\Diamond\neg\phi \quad (\phi \in \Phi_m) \tag{1.1.6}$$

Inference rules

All inference rules of LK +

Logical rules

($\Box$)

$$\frac{\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}$$

A.1.3 Modal logic : D, T, B, S4, S5

Each logic among D, T, B, S4, S5 has same the formula set and inference rules as K but axioms are different as follows:

Axioms of modal logic D

All axioms of K ((1.1.2), (1.1.3), (1.1.4), (1.1.6)) +

$$\text{(D)} \qquad \Box\phi \Rightarrow \Diamond\phi \quad (\phi \in \Phi_m) \tag{1.1.7}$$

Axioms of modal logic T

All axioms of K ((1.1.2), (1.1.3), (1.1.4), (1.1.6)) +

$$\text{(T)} \qquad \Box\phi \Rightarrow \phi \quad (\phi \in \Phi_m) \tag{1.1.8}$$

Axioms of modal logic B

All axioms of K ((1.1.2), (1.1.3), (1.1.4), (1.1.6)) +

$$\text{(B)} \qquad \phi \Rightarrow \Box\Diamond\phi \quad (\phi \in \Phi_m) \tag{1.1.9}$$

Axioms of modal logic S4

All axioms of T ((1.1.2), (1.1.3), (1.1.4), (1.1.6), (1.1.8)) +

$$\text{(4)} \qquad \Box\phi \Rightarrow \Box\Box\phi \quad (\phi \in \Phi_m) \tag{1.1.10}$$

Axioms of modal logic S5

All axioms of T ((1.1.2), (1.1.3), (1.1.4), (1.1.6), (1.1.8)) +

$$(5) \qquad \Box\phi \Rightarrow \Box\Diamond\phi \quad (\phi \in \Phi_m) \tag{1.1.11}$$

A.1.4 Minimal temporal logic : $\mathbf{K}_t$

Formulas

$$\Phi_t = \Phi(\Psi, G, F, H, P). \tag{1.1.12}$$

Axioms

All axioms of LK ((1.1.2), (1.1.3), (1.1.4)) +

$$F\phi \Leftrightarrow \neg G \neg \phi \quad (\phi \in \Phi_m) \tag{1.1.13}$$
$$P\phi \Leftrightarrow \neg H \neg \phi \quad (\phi \in \Phi_m) \tag{1.1.14}$$
$$\phi \Rightarrow GP\phi \quad (\phi \in \Phi_m) \tag{1.1.15}$$
$$\phi \Rightarrow HF\phi \quad (\phi \in \Phi_m) \tag{1.1.16}$$

Inference rules

All inference rules of LK +

Logical rules

$([G])$

$$\frac{\Gamma \Rightarrow \phi}{G\Gamma \Rightarrow G\phi}$$

$([H])$

$$\frac{\Gamma \Rightarrow \phi}{H\Gamma \Rightarrow H\phi}$$

A.1.5 Intuitionistic logic : LJ

Formulas

$$\Phi_i = \Phi(\Psi). \tag{1.1.17}$$

Axioms

All axioms of LK ((1.1.2), (1.1.3), (1.1.4))

Inference rules

In intuitionistic calculus, any sequent is assumed to be the form

$$\Gamma \Rightarrow \Delta \tag{1.1.18}$$

with Δ is at most one formula.

Structural rules

(weakening left)

$$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}$$

(weakening right)

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \phi}$$

(contraction left)

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}$$

(exchange left)

$$\frac{\Gamma, \phi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \phi, \Pi \Rightarrow \Delta}$$

(cut)

$$\frac{\Gamma \Rightarrow \phi \qquad \phi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Lambda}$$

Logical rules

($\wedge$ left 1)

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}$$

($\wedge$ left 2)

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}$$

($\wedge$ right)

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi}$$

($\vee$ left)

$$\frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}$$

($\vee$ right 1)

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi}$$

($\vee$ right 2)

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi}$$

(→ left)

$$\frac{\Gamma \Rightarrow \phi \qquad \psi, \Pi \Rightarrow \Lambda}{\phi \to \psi, \Gamma, \Pi \Rightarrow \Lambda}$$

(→ right)

$$\frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi}$$

(¬ left)

$$\frac{\Gamma \Rightarrow \phi}{\neg\phi, \Gamma \Rightarrow}$$

(¬ right)

$$\frac{\phi, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\phi}$$

A.1.6 Intuitionistic linear logic : ILL

Formulas

$$\Phi_{i\ell} = \Phi^L(\Psi). \tag{1.1.19}$$

Axioms

$$\phi \Rightarrow \phi \quad (\phi \in \Phi_\ell) \tag{1.1.20}$$

$$\Gamma, \mathbf{0} \Rightarrow \Delta \tag{1.1.21}$$

$$\Gamma \Rightarrow \top \tag{1.1.22}$$

$$\Rightarrow \mathbf{1} \tag{1.1.23}$$

Inference rules

As well as in intuitionistic calculus, in intuitionistic linear calculus, any sequent is assumed to be of the form

$$\Gamma \Rightarrow \Delta \tag{1.1.24}$$

with Δ is at most one formula.

Structural rules

(exchange)

$$\frac{\Gamma, \phi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \phi, \Pi \Rightarrow \Delta}$$

(cut)

$$\frac{\Gamma \Rightarrow \phi \qquad \phi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Lambda}$$

Logical rules

($\otimes$ left 1)

$$\frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \otimes \psi, \Gamma \Rightarrow \Delta}$$

($\otimes$ right)

$$\frac{\Gamma \Rightarrow \phi \qquad \Pi \Rightarrow \psi}{\Gamma, \Pi \Rightarrow \phi \otimes \psi}$$

(& left 1)

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\phi \,\&\, \psi, \Gamma \Rightarrow \Delta}$$

(& left 2)

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\phi \,\&\, \psi, \Gamma \Rightarrow \Delta}$$

(& right)

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \,\&\, \psi}$$

($\oplus$ left)

$$\frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \oplus \psi, \Gamma \Rightarrow \Delta}$$

($\oplus$ right 1)

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \oplus \psi}$$

($\oplus$ right 2)

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \oplus \psi}$$

($\multimap$ left)

$$\frac{\Gamma \Rightarrow \phi \qquad \psi, \Pi \Rightarrow \Lambda}{\phi \multimap \psi, \Gamma, \Pi \Rightarrow \Lambda}$$

($\multimap$ right)

$$\frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \multimap \psi}$$

(**1** left)

$$\frac{\Gamma \Rightarrow \Delta}{\mathbf{1}, \Gamma \Rightarrow \Delta}$$

A.1.7 Linear logic : LL

Formulas

$$\Phi_\ell = \Phi^L(\Psi, \bot, {}^\bot, \invamp). \tag{1.1.25}$$

Axioms

$$\phi \Rightarrow \phi \quad (\phi \in \Phi_\ell) \tag{1.1.26}$$

$$\Rightarrow \mathbf{1} \tag{1.1.27}$$

$$\Gamma, \mathbf{0} \Rightarrow \Delta \tag{1.1.28}$$

$$\Gamma \Rightarrow \top, \Delta \tag{1.1.29}$$

$$\bot \Rightarrow \tag{1.1.30}$$

Inference rules

Structural rules

(exchange left)

$$\frac{\Gamma, \phi, \psi, \Pi \Rightarrow \Delta}{\Gamma, \psi, \phi, \Pi \Rightarrow \Delta}$$

(exchange right)

$$\frac{\Gamma \Rightarrow \Delta, \phi, \psi, \Lambda}{\Gamma \Rightarrow \Delta, \psi, \phi, \Lambda}$$

(cut)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \phi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Logical rules

($\otimes$ left 1)

$$\frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \otimes \psi, \Gamma \Rightarrow \Delta}$$

($\otimes$ right)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \Pi \Rightarrow \Lambda, \psi}{\Gamma, \Pi \Rightarrow \Delta, \Lambda, \phi \otimes \psi}$$

(& left 1)

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\phi \,\&\, \psi, \Gamma \Rightarrow \Delta}$$

(& left 2)

$$\frac{\psi, \Gamma \Rightarrow \Delta}{\phi \,\&\, \psi, \Gamma \Rightarrow \Delta}$$

(& right)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \,\&\, \psi}$$

($\parr$ left)

$$\frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Pi \Rightarrow \Lambda}{\phi \parr \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

($\parr$ right)

$$\frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \parr \psi}$$

($\oplus$ left)

$$\frac{\phi, \Gamma \Rightarrow \Delta \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \oplus \psi, \Gamma \Rightarrow \Delta}$$

($\oplus$ right 1)

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \phi \oplus \psi}$$

($\oplus$ right 2)

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \oplus \psi}$$

($\multimap$ left)

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Pi \Rightarrow \Lambda}{\phi \multimap \psi, \Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

($\multimap$ right)

$$\frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \multimap \psi}$$

($^\perp$ left)

$$\frac{\Gamma \Rightarrow \Delta, \phi}{\phi^\perp, \Gamma \Rightarrow \Delta}$$

($^\perp$ right)

$$\frac{\phi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi^\perp}$$

($\mathbf{1}$ left)

$$\frac{\Gamma \Rightarrow \Delta}{\mathbf{1}, \Gamma \Rightarrow \Delta}$$

($\perp$ right)

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \perp}$$

A.2 Possible world semantics of n-modal logic

Let $\Phi_{n,m} = \Phi(\Psi, \Box_1, \Diamond_1, \ldots, \Box_n, \Diamond_n)$ be the set of n-modal formulas and $\mathscr{M} = (\mathscr{F}, V)$ be a standard model based on an n-frame $\mathscr{F} = (\Omega, R_1, \ldots, R_n)$. The truth set of a n-modal formula $\varphi \in \Phi_{n,m}$ in the standard model $\mathscr{M}$ is defined as follows:

(1) $V(p_i)$ for $p_i \in \Psi$.

(2) $V(\top) = \Omega$.

(3) $V(\perp) = \emptyset$.

(4) $V(\neg\varphi) = V(\varphi)^c$.

(5) $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$.

(6) $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$.

(7) $V(\varphi \to \psi) = V(\varphi)^c \cup V(\psi)$.

(8) $V(\varphi \leftrightarrow \psi) = (V(\varphi)^c \cup V(\psi)) \cap (V(\varphi) \cup V(\psi)^c)$.

(9) $V(\Box_j \varphi) = \{\omega \in \Omega \mid \forall \varpi \in \Omega\, (\varpi \in R_j(\omega) \Rightarrow \varpi \in V(\varphi))\}\ (j = 1, \ldots, n)$.

(10) $V(\Diamond_j \varphi) = \{\omega \in \Omega \mid \exists \varpi \in \Omega\, (\varpi \in R_j(\omega) \text{ and } \varpi \in V(\varphi))\}\ (j = 1, \ldots, n)$.

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