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Intelligent Systems for Information Processing:
From Representation to Applications

On the Representation of Fuzzy Spatial Relations in Robot Maps

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Abstract

Spatial directional relations, like "north of," play an important role in the modeling of the environment by an autonomous robot. We propose an approach to represent spatial relations grounded in fuzzy set theory and fuzzy mathematical morphology. We show how this approach can be applied to robot maps, and suggest that these relations can be used for self-localization and for reasoning about the environment. We illustrate our approach on real data collected by a mobile robot in an office environment.

Keywords: autonomous robots, occupancy grids, topological maps, fuzzy spatial relations, fuzzy mathematical morphology.

1 Introduction

Autonomous robots need the ability to perceive their environment, build a model of it, and use this model to effectively navigate and operate in that environment. One important aspect of these models is the ability to incorporate spatial directional relations, like "north of." These relations are inherently vague, since they depend on how much of an object is in the specified direction with respect to the reference object.

Relative directional relations have not been extensively studied in the mobile robotics literature. The field of image processing contains a comparatively larger body of work on spatial relations, although directional positions have received much less attention in that field than topological relations like set relationships, part-whole relationships, and adjacency. Most non-fuzzy approaches use a set of basic relations based on Allen's interval relations [1] (e.g., [22]) or on simplifications of objects (e.g. [10]). Some approaches use intervals to represent qualitative expressions about angular positions [15]. Stochastic approaches have also been proposed for representing spatial uncertainty in robotics, e.g., [24]. Most of the above approaches, however, suffer from a somehow simplified treatment of the uncertainty and vagueness which is intrinsic in spatial relations. Concepts related to directional relative position are rather ambiguous, and defy precise definitions. However, humans have a rather intuitive and common way of understanding and interpreting them. From our everyday experience, it is clear that any "all-or-nothing" definition of these concepts leads to unsatisfactory results in several situations of even moderate complexity. Fuzzy set theory appears then as an appropriate tool for such modeling since it allows to integrate both quantitative and qualitative knowledge, using semiquantitative interpretation of fuzzy sets. As noted by Freeman in [9], this allows us to provide a computational representation and interpretation of imprecise spatial relations, expressed in a linguistic way, possibly including quantitative knowledge.

In this paper, we show how fuzzy mathematical morphology can be used to compute approximate spatial relations between objects in a robot map. The key step is to represent the space in the robot's environment by an occupancy grid [7, 20], and to treat this grid as a grey-scale image. This allows us to apply techniques from the field of image processing to extract spatial information from this grid. In particular, we are interested in the spatial relations between rooms and corridors in the environment.

In the rest of this paper, we briefly introduce fuzzy mathematical morphology and we show how it can be used to define fuzzy spatial relations. We then discuss the use of this approach in the context of one particular type of robot maps, called *topology-based maps*, which are built from occupancy grids [8]. We illustrate our approach on real data collected by a mobile robot in an office environment. Finally, we discuss a few possible applications of fuzzy spatial relations to robot navigation.

2 Fuzzy mathematical morphology

Mathematical morphology is originally based on set theory. Introduced in 1964 by Matheron [16] to study porous media, mathematical morphology has rapidly evolved into a general theory of shape and its transformations, and it has found wide applications in image processing and pattern recognition [21].

The four basic operations of mathematical morphology are dilation, erosion, opening and closing. The *dilation* of a set X of an Euclidean space $\mathcal{S}$ (typically $\mathbb{R}^n$ or $\mathbb{Z}^n$) by a set B is defined by [21]:

$$D_B(X) = \{x \in \mathcal{S} \mid B_x \cap X \neq \emptyset\}, \tag{1}$$

where B_x denotes the translation of B at x. Similarly the *erosion* of X by B is defined by:

$$E_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\}. \tag{2}$$

The set B, called *structuring element*, defines the neighborhood that is considered at each point. It controls the spatial extension of the operations: the result at a point only depends on the neighborhood of this point defined by B.

From these two operators, *opening* and *closing* are defined respectively by: $O(X) = D_{\check{B}}[E_B(X)]$, and $C(X) = E_{\check{B}}[D_B(X)]$, where $\check{B}$ denotes the symmetrical of B with respect to the origin of the space.

The above operators satisfy a number of algebraic properties [21]. Among the most important ones are commutativity of dilation (respectively erosion) with union or sup (respectively intersection or inf), increasingness[1] of all operators, iteration properties of dilation and erosion, idempotency of opening and closing, extensivity[2] of dilation (if the origin belongs to the structuring element) and of closing, anti-extensivity of erosion (if the origin belongs to the structuring element) and of opening.

Mathematical morphology has been extended in many ways. In the following, we make use of *fuzzy morphology*, where operations are defined on fuzzy sets (representing spatial entities along with their imprecision) with respect to fuzzy structuring elements. Several definitions of fuzzy mathematical morphology have been proposed (e.g. [3, 5, 23]). Here, we define dilation and erosion of a fuzzy set μ by a structuring element ν for all $x \in \mathcal{S}$ by, respectively:

$$\begin{aligned} D_\nu(\mu)(x) &= \sup_y\{t[\nu(y-x), \mu(y)]\}, \\ E_\nu(\mu)(x) &= \inf_y\{T[c(\nu(y-x)), \mu(y)]\} \end{aligned}$$

where y ranges over the Euclidean space $\mathcal{S}$ where the objects are defined, t is a t-norm, and T its associated t-conorm with respect to the complementation c [27]. In these equations, fuzzy sets are assimilated to their membership functions. These definitions extend classical morphology in a natural way, providing similar properties as in the crisp case [3, 19].

Through the notion of structuring element, mathematical morphology can deal with local or regional spatial context. It also has some features that allow us to include more global information, which is particularly important when the spatial arrangement of objects in a scene has to be assessed. This fact is exploited in the following.

3 Spatial relations from fuzzy mathematical morphology

Spatial relationships between the objects in the environment carry structural information about the environment, and provide important information for object recognition and for self localization [11]. Fuzzy mathematical morphology can be used here to represent and compute in a uniform setting several types of relative position information, like distance, adjacency and directional relative position. In this section, we explain how we can use it to deal with directional relations.

A few works propose fuzzy approaches for assessing the directional relative position between objects, which is an intrinsically vague relation [2, 12, 13, 17, 18]. The approach used here and described in more details in [2] relies on a fuzzy dilation that provides a map (or fuzzy landscape) where the membership value of each point represents the degree of the satisfaction of the relation to the reference object. This approach has interesting features: it works directly in the image space, without reducing the objects to points or histograms, and it takes the object shape into account.

We consider a (possibly fuzzy) object R in the space $\mathcal{S}$, and denote by $\mu_\alpha(R)$ the fuzzy subset of $\mathcal{S}$ such that points of areas which satisfy to a high degree the relation "to be in the direction $\vec{u}_\alpha$ with respect to object R" have high membership values, where $\vec{u}_\alpha$ is a vector making an angle α with respect to a reference axis.

[1] An operation ψ is increasing if $\forall X, Y\ X \subseteq Y \Rightarrow \psi(X) \subseteq \psi(Y)$.

[2] An operation ψ is extensive if $\forall X,\ X \subseteq \psi(X)$ and anti-extensive if $\forall X, \psi(X) \subseteq X$.

The form of $\mu_\alpha(R)$ may depend on the application domain. Here, we use the definition proposed in [2], which considers those parts of the space that are visible from a reference object point in the direction $\vec{u}_\alpha$. This can be expressed formally as the fuzzy dilation of μ_R by ν, where ν is a fuzzy structuring element depending on α: $\mu_\alpha(R) = D_\nu(\mu_R)$ where μ_R is the membership function of the reference object R. This definition applies both to crisp and fuzzy objects and behaves well even in case of objects with highly concave shape [2]. In polar coordinates, ν is defined by: $\nu(\rho,\theta) = f(\theta-\alpha)$ and $\nu(0,\theta) = 1$, where $\theta - \alpha$ is defined modulo π and f is a decreasing function. In the experiments reported here, we have used $f(x) = \max[0, \cos x]^2$ for $x \in [0,\pi]$ — see Figure 1. Techniques for reducing the computation cost have been proposed in [2].

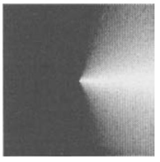


Figure 1: Structuring element ν for $\alpha = 0$ (high grey values correspond to high membership values).

Once we have defined $\mu_\alpha(R)$, we can use it to define the degree to which a given object A is in direction $\vec{u}_\alpha$ with respect to R. Let us denote by μ_A the membership function of the object A. The evaluation of relative position of A with respect to R is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all x in $\mathcal{S}$. The histogram of $\mu_\alpha(R)$ conditionally to μ_A is such a function. If A is a binary object, then the histogram of $\mu_\alpha(R)$ in A is given by:

$$h(z) = \mathrm{Card}\left(\{x \in A \mid \mu_\alpha(R)(x) = z\}\right),$$

where $z \in [0,1]$. This extends to the fuzzy case by:

$$h(z) = \sum_{x\,:\,\mu_\alpha(R)(x)=z} \mu_A(x).$$

While this histogram gives the most complete information about the relative spatial position of two objects, it is difficult to reason in an efficient way with it. A summary of the contained information could be more useful in practice. An appropriate tool for defining this summary is the fuzzy pattern matching approach [6]. Following this approach, the matching between two possibility distributions is summarized by two numbers, a necessity degree N (a pessimistic evaluation) and a possibility degree Π (an optimistic evaluation), as often used in the fuzzy set community. In our application, they take the following forms:

$$\Pi_\alpha^R(A) = \sup_{x\in\mathcal{S}} t[\mu_\alpha(R)(x), \mu_A(x)], \tag{3}$$

$$N_\alpha^R(A) = \inf_{x\in\mathcal{S}} T[\mu_\alpha(R)(x), 1-\mu_A(x)], \tag{4}$$

where t is a t-norm and T a t-conorm. The possibility corresponds to a degree of intersection between the fuzzy sets A and $\mu_\alpha(R)$, while the necessity corresponds to a degree of

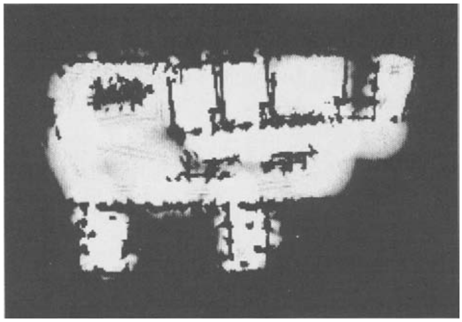


Figure 2: The occupancy grid built by the robot from sensor data in a test environment.

inclusion of A in $\mu_\alpha(R)$. These operations can also be interpreted in terms of fuzzy mathematical morphology, since $\Pi_\alpha^R(A)$ is equal to the dilation of μ_A by $\mu_\alpha(R)$ at the origin of $\mathcal{S}$, while $N_\alpha^R(A)$ is equal to the erosion at the origin [3]. The set-theoretic and the morphological interpretations indicate how the shape of the objects is taken into account.

It should be emphasized that, since the aim of these definitions is not to find only the dominant relationship, an object may satisfy several different relationships, for different angles, with high degrees. Therefore, "to be to the right of R" does not mean that the object should be completely to the right of the reference object, but only that it is at least to the right of some part of it.

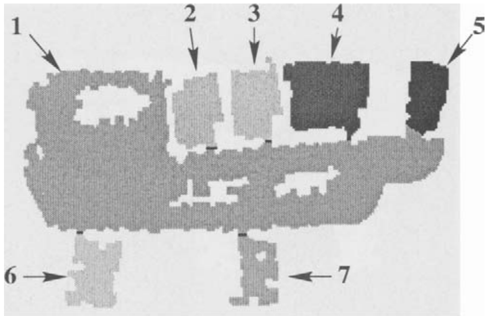
The defined directional relations are symmetrical (only for Π), invariant with respect to translation, rotation and scaling, both for crisp and for fuzzy objects, and when the distance between the objects increases, the shape of the objects plays a smaller and smaller role in the assessment of their relative position [2].

4 Robot maps

We now study how fuzzy spatial relations can be used to enrich the spatial representations used by a mobile robot, or *robot maps*. A number of different representations of space have been proposed in the literature on mobile robotics. Most of these fall into two categories: *metric maps*, which represent the environment according to the absolute geometric position of objects (or places); and *topological maps*, which represent the environment according to the relationships among objects (or places) without an absolute reference system (e.g., [14, 25]).

In this work, we consider robot maps in the form of digital grids ($\mathcal{S}$ is therefore a 2D discrete space) on which certain objects, corresponding to the sub-spaces of interest (rooms and corridors), have been isolated. The reason for this is that we can directly apply the above methods to these representations.

More precisely, we consider the particular type of maps, called *topology-based maps*, proposed by Fabrizi and Saffiotti [8]. These maps represent the environment as a graph of rooms and corridors connected by doors and passages. The authors use image processing techniques to automatically extract regions that correspond to large open spaces (rooms and corridors) from a fuzzy occupancy grid that represents the free space in the



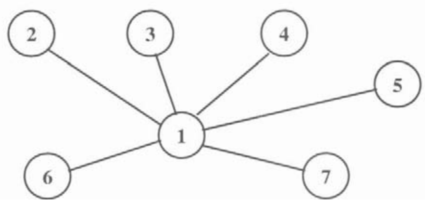


Figure 3: (top) regions extracted from the above occupancy grid; (bottom) the corresponding topology-based map.

environment. This grid is built by the robot itself using the technique described in [20].

Figure 2 shows a fuzzy occupancy grid built by a Nomad 200 robot in an office environment of 21×14 meters using sonar sensors. The environment consists of six rooms connected to a large corridor, which expands to a hall on the left hand side of the map. The dark areas in the corridor correspond to pieces of furniture. Each cell in the grid represents a square of side 10 cm, and its value, in the $[0, 1]$ interval, represents the degree of necessity of that space being empty. White cells have received sensor evidence of being empty; darker cells have not—they are either occupied or unexplored. (A dual grid, not used here, represents the occupied space.)

In order to extract the desired rooms and corridors, the authors in [8] regard this occupancy grid as a grey-scale image and process it using a technique based on fuzzy mathematical morphology. The open spaces can be extracted from the grid by performing a morphological opening by a fuzzy structuring element of a conic shape that represents the fuzzy concept of a large space. The result of the opening is then segmented by a watershed algorithm [26] in order to separate these spaces. Figure 3 (top) shows the result obtained by applying this procedure to our occupancy grid. The extracted regions correspond to the open spaces in the environment. These regions, together with the adjacency relation, constitute a topology-based map for our environment, summarized in graph form in Figure 3 (bottom). This graph provides an abstract representation that captures the structure of the space with a reduced number of parameters.

5 Adding fuzzy spatial relations to a robot map

Once we have segmented the environment into regions (rooms and corridors) we can use the technique described in the previous section to compute directional spatial relationships between these regions. These relations provide important information for object recognition and for self localization [11].

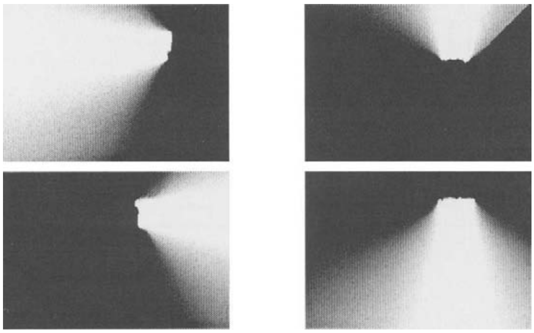


Figure 4: Fuzzy landscapes for being West, North, East and South of fuzzy region 4.

Figure 4 shows the *fuzzy landscapes* for the fuzzy notions of being, respectively, West, North, East, and South of the fuzzy region number 4 in Figure 3. These landscapes represent the $\mu_\alpha(R)$ fuzzy sets (see Section 3 above) with R being the fuzzy occupancy grid restricted to region number 4, and α taking the values 0, $\frac{1}{2}\pi$, π and $\frac{3}{2}\pi$, respectively.

We can use these landscapes to compute the relative directional position of any other region in our map with respect to region 4. For instance, Figure 5 shows the histograms of these fuzzy landscapes computed conditionally to region 1. These histograms represent the satisfaction of the relationships "region 1 is to the West (respectively, North) of region 4".

It should be noted that the direct computation of $\mu_\alpha(R)$ can be very expensive. Interestingly, the interpretation of that definition as a fuzzy dilation may suggest a few ways to reduce the computation time by reducing the precision of $\mu_\alpha(R)$: e.g., we can perform the dilation with a limited support for the structuring element, which corresponds to using a rough quantification of angles.

The above histograms can give the robot important information about the environment. In practice, however, storing and manipulating the whole histograms for each pair of regions may be prohibitive, and in real applications it is convenient to summarize the information contained in the histograms by a few parameters. A common choice is to use a pair of necessity and possibility degrees, computed according to equations (3) and (4) above.

The following table shows, for each region in our example, the degrees of necessity and possibility of being West, East, South and North of region 4. Degrees are written as a $[N, \Pi]$ interval.

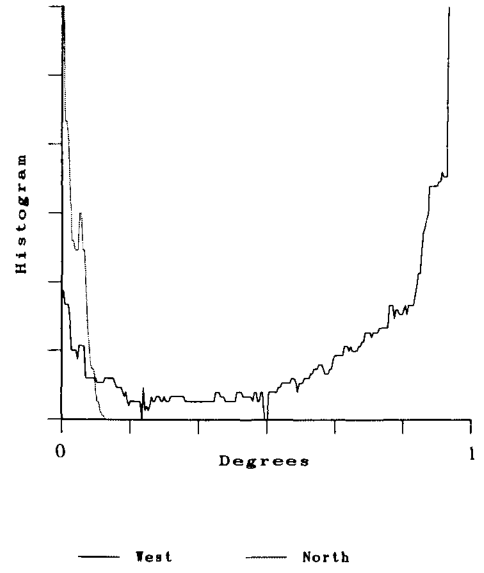


Figure 5: Histograms of the fuzzy landscapes of region 4 (west and north) conditionally to region 1.

	West	East	South	North
1	[0.0, 1.0]	[0.00, 0.99]	[0.0, 1.0]	[0.0, 0.11]
2	[0.99, 1.0]	[0.0, 0.0]	[0.0, 0.36]	[0.0, 0.24]
3	[0.92, 1.0]	[0.0, 0.0]	[0.00, 0.85]	[0.0, 0.83]
4	[0.55, 1.0]	[0.51, 1.0]	[0.50, 1.0]	[0.55, 1.0]
5	[0.0, 0.0]	[0.98, 1.0]	[0.02, 0.40]	[0.00, 0.59]
6	[0.65, 0.87]	[0.0, 0.0]	[0.30, 0.56]	[0.0, 0.0]
7	[0.17, 0.54]	[0.0, 0.0]	[0.86, 0.99]	[0.0, 0.0]

These results correspond well to intuition. For instance, regions 2 and 3 are found to be fully West of region 4, and totally not East of it; while region 5 is fully East of it and totally not West. Region 1 offers an interesting example. This region surrounds region 4 on the West and South side, and extends further East from it. Correspondingly, it has full possibility of being considered West, South and East of region 4, although no one of these relations is necessary. Its possibility of being considered North of region 4 is, however, neglectable, which is consistent with intuition. This can also be seen in the histogram, where no high degrees are obtained for the North direction, while many points satisfy the West relation to a degree close to 1. Finally, regions 6 and 7 are, at different degrees, both South and West of region 4, again conforming with intuition.

6 Discussion and conclusions

The proposed approach to represent directional relations has several interesting features. The interval representation allows us to capture the ambiguity of some relations, like in the case of the relation between region 1 and region 4 in the above example. The formal properties listed at the end of Section 3 are also of direct interest for applications in autonomous robotics. For instance, the invariance with respect to geometrical transformations is needed to guarantee that localization and recognition are independent of the frame of reference used to define directions. The fact that the shape of an object plays a smaller and smaller role as the distance of that object increases is useful when considering relationships to the robot itself: far away objects are seen by the robot as points, which is consistent with the idea that the spatial extent of these objects becomes irrelevant. The behavior of our definitions in case of concave objects agrees with intuition: an object can satisfy several relations with respect to a concave one at a high degree. In the above example, regions 2, 3, 4, 5 are all both East and North of region 1 to a high degree, which expresses that they are in the upper-right concave area of region 1. This is a way to express more complex relationships.

The computed fuzzy directional information can be used in several ways during autonomous navigation. Perhaps the most direct application is to improve the self-localization ability of the robot. The robot can perform coarse self-localization on the topological map by estimating, at every moment, the node (room) in which it is. Markov techniques can be used to update this estimate when the robot detects a transition from one node to the next: directional information can then be used to produce an expectation about the next node, by comparing the direction of travel with the distribution of possible directions associated to the outgoing links from the current node.

The ability to produce a fuzzy landscape for a given direction with respect to a node opens the possibility of additional applications. For instance, the robot can use linguistic directional information to identify important areas in the environment. As an example, we can tell the robot that the door to a given room is North with respect to the room where it currently is: the corresponding fuzzy landscape limits the area where the door should be looked for. Alternatively, we can tell the robot that the area North of a given corridor is dangerous (e.g., there is a staircase) and it should be avoided. A similar use of fuzzy logic to incorporate linguistic information in a robot map has been proposed in [11].

The proposed method to define directional information and fuzzy directional landscapes is not limited to a fixed set of directions (e.g., North, South, West, East), but can be applied to any desired angle. Also, we can tune the f function used in the definition of the structuring element ν in order to define directions which are more or less vague, depending on the application needs. The definition of fuzzy landscapes makes it easy to define complex directional relations by combining elementary relations using fuzzy operations. For instance, we can define a landscape for "North but not East" by fuzzy intersection of the landscape for North and the complement of the one for East.

Finally, it should be noted that fuzzy mathematical morphology can be used to solve several other problems in mobile robot navigation, including self-localization and spatial object processing (see [4]).

While the initial results reported in this paper show the viability of our technique, more experiments on real robotic applications are needed in order to establish the actual utility of this technique, for instance for robot self-localization or for human-robot interaction by linguistic expressions. These experiments are part of our current work.

Acknowledgments

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