PERGAMON

Pattern Recognition 33 (2000) 1487–1496

PATTERN RECOGNITION
THE JOURNAL OF THE PATTERN RECOGNITION SOCIETY

www.elsevier.com/locate/patcog

On links between mathematical morphology and rough sets

Isabelle Bloch*

Ecole Nationale Supérieure des Télécommunications, Département TSI – CNRS URA 820, 46 rue Barrault, 75013 Paris, France

Received 28 September 1998; accepted 10 May 1999

Abstract

Based on the observation that rough sets and mathematical morphology are both using dual operators sharing similar properties, we investigate more closely the links existing between both the domains. We establish the equivalence between some morphological operators and rough sets defined from either a relation, or a pair of dual operators or a neighborhood system. Then we suggest some extensions using morphological thinning and thickening, and using algebraic operators. We propose to define rough functions and fuzzy rough sets using mathematical morphology on functions and fuzzy mathematical morphology. © 2000 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

Keywords: Mathematical morphology; Rough sets; Approximation spaces; Topology; Similarity relations; Fuzzy sets; Fuzzy mathematical morphology; Rough functions; Fuzzy rough sets

1. Introduction

Rough set theory has been introduced in 1982 [1], as an extension of set theory, mainly in the domain of intelligent systems. The objective was to deal with incomplete information, leading to the idea of indistinguishability of objects in a set. It is therefore related to the concept of approximation, and of granularity of information (in the sense of Zadeh [2]). This theory was applied successfully in several applications, e.g. information analysis, data analysis and data mining, knowledge discovery (for instance, discovery of which features are relevant for data description), i.e. all those applications in which a need arises for intelligent decision support.

Mathematical morphology is originally also based on set theory. It has been introduced in 1964 by Matheron [3,4], in order to study porous media. But this theory evolved rapidly to a general theory of shape and its transformations, and was applied in particular to image processing and pattern recognition [5]. In addition to its set theoretical foundations, it relies on topology on sets, on random sets, on topological algebra, on integral geometry, on lattice theory. The basic idea in mathematical morphology is to study shapes by transforming them using some interaction with a set called structuring element and which is chosen by the user (the observer).

Rough set theory [1] is an extension of set theory for dealing with coarse information. In this framework, a set X is approximated by two sets, called upper and lower approximations, and denoted by $\bar{A}(X)$ and $\underline{A}(X)$, such that $\underline{A}(X) \subset X \subset \bar{A}(X)$. On the other hand, mathematical morphology [5,6] provides operators that are either extensive or anti-extensive, such as dilation D_B and erosion E_B (if the origin of the space belongs to the chosen structuring element B), or closing C_B and opening O_B. We have: $E_B(X) \subset X \subset D_B(X)$ and $O_B(X) \subset X \subset C_B(X)$, i.e. similar relations as the one for rough sets. One of the basic properties of upper and lower set approximations is duality. A similar property holds for mathematical morphology between dilation and erosion, and between opening and closing. In fact, most of the morphological operators go by pairs of dual operators.

Based on these elementary observations, it is tempting to look at closer links between both domains. To our knowledge, the only work that puts together both domains is the one of Polkowski [7], where a hit-or-miss topology is defined on rough sets, similar to what is used

* Tel.: + 33-1-45-81-75-85; fax: + 33-1-45-81-37-94.
E-mail address: isabelle.bloch@enst.fr (I. Bloch).

PII: S0031-3203(99)00129-6

in mathematical morphology. Here we take another point of view and try to link lower and upper approximations directly to morphological operators. To our knowledge, it is the first time that such links are established,

We first recall in Section 2 the basic definitions of rough sets, in particular those based on a similarity relation, and of mathematical morphology, in particular its four basic operators. Then we compare both theories in light of a list of properties that are commonly used in rough set theory (Section 3). Then we establish in Section 4 formal links between upper and lower approximations on the one hand, and dilation and erosion (respectively opening and closing) on the other hand. In Section 5 we take a closer look on some topological aspects. Then we propose in Section 6 some extensions of these links, using other operators like thinning and thickening, or algebraic operators. We also extend this work to functions and to fuzzy sets, and show how mathematical morphology on functions and on fuzzy sets can be used for defining rough functions and rough fuzzy sets. This brings together three different aspects of the information: vagueness (through fuzzy sets), coarseness (through rough sets) and shape (through mathematical morphology). Finally, we conclude with some insights on the respective contributions of each domain to the other, those that can be anticipated from this work.

2. Definitions of rough sets and basic morphological operators

2.1. Rough sets from relations

In rough set theory [1], the two sets $\bar{A}(X)$ and $\underline{A}(X)$ such that $\underline{A}(X) \subset X \subset \bar{A}(X)$ are defined from an equivalence relation. Let $\mathscr{U}$ denote the universe of discourse, X being a subset of $\mathscr{U}$. Each element of $\mathscr{U}$ is known through its attributes a. The set of attributes A is a set of functions defined on $\mathscr{U}$. Let $Inf(x)$ be the information vector of x:

$$Inf(x) = \{a(x) | a \in A\}. \quad (1)$$

An equivalence relation R_A is defined with respect to the set of attributes on $\mathscr{U}$ as

$$xR_Ay \Leftrightarrow Inf(x) = Inf(y). \quad (2)$$

This relation characterizes the elements that are indistinguishable from each other based on the available information. The pair $(\mathscr{U}, R_A)$ is called an approximation space. Let $[x]_A$ denote the class of x. Then lower and upper approximations of a subset X of $\mathscr{U}$ are defined as

$$\underline{A}(X) = \{x \in \mathscr{U} | [x]_A \subset X\}, \quad (3)$$

$$\bar{A}(X) = \{x \in \mathscr{U} | [x]_A \cap X \neq \emptyset\}. \quad (4)$$

A rough set is the pair $(\underline{A}(X), \bar{A}(X))$. Obviously, we have

$$\underline{A}(X) \subset X \subset \bar{A}(X). \quad (5)$$

The lower approximation of X contains the elements x such that all the elements that are indistinguishable from x (according to the considered attributes) are in X. The upper approximation of X contains the elements x such that at least one element that is indistinguishable from x belongs to X.

This definition can be extended to any relation R, leading to the notion of generalized approximate space (see e.g. Ref. [8]). Let $r(x)$ be the set defined as

$$r(x) = \{y \in \mathscr{U} | xRy\}. \quad (6)$$

The lower and upper approximations of X according to R are then defined as

$$\underline{R}(X) = \{x \in \mathscr{U} | r(x) \subset X\}, \quad (7)$$

$$\bar{R}(X) = \{x \in \mathscr{U} | r(x) \cap X \neq \emptyset\}. \quad (8)$$

Conversely $r(x)$ can be obtained from the upper approximation of X as

$$r(x) = \{y \in \mathscr{U} | x \in \bar{R}(\{y\})\}. \quad (9)$$

Obviously, if R is an equivalence relation, $r(x) = [x]_R$ and these definitions are equivalent to the original Pawlak's definitions. If R is a tolerance relation (i.e. reflexive and symmetrical), these equations define tolerance rough sets. The properties of $\bar{R}(X)$ and $\underline{R}(X)$ depend on the properties of R, as will be seen in Section 3.

2.2. Mathematical morphology: basic operators

Mathematical morphology is basically a set theory [5], that has extensions to functions [5], vectors, fuzzy sets [9]. We just recall here the definitions of the four basic operations for sets. Let X be a set of $\mathscr{U}$, and B a set called structuring element. The morphological dilation of X by B is defined as

$$D_B(X) = \{x \in \mathscr{U} | B_x \cap X \neq \emptyset\}, \quad (10)$$

where B_x denotes the translation of the structuring element at point x. The morphological erosion of X by B is defined as

$$E_B(X) = \{x \in \mathscr{U} | B_x \subset X\}. \quad (11)$$

Morphological opening and closing are defined, respectively, by

$$O_B(X) = D_{\check{B}}[E_B(X)], \quad (12)$$

$$C_B(X) = E_{\check{B}}[D_B(X)], \quad (13)$$

where $\check{B}$ denotes the symmetrical of B with respect to the origin of the space. Opening and closing can be rewritten as

$$O_B(X) = \{x \in \mathcal{U} | \exists y \in \mathcal{U} | x \in B_y \text{ and } B_y \subset X\}, \tag{14}$$

$$C_B(X) = \{x \in \mathcal{U} | \forall y \in \mathcal{U},\ x \in B_y \Rightarrow B_y \cap X \neq \emptyset\}. \tag{15}$$

For any structuring element that contains the origin of the space, the following property holds:

$$E_B(X) \subset X \subset D_B(X). \tag{16}$$

For any structuring element (without restriction), the following property holds:

$$O_B(X) \subset X \subset C_B(X). \tag{17}$$

In the following, we assume that the origin of $\mathcal{U}$ belongs to the structuring element B.

2.3. First conclusion

A first conclusion that can be drawn from these definitions is the similarity between the operators involved in both the domains. Lower approximations involve subsethood, as do erosion and opening, while upper approximations involve set intersection, as do dilation and closing.

Moreover, the inclusion properties are similar.

These remarks lead to a first parallelism between lower approximation and erosion or opening on the one hand, and between upper approximation and dilation or closing on the other hand. Further similarities call for a closer look at the properties satisfied by the operators in both the domains.

3. Comparison of basic properties

3.1. A list of properties of interest

In this section, we list the properties that are of interest in the theory of rough sets. We follow here the presentation provided in Ref. [8]. The satisfaction of these properties depending on the chosen definition is detailed in the next subsection.

- L1. $\underline{R}(X) = [\bar{R}(X^C)]^C$, where X^C denotes the complementation of X in $\mathcal{U}$.
- L2. $\underline{R}(\mathcal{U}) = U$.
- L3. $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$.
- L4. $\underline{R}(X \cup Y) \supset \underline{R}(X) \cup \underline{R}(Y)$.
- L5. $X \subset Y \Rightarrow \underline{R}(X) \subset \underline{R}(Y)$.
- L6. $\underline{R}(\emptyset) = \emptyset$.
- L7. $\underline{R}(X) \subset X$.
- L8. $X \subset \underline{R}(\bar{R}(X))$.
- L9. $\underline{R}(X) \subset \underline{R}(\underline{R}(X))$.
- L10. $\bar{R}(X) \subset \underline{R}(\bar{R}(X))$.
- U1. $\bar{R}(X) = [\underline{R}(X^C)]^C$.
- U2. $\bar{R}(\emptyset) = \emptyset$.
- U3. $\bar{R}(X \cup Y) = \bar{R}(X) \cup \bar{R}(Y)$.
- U4. $\bar{R}(X \cap Y) \subset \bar{R}(X) \cap \bar{R}(Y)$.
- U5. $X \subset Y \Rightarrow \bar{R}(X) \subset \bar{R}(Y)$.
- U6. $\bar{R}(\mathcal{U}) = \mathcal{U}$.
- U7. $X \subset \bar{R}(X)$.
- U8. $\bar{R}(\underline{R}(X)) \subset X$.
- U9. $\bar{R}(\bar{R}(X)) \subset \bar{R}(X)$.
- U10. $\bar{R}(\underline{R}(X)) \subset \underline{R}(X)$.
- K. $\underline{R}(X^C \cup Y) \subset \underline{R}(X)^C \cup \underline{R}(Y)$.
- LU. $\underline{R}(X) \subset \bar{R}(X)$.

Properties L1 and U1 express the duality between lower and upper approximations. These properties allow to derive relations U1–U10 from relations L1–L10. Properties L2, L6, U2 and U6 express limit conditions for the empty set and the whole space. Compatibility with union and intersection is expressed by L3, L4, U3 and U4. Properties L5 and U5 express the increasingness with respect to set inclusion. The basic notions of lower and upper approximations can be found in properties L7, U7 and LU. Properties L8–L10 and U8–U10 concern the composition of approximations. Note that if L7 and L9 are simultaneously satisfied, we have, due to L5

$$\underline{R}(X) = \underline{R}(\underline{R}(X)),$$

i.e. lower approximation is idempotent. In the same way, if U7 and U9 are simultaneously satisfied, then upper approximation is idempotent. If L7 and L10 are simultaneously satisfied, then we have

$$\underline{R}(X) = \underline{R}(\bar{R}(X))$$

and a similar expression for the upper approximation.

3.2. Which properties do the rough sets and mathematical morphology have in common?

In Table 1 we compare the properties that are satisfied by the different definitions of rough sets with those satisfied by the four basic morphological operators.

3.3. Second conclusion

From the results in Table 1, it appears clearly that lower approximations share many properties with erosion and with opening, while upper approximations share many properties with dilation and closing. These algebraic properties make rough set algebra similar to mathematical morphology algebra.

Having made these observations, we can now establish formal links between set approximations and morphological operators.

Table 1
Comparison between the properties of rough sets depending on the properties of R with those of mathematical morphology operators. A cross ($\times$) indicates that the property is satisfied. The first column contains the list of properties, according to the notations given in Section 3.1. The next four columns are for rough sets, defined from any relation, a tolerance relation, a relation that is reflexive and transitive, and an equivalence relation, respectively. The two last columns are for morphological operators, erosion and opening in the upper part (corresponding to the properties of lower approximation) and dilation and closing in the lower part (corresponding to the properties of upper approximation)

Property	Any R	Tolerance rel.	R reflex. and trans.	Equivalence rel.	Erosion/Dilation	Opening/Closing
L1	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
L2	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
L3	$\times$	$\times$	$\times$	$\times$	$\times$	only $\subset$
L4	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
L5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
L6		$\times$	$\times$	$\times$	$\times$	$\times$
L7		$\times$	$\times$	$\times$	$\times$	$\times$
L8		$\times$		$\times$	$\times$	
L9			$\times$	$\times$		$\times$
L10				$\times$		
U1	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
U2	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
U3	$\times$	$\times$	$\times$	$\times$	$\times$	only $\subset$
U4	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
U5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
U6		$\times$	$\times$	$\times$	$\times$	$\times$
U7		$\times$	$\times$	$\times$	$\times$	$\times$
U8		$\times$		$\times$	$\times$	
U9			$\times$	$\times$		$\times$
U10				$\times$		
K	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
LU		$\times$	$\times$	$\times$	$\times$	$\times$

4. Formal links between rough sets and mathematical morphology

Lower and upper approximations can be obtained from erosion and dilation. For a given structuring element, the corresponding relation is then as follows:

$$xRy \Leftrightarrow y \in B_x. \tag{18}$$

From R, we derive $r(x)$ as

$$\forall x \in \mathscr{U}, \quad r(x) = \{y \in \mathscr{U} | y \in B_x\} = B_x. \tag{19}$$

We always assume that the origin of $\mathscr{U}$ belongs to the structuring element B. It follows that:

$$\forall x \in \mathscr{U}, \quad x \in B_x \tag{20}$$

and therefore,

$$\forall x \in \mathscr{U}, \quad xRx, \tag{21}$$

i.e. R is reflexive. Moreover, if B is symmetrical (i.e. $B = \check{B}$), we have

$$\forall (x, y) \in \mathscr{U}^2, \quad xRy \Leftrightarrow y \in B_x, \tag{22}$$

$$\Leftrightarrow y - x \in B, \tag{23}$$

$$\Leftrightarrow x - y \in \check{B} (= B), \tag{24}$$

$$\Leftrightarrow x \in B_y, \tag{25}$$

$$\Leftrightarrow yRx \tag{26}$$

which proves that R is symmetrical. If follows that R is a tolerance relation.

Let us show that for this relation, erosion and lower approximation coincide:

$$\forall X \subset \mathscr{U}, \underline{R}(X) = \{x \in \mathscr{U} | r(x) \subset X\}, \tag{27}$$

$$= \{x \in \mathscr{U} | B_x \subset X\}, \tag{28}$$

$$= E_B(X). \tag{29}$$

In a similar way, dilation and upper approximation coincide, since we have

$$\forall X \subset \mathscr{U}, \bar{R}(X) = \{x \in \mathscr{U} | r(x) \cap X \neq \emptyset\} \tag{30}$$

$$= \{x \in \mathscr{U} | B_x \cap X \neq \emptyset\} \tag{31}$$

$$= D_B(X). \tag{32}$$

These results are comfirmed by the properties shown in Table 1, that are the same for lower and upper approximations derived from a tolerance relation, and for erosion and dilation.

These equivalences are in accordance with the operator-oriented view of rough sets [10,8]. Let L and H be two dual operators, such that

$$\forall X \subset \mathscr{U}, L(X) = H(X^C)^C \tag{33}$$

and satisfy the following properties:

(1) $H(\emptyset) = \emptyset$,
(2) H commutes with union: $\forall X \subset \mathscr{U}$, $\forall Y \subset \mathscr{U}$, $H(X \cup Y) = H(X) \cup H(Y)$.

Then there exists a relation R such that $L(X) = \underline{R}(X)$ and $H(X) = \bar{R}(X)$. This relation is defined by

$$xRy \Leftrightarrow x \in H(\{y\}). \tag{34}$$

The results proved in this section provide concrete examples of operators L and H, which are morphological erosions and dilations. Actually, a family of operators is obtained, indexed by the structuring element. The derived relation R is exactly the one introduced in Eq. (18), since we have for a symmetrical structuring element B:

$$xRy \Leftrightarrow x \in H(\{y\}) \Leftrightarrow x \in D_B(\{y\}) \Leftrightarrow x \in B_y \Leftrightarrow y \in B_x. \tag{35}$$

Let us now consider opening and closing. Using the operator-oriented point of view, they can be used, respectively, as lower and upper approximations, since they have most of the required properties as shown in Table 1. However, since closing does not commute in general with union (only a inclusion holds for property U3), the direct derivation of R as in Eq. (34) cannot be applied. Even if it is not as obvious as for dilation and erosion to find an expression for opening and closing based on a relation, these operators have interesting properties that deserve to consider them as good operators for constructing rough sets. In particular, they are idempotent, which is particularly useful if we take the topology-oriented point of view. This is the scope of the next section.

5. Topological aspects

Important notions in topology are interior and closure operators. More local information is the notion of neighborhood. These two aspects will be dealt within the two parts of this section.

5.1. Topology and pre-topology

The idea is that lower and upper approximations can be interpreted as interior and closure. Morphological operators also receive similar interpretations.

Let us consider again two dual operators L and H, but satisfying some more axioms:

(1) $H(\emptyset) = \emptyset$,
(2) H commutes with union: $\forall X \subset \mathscr{U}$, $\forall Y \subset \mathscr{U}$, $H(X \cup Y) = H(X) \cup H(Y)$.
(3) H is extensive: $\forall X \subset \mathscr{U}$, $X \subset H(X)$.
(4) H is idempotent: $\forall X \subset \mathscr{U}$, $H(H(X) = H(X)$.
(5) $\forall X \subset \mathscr{U}$, $X \subset L(H(X))$.

If properties (1)–(4) are satisfied, then the relation R that is derived from H using Eq. (34) is reflexive and transitive, and this defines a topological approximation space. Indeed, properties (1)–(4) are the properties of a common closure operator.

Except for property (2) where we generally have only an inclusion, properties (1)–(4) are also satisfied by closing (and the dual operator opening). Therefore, these morphological operators define a topological approximation space.

If properties (1)–(5) are satisfied, then R is an equivalence relation. Property (5) is in general not satisfied by opening and closing. If the set of considered objects is restricted to the objects that are opened by B (i.e. they do not contain details smaller than B), then property (5) holds. However, in such a case, the lower approximation does not modify the set.

Let us now consider erosion and dilation. They do not satisfy property (4), but satisfy all others. The loss of idempotence for the closure operator corresponds to a pre-topology [11]. Therefore, using erosion and dilation introduces the notion of pretopological approximation space. This may be of interest for pattern recognition purposes, since non-idempotent closure allows to aggregate patterns using iterated closure operations.

The basic topology on sets in mathematical morphology is the hit-or-miss topology, that is based on the intersection of a closed set with some open sets (the "hit" part) and on the non-intersection of a closed set with some compact sets (the "miss" part) [5]. It appears that the relations that define this topology are the same as the ones defining lower and upper approximations. This leads as given in Ref. [7], to a construction of hit-or-miss topology on rough sets.

5.2. Neighborhood systems

Let us now consider a topology defined through a neighborhood system. Let $n(x)$ be a neighborhood of x and $N(x)$ be a neighborhood system for x. Lower and upper approximations are then defined as [12]:

$$\underline{N}(X) = \{x \in \mathscr{U} | \exists n(x) \in N(X) | n(x) \subset X\}, \tag{36}$$

$$\bar{N}(X) = \{x \in \mathscr{U} | \forall n(x) \in N(X) | n(x) \cap X \neq \emptyset\}. \tag{37}$$

The definitions presented in Section 2 correspond to the case where only one neighborhood is considered, i.e. $N(x) = \{n(x)\}$.

The analogy with mathematical morphology is straightforward, if we consider that the structuring element translated at a point x of $\mathscr{U}$ is nothing but a neighborhood of x. If we set $N(x) = \{B_x\}$, we obtain

$$\underline{N}(X) = E_B(X), \quad (38)$$

$$\bar{N}(X) = D_B(X). \quad (39)$$

Moreover, if we consider a family of structuring elements $B^1, \ldots, B^k$, and if we set $N(x) = \{B^1_x, \ldots, B^k_x\}$, the B^i_x being considered as different neighborhoods of x whose union builds the neighborhood system, we obtain

$$\underline{N}(X) = \bigcup_{i=1\ldots k} E_{B^i}(X), \quad (40)$$

$$\bar{N}(X) = \bigcup_{i=1\ldots k} D_{B^i}(X). \quad (41)$$

Let us now consider opening and closing. Similar relations are obtained, by setting this time $N(x) = \{B_y | y \in \mathscr{U}$ and $x \in B_y\}$. Then we obtain

$$\underline{N}(X) = O_B(X), \quad (42)$$

$$\bar{N}(X) = C_B(X). \quad (43)$$

The proof of these results comes from the writing of an opening as

$$O_B(X) = \{x \in \mathscr{U} | \exists y \in \mathscr{U} | x \in B_y \text{ and } B_y \subset X\}. \quad (44)$$

As for erosion and dilation, we can consider a family of structuring elements for opening and closing.

This view is particularly interesting for shape recognition, since in morphological recognition, an object has often to be tested or matched with a set of patterns, like directional structuring elements. This set of patterns is interpreted as a neighborhood system.

6. Extensions

In this section, we give some hints on possible extensions of the results we obtained in this paper. These extensions concern the choice of the dual operators and the objects on which they are applied.

6.1. Thinning and thickening

Among the dual operators used in mathematical morphology, thinning and thickening are of particular interest, since they allow to perform operations depending on various local configurations. The main difference with erosion, dilation, opening and closing, is that the structuring element is not only tested against object points ($B_x \subset X, B_x \cap X \neq \emptyset$, etc.), but it is also tested against background points (i.e. points of X^C). Let us first recall the definitions of these operations (the reader may refer to Refs. [5,6] for more details). The structuring element is divided into two disjoint parts (T_1, T_2), where T_1 is tested against points of X, while T_2 is tested against points of X^C. The hit-or-miss transformation is defined as

$$HMT_{(T_1, T_2)}(X) = E_{T_1}(X) \cap E_{T_2}(X^C). \quad (45)$$

From this operation, thinning and thickening are defined as

$$Thin_{(T_1, T_2)}(X) = X - HMT_{(T_1, T_2)}(X), \quad (46)$$

$$Thick_{(T_1, T_2)}(X) = X \cup HMT_{(T_1, T_2)}(X). \quad (47)$$

Since $T_1 \cap T_2 = \emptyset$, the origin of the space belongs either to T_1 or to T_2. In the first case, the hit-or-miss transformation provides a subset of X and it is meaningful to perform thinning. In the second case, the hit-or-miss transformation provides a subset of X^C and it is meaningful to perform thickening.

The duality that holds between thinning and thickening takes the following form:

$$Thin_{(T_1, T_2)}(X) = [Thick_{(T_2, T_1)}(X^C)]^C. \quad (48)$$

Therefore, the possible pairs that can be defined from these operators as lower and upper approximations can be of the following types:

(1) $(Thin_{(T_1, T_2)}(X), Thick_{(T_2, T_1)}(X))$ if the origin of $\mathscr{U}$ belongs to T_1,
(2) $(Thin_{(T_2, T_1)}(X), Thick_{(T_1, T_2)}(X))$ if the origin of $\mathscr{U}$ belongs to T_2,
(3) $(Thin_{(T_1, T_2)}(X), X)$ if the origin of $\mathscr{U}$ belongs to T_1,
(4) $(X, Thick_{(T_1, T_2)}(X))$ if the origin of $\mathscr{U}$ belongs to T_2.

In the following, we restrict our study to the first case, where the origin of $\mathscr{U}$ belongs to T_1. The second case is similar.

Taking the operator-oriented point of view, the rough sets that can be built from thinning and thickening according to the first pair are obtained for the following operators:

$$L = Thin_{(T_1, T_2)}, H = Thick_{(T_2, T_1)}. \quad (49)$$

Since thickening generally does not commute with union (except for particular structuring elements where it is equivalent to dilation), it is not possible to derive directly a relation according to which the rough sets are defined.

Among the properties listed in Table 1, L1, L2, L6, L7, U1, U2, U6, U7 and LU are always satisfied. This shows that several properties are lost, and therefore we call "generalized rough sets" the pairs obtained from thinning and thickening. The ones obtained using erosion and dilation by a structuring element B are particular cases, corresponding to T_1 defined as the family of structuring elements containing the origin and at least another point of B, and $T_2 = B - T_1$.

Let us consider now the third and fourth possible pairs of approximations. They bring an original aspect which is a kind of assymetry between lower and upper approximations. Taking the operator-oriented point of view, we have for the third pair:

$$L(X) = Thin_{(T_1, T_2)}(X), \quad H(X) = X, \tag{50}$$

$$L'(X) = X, \quad H'(X) = Thick_{(T_2, T_1)}(X). \tag{51}$$

Here the duality is not directly between L and H, or between L' and H', but between L and H' and between L' and H, since we have

$$L(X^C) = Thin_{(T_1, T_2)}(X^C) = [Thick_{(T_2, T_1)}(X)]^C$$
$$= [H'(X)]^C, \tag{52}$$

$$L'(X^C) = X^C = [H(X)]^C. \tag{53}$$

The topological interpretation is particularly interesting when using thinning and thickening. Indeed, the pair (T_1, T_2) defines a neighborhood around a point, describing which points of the neighborhood should belong to the set X and which ones should belong to its complement. For instance, taking just the origin for T_1 and $B - T_1$ for T_2 (B being any structuring element, or neighborhood), the hit-or-miss transformation using (T_1, T_2) selects the points that are isolated in the background. The thinning by (T_1, T_2) removes such points, leading to a lower approximation of X that has no isolated points, while the thickening by (T_2, T_1) fills up isolated points of the background, leading to an upper approximation that has no holes constituted by only one point. This shows that very fine operations can be obtained using these operators.

Another interesting point is that these operations can be iterated by using families of structuring elements [5] (for instance, rotations of a generic structuring element). In this way, we can use for instance the skeleton as the lower approximation, the convex hull as the upper approximation, etc., which are useful tools in shape representation and recognition. Moreover, several thinnings and thickenings are homotopic operators, i.e. that deform shapes while preserving their homotopy. This leads to homotopic rough sets, that deserve probably a deeper study.

6.2. Algebraic rough sets using algebraic operations

Another possible extension may be derived from algebraic operators. Algebraic erosions and dilations are defined on complete lattices as operators that commute with intersection and union, respectively [5,6]. Therefore L3 and U3 are directly satisfied. Properties L2, L5, L6, U2, U5 and U6 are also satisfied by these operators. Note that morphological erosions and dilations are particular cases of algebraic operators, if translation invariance holds.

Algebraic openings and closings are defined as increasing, anti-extensive (respectively extensive) and idempotent operators [5,6]. Therefore properties L5, L7, L9 and U5, U7, U9 are automatically satisfied.

The use of algebraic erosion/dilation or opening/closing for defining lower and upper approximations lead to what we call "algebraic rough sets".

6.3. Rough functions

Since mathematical morphology also applies to functions [5,6], we can use the definitions of dilation, erosion, opening and closing on functions to define lower and upper approximations of functions. This seems to be a natural extension of rough sets.

Let f be a function defined on $\mathcal{U}$, and let B be a set of $\mathcal{U}$ (structuring element), that we consider here symmetrical, containing the origin of $\mathcal{U}$. Using erosion and dilation, we define lower and upper approximations of f as

$$\forall x \in \mathcal{U}, \underline{B}(f)(x) = E_B(f)(x) = \inf_{y \in B_x} f(y), \tag{54}$$

$$\forall x \in \mathcal{U}, \bar{B}(f)(x) = D_B(f)(x) = \sup_{y \in B_x} f(y). \tag{55}$$

Using opening and closing, we define lower and upper approximations of f as

$$\forall x \in \mathcal{U}, \underline{B}(f)(x) = O_B(f)(x) = D_B[E_B(f)](x), \tag{56}$$

$$\forall x \in \mathcal{U}, \bar{B}(f)(x) = C_B(f)(x) = E_B[D_B(f)](x). \tag{57}$$

The properties of these rough functions are direct transpositions of the ones of rough sets:

L′1. $\underline{B}(f) = -[\bar{B}(-f)]$.
L′2. $\underline{B}(f_c) = f_c$, where f_c is any constant function.
L′3. $\underline{B}(\min(f, g)) = \min[\underline{B}(f), \underline{B}(g)]$.
L′4. $\underline{B}(\max(f, g)) \geqslant \max[\underline{B}(f), \underline{B}(g)]$.
L′5. $f \leqslant g \Rightarrow \underline{B}(f) \leqslant \underline{B}(g)$.
L′6. $\underline{B}(f_0) = f_0$, where f_0 is identically zero.
L′7. $\underline{B}(f) \leqslant f$.
L′8. $f \leqslant \underline{B}(\bar{B}(f))$.
L′9. $\underline{B}(f) \leqslant \underline{B}(\underline{B}(f))$.
L′10. $\bar{B}(f) \leqslant \underline{B}(\bar{B}(f))$.
U′1. $\bar{B}(f) = -[\underline{B}(-f)]$.
U′2. $\bar{B}(f_0) = f_0$.
U′3. $\bar{B}(\max(f, g)) = \max[\bar{B}(f), \bar{B}(g)]$.
U′4. $\bar{B}(\min(f, g)) \leqslant \min[\bar{B}(f), \bar{B}(g)]$.
U′5. $f \leqslant g \Rightarrow \bar{B}(f) \leqslant \bar{B}(g)$.
U′6. $\bar{B}(f_c) = f_c$.
U′7. $f \leqslant \bar{B}(f)$.
U′8. $\bar{B}(\underline{B}(f)) \leqslant f$.

U′9. $\bar{B}(\bar{B}(f)) \leqslant \bar{B}(f)$.
U′10. $\bar{B}(\underline{B}(f)) \leqslant \underline{B}(f)$.
K′. $\underline{B}(\max(-f, g)) \leqslant \max[-\underline{B}(f), \underline{B}(g)]$.
LU′. $\underline{B}(f) \leqslant \bar{B}(f)$.

Using erosion and dilation, properties L′1–L′8, U′1–U′8, K′ and LU′ hold, as for the case of sets. Using opening and closing, we usually have an inequality only for L′3 and U′3, properties L′8, L′10, U′8 and U′10 being generally not satisfied, but L′9 and U′9 are always satisfied.

This construction can be further extended using *a* function g as structuring element:

$$\forall x \in \mathcal{U}, \underline{g}(f)(x) = E_g(f)(x) = \inf\{f(y) - g(y-x), y \in \mathcal{U}\}, \quad (58)$$

$$\forall x \in \mathcal{U}, \bar{g}(f)(x) = D_g(f)(x) = \sup\{f(y) + g(y-x), y \in \mathcal{U}\}. \quad (59)$$

Similar properties are obtained.

6.4. Fuzzy rough sets

In Ref. [9], we defined erosion and dilation of a fuzzy set μ by a fuzzy structuring element ν as follows:

$$\forall x \in \mathcal{U}, E_\nu(\mu)(x) = \inf_{y \in \mathcal{U}} T[c(\nu(y-x)), \mu(y)], \quad (60)$$

$$\forall x \in \mathcal{U}, D_\nu(\mu)(x) = \sup_{y \in \mathcal{U}} t[\nu(y-x), \mu(y)], \quad (61)$$

where t is a t-norm (fuzzy intersection), T a t-conorm (fuzzy union) and c a fuzzy complementation. The reader may refer to Ref. [13] for more details about fuzzy connectives.

Fuzzy opening and closing are defined for crisp sets as combinations of erosion and dilation.

Fuzzy morphological operations have the same properties as that of crisp ones, as shown in Ref. [9]. Most of the properties hold for any t-norm and t-conorm. Only idempotence and extensivity (respectively, anti-extensivity) of closing (respectively opening) are satisfied for particular t-norms and t-conorms only, as for instance Lukasiewicz operators, defined as: $t(a, b) = \max(0, a + b - 1)$ and $T(a, b) = \min(1, a + b)$.

Therefore, fuzzy rough sets defined from these morphological operators have exactly the same properties as crisp rough sets, at least for particular t-norms and t-conorms.

It turns out that these definitions using fuzzy erosion and dilation are generalizations of the ones proposed in Ref. [14], for $t = \min$ and $T = \max$ in a completely different context, using a fuzzy relation μ_R. The equivalence is obtained as in the crisp case by setting

$$\mu_R(x, y) = \nu(y - x). \quad (62)$$

The interpretation is similar to that in the crisp case: the degree of relation between x and y is equal to the degree to which $y - x$ belongs to the structuring element, i.e. to the degree to which y belongs to the structuring element translated at x.

This extension brings together three different aspects of the information: rough sets represent coarseness, fuzzy sets represent vagueness and mathematical morphology brings a geometrical, topological and morphological aspect. The conjunction of vagueness and coarseness had already been pointed out in Ref. [14]. In this paper, we bring an additional morphological point of view, by defining fuzzy rough sets using fuzzy mathematical morphology.

7. Discussion

The main results established in this paper are that morphological operators happen to be good tools for defining lower and upper approximations of sets, in the theory of rough sets, with the appropriate properties. Moreover, these operators lead to a generalization of rough sets to functions and to fuzzy sets. Several mathematical aspects are common to both theory, as set theoretical, algebraic and topological aspects. From an information point of view, several aspects are merged: coarseness, morphology, and vagueness for the extension to fuzzy sets.

In addition to the formal similarities between both domains, contributions can be brought from each domain to the other.

Mathematical morphology brings tools for analyzing shapes, the approximations it provides contain a regularization and filtering aspect (in particular using opening and closing, or their combinations). Some formulations are particular cases of general lower and upper approximations, but others may bring some generalizations (for instance in the framework of pre-topology instead of topology, or using thinning and thickening). The use of fuzzy morphology for defining fuzzy rough sets provide formulations that are more general than the ones originally proposed in Ref. [14] using fuzzy relations. Mathematical morphology operators have a lot of properties that can from now on be used also in the theory of rough sets. One example of useful property is iterativity and combination (for instance dilating a set n times by a ball of radius 1 is equivalent to dilating the set once with a ball of radius n). Such properties allow to perform successive approximations, using the same operator or different ones, in a controlled way, leading to different levels of representation, or of precision. In the same way, compatibility with geometric transformations is useful for spatial applications. Also the choice of the structuring element B has a direct impact on the approximations that can be more or less strong depending on the size of B.

Since mathematical morphology is issued from a completely different domain from the theory of rough sets, it also brings a large variety of applications, particularly in image processing. Until now, very few attempts have been done to use rough sets in image processing (see Refs. [15,16]). The results shown in this paper might provide a bridge to fill this gap.

Conversely, since the theory of rough sets has been mainly developed in the domain of artificial intelligence, it brings a new look to mathematical morphology, in particular for approximate reasoning and logics. What seems to deserve further study is for instance to build a possibilistic modal logic based on morphological operators. A modal logic based on rough sets is described e.g. in Ref. [8], where the connectives used are in a classical way negation, conjunction, disjunction, implication and equivalence, but also two modal operators, necessity $\Box$ and possibility $\Diamond$, that are defined from lower and upper approximations. The properties of Table 1 have equivalents in logical terms using these connectives, leading to reasoning rules like $\Box p \rightarrow \Diamond p$, $\Box p \rightarrow p$ and several others. If we use morphological operators for defining lower and upper approximations, we have for instance: $\Box X = E_B(X)$, $\Diamond X = D_B(X)$. A set X may correspond to a proposition like "this object is X", or "this object is in X". Then $\Box X$ and $\Diamond X$ represent, respectively, the area where it is necessary that such a proposition holds, and the area where it is just possible. These two areas represent approximations of the location of X for instance, due to imprecision, incomplete knowledge, etc. The logic derived from rough sets provides then some tools for reasoning under imprecision in a morphological context.

8. Summary

Rough set theory has been introduced in 1982, as an extension of set theory, mainly in the domain of intelligent systems. The objective was to deal with incomplete information, leading to the idea of indistinguishability of objects in a set. It is therefore related to the concept of approximation. In this framework, a set is approximated by two sets, called upper and lower approximations, that respectively contains and is included in the initial set. Mathematical morphology is originally also based on set theory. It has been introduced in 1964 by Matheron, in order to study porous media. But this theory evolved rapidly to a general theory of shape and its transformations, and was applied particularly in image processing and pattern recognition. Mathematical morphology provides operators that are extensive or anti-extensive, such as dilation and erosion (if the origin of the space belongs to the chosen structuring element), or closing and opening. One of the basic properties of upper and lower set approximations is duality. A similar property holds for mathematical morphology between dilation and erosion, and between opening and closing. In fact, most of the morphological operators go by pairs of dual operators.

Based on these elementary observations, it is tempting to look at closer links between both domains. Here we try to link lower and uper approximations directly to morphological operators. To our knowledge, this is the first time that such links are established.

We first start from the basic definitions of rough sets, in particular those based on a similarity relations, and of mathematical morphology, in particular its four basic operators. Then we compare both the theories in the light of a list of properties that are commonly used in rough set theory. Then we establish formal links between upper and lower approximations on the one hand, and dilation and erosion (respectively opening and closing) on the other hand. We then look more closely on some topological aspects (topology and pre-topology defined from a closure operator, and neighbourood systems). Then, we propose some extensions of these links, using other operators like thinning and thickening, or algebraic operators. We also extend this work to funtions and on fuzzy sets, and show how mathematical morphology on functions and on fuzzy sets can be used for defining rough functions and rough fuzzy sets. This brings together three different aspects of the information: vagueness (through fuzzy sets), coarseness (through rough sets) and shape (through mathematical morphology). Finally we provide some insights on the respective contributions of each domain to the other, those that can be anticipated from this work.

References

[1] Z. Pawlak, Rough sets, Int. J. Inform. Comput. Sci. 11 (5) (1982) 341–356.
[2] L. Zadeh, Fuzzy sets and information granularity, in: M. Gupta, R. Ragade, R. Yager (Eds.), Advances in Fuzzy Set Theory and Applications, North-Holland, Amsterdam, 1979, pp. 3–18.
[3] G. Matheron, Eléments pour une théorie des milieux poreux, Masson, Paris, 1967.
[4] G. Matheron, Random Sets and Integral Geometry, Wiley, New York, 1975.
[5] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.
[6] J. Serra, in: J. Serra (Ed.), Image Analysis and Mathematical Morphology, Part II: Theoretical Advances, Academic Press, London, 1988.
[7] L. Polkowski, Rough set approach to mathematical morphology: approximate compression of data, in: Information Processing and Management of Uncertainty IPMU'98, Paris, 1998, pp. 1183–1189.
[8] Y.Y. Yao, Two views of the theory of rough sets in finite universes, Int. J. Approximate Reasoning 15 (4) (1996) 291–310.
[9] I. Bloch, H. Maître, Fuzzy mathematical morphologies: a comparative study, Pattern Recognition 28 (9) (1995) 1341–1387.

[10] T.Y. Lin, Q. Liu, Rough approximate operators: axiomatic rough sets theory, in: W.P. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer, London, 1994, pp. 256–260.
[11] H. Emptoz, Modèle prétopologique pour la reconnaissance des formes. Applications en neurophysiologie, Thèse de Doctorat d'Etat, Univ. Claude Bernard, Lyon I, Lyon, France, 1983.
[12] T.Y. Lin, Neighborhood systems: a qualitative theory for fuzzy and rough sets. in: Second Annual Joint Conference on Information Science, Wrightsville Beach, NC, 1995, pp. 255–258.
[13] D. Dubois, H. Prade, A review of fuzzy set aggregation connectives, Inform. Sci. 36 (1985) 85–121.
[14] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. General Systems 17 (1990) 191–209.
[15] Z. Wojcik, Rough approximation of shapes in pattern recognition, Comput. Vision Graphics Image Process. 40 (1987) 228–249.
[16] Z.M. Wojcik, Application of rough sets for edge enhancing image filters, in: IEEE International Conference on Image Processing ICIP'94, Vol. II, Austin, Texas, 1994, pp. 525–529.

About the Author—ISABELLE BLOCH is a professor at ENST Paris (Signal and Image Department), and is in charge of the Image Processing and Interpretation Group. She is graduated from Ecole des Mines de Paris in 1986, received a Ph.D. from ENST Paris in 1990, and the "Habilitation à Diriger des Recherches" from University Paris 5 in 1995. Her research interests include 3D image and object processing, structural pattern recognition, 3D and fuzzy mathematical morphology, decision theory, data fusion in image processing, fuzzy set theory, evidence theory, medical imaging, aerial and satellite imaging.