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Defining and computing Hausdorff distances between distributions on the real line and on the circle: link between optimal transport and morphological dilations

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Abstract: Comparing probability or possibility distributions is important in many fields of information processing under uncertainty. In this paper we address the question of defining and computing Hausdorff distances between distributions in a general sense. We propose several dilations of distributions, and exhibit some links between Lévy-Prokhorov distances and dilation-based distances. In particular, mathematical morphology provides an elegant way to deal with periodic distributions. The case of possibility distributions is addressed using fuzzy mathematical morphology. As an illustration, the proposed approaches are applied to the comparison of spatial relations between objects in an image or a video sequence, when these relations are represented as distributions.

Keywords: Comparison of probabilistic or possibilistic distributions, optimal transport, mathematical morphology, fuzzy mathematical morphology, Hausdorff, Prokhorov, Lévy distances, spatial relations.

1 Introduction

Comparing probability or possibility distributions is important in many fields of information processing under uncertainty. For instance distributions may represent uncertain measurements, imprecise preferences, membership to a class, etc. The comparison then aims at assessing the evolution of these distributions over time in dynamic systems, or their similarities between different situations or in different scenarios. As an example, comparing distributions is important in image processing and understanding, and typical applications concern the comparison of histograms of gray levels or colors, or of key points [15, 25]. At a more structural level, spatial relations between objects, or between instances of objects at different times, are important to assess the spatial arrangement of objects on a scene and its evolution, thus requiring also comparison between representations, e.g. as distributions, of such spatial relations [4, 5].

In this paper we consider the general framework of comparison of distributions in a general sense (related to image information or not), that can have a probabilistic or a possibilistic and fuzzy meaning. We focus on links between dilation-based distances and optimal transport ones.

The Hausdorff distance is a good choice for comparing sets or functions, since it has all the properties of a metric on compact sets. In this paper, we study this distance between distributions, from a mathematical morphology perspective. In particular we highlight links between existing metrics such as Prokhorov and Lévy ones, and existing or newly proposed expressions of the Hausdorff distance derived from morphological dilations. We consider distributions on the real line, as well as periodic distributions, which are important

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for comparing histograms of orientations or of colors in some specific color spaces, or directional spatial relations. This problem has been addressed using the Wasserstein distance in [19], but not using the Hausdorff distance.

The Hausdorff distance has been defined between functions in [20], and by considering 1D functions as subsets of $\mathbb{R}^2$ in [22]. We will also investigate a similar approach in this work. This idea was then further studied in [9] by considering truncated umbras and dilations by a half ball, and in [16], where the case of discontinuous functions was also addressed.

When functions are membership functions of fuzzy sets or possibility distributions, different approaches for defining the Hausdorff distance have been proposed. Some of them define the distance as a number, by combining the values of the Hausdorff distances computed between α-cuts (thresholds of the functions, hence sets), either as a weighted sum, or using the extension principle [7, 8, 18, 26]. Several generalizations of the Hausdorff distance have also been proposed under the form of fuzzy numbers [2, 10]. Extensions of the Hausdorff distance based on fuzzy mathematical morphology have been developed, either as a number in [13] from the distance from a point to a fuzzy set [3], or as a fuzzy number [3]. This last approach will be exploited in the present work too.

Some preliminaries on periodic and non periodic distributions are first given in Section 2, as well as some considerations about the ground distance. Indeed, existing methods for comparing histograms or probability distributions [11] are usually categorized into two classes: (i) bin-to-bin distances, and (ii) cross-bin distances, involving the distance on the support (or ground distance) [11, 19, 25]. In this paper, we only consider distances of the second class, keeping in mind the application to spatial relations. For instance, if two distributions are identical up to a translation and with disjoint supports, the distances of the first class will always provide the same value, while the second ones will differentiate situations with different translations.

The contributions of the paper are then presented in the following sections. Several types of dilations are proposed in Section 3. Then we propose Hausdorff distances on distributions based on optimal transport and morphological methods in Section 4. The links between these two types of approaches allow us to address the case of non periodic distributions in Section 5, which is another important contribution of this paper. This case is illustrated in Section 6 for comparing directional relations between objects and their change in a synthetic video sequence.

This work extends the one in [6], in particular by providing complete proofs of the main results as well as additional illustrative examples.

2 Preliminaries

2.1 Distributions and cumulative distributions

Let f and g denote the distributions (in a broad sense) to be compared, via the computation of a distance between them. We denote by M the definition domain of these distributions. In this paper, we consider only one-dimensional domains, and M can be $\mathbb{R}$ or $\mathbb{R}^+$ for non-periodic distributions, and $[0, \rho]$ for periodic distributions of period ρ (for instance $[0, 2\pi]$ for the example of relative direction in Section 6). We denote points of M by $x, y\ldots$, or $\theta, \alpha\ldots$ when they are angles.

Normalized distributions are assumed in this paper. Two types of normalization are considered: by the sup or max, or by the sum. The first case goes with a fuzzy or possibilistic interpretation, while the second one corresponds to a probabilistic interpretation. By convention all distributions take values in $[0, 1]$. In the probabilistic interpretation, $f(x)$ represents the probability that a random variable takes the value x. In the fuzzy or possibilistic interpretation, $f(x)$ represents the membership degree of x to some set (which is imprecisely defined), or the possibility degree that a variable takes the value x.

The cumulative distribution of f defined on the real line, denoted by F, is defined as :

$$\forall x \in \mathbb{R}, F(x) = \int_{-\infty}^{x} f(t)\,dt,$$

and for f defined on $[0, 2\pi]$, taking arbitrarily 0 as origin:

$$\forall \theta \in [0, 2\pi], F(\theta) = \int_{0}^{\theta} f(t)\,dt.$$

The cumulative distribution of g is denoted by G and is defined similarly. Note that defining a distance between f and g from a distance between F and G actually provides a distance between distributions (the proof is immediate). For some definitions, we will consider F and G as sets in a 2D space, denoted by SF and SG. Cumulative distributions are right continuous and jumps correspond to discontinuities in the underlying distributions. In such cases, SF and SG are completed by vertical segments corresponding to these jumps:

$$SF = \{(x, F(x)) \mid x \in M\} \cup \{(x, y) \mid x \in J(F) \text{ and } \lim_{x' \to x^-} F(x') \leq y \leq F(x)\}$$

where $J(F)$ denotes the set of points at which jumps occur (i.e. where the left limit of F at x is not equal to $F(x)$). In the sequel, we always assume that SF and SG are completed graphs.

2.2 Ground distance

As mentioned in the introduction, we only consider here cross-bin distances between distributions, involving the ground distance on the support M.

Let us denote by d the ground distance on M. Its definition depends on M. If M is equal to $\mathbb{R}$ or $\mathbb{R}^+$, then d is defined from an L_p norm, for instance in 1D:

$$d(x, y) = |x - y|.$$

For periodic distributions (or defined on a circle), the geodesic distance is used. If the period is ρ, we will use:

$$d(x, y) = \min(|x - y|, \rho - |x - y|) = \frac{\rho}{2} - ||x - y| - \frac{\rho}{2}|.$$

In the case of distributions on the circle, with $\rho = 2\pi$, this ground distance is expressed as:

$$d(\theta, \theta') = \min(|\theta - \theta'|, 2\pi - |\theta - \theta'|) = \pi - ||\theta - \theta'| - \pi|. \tag{1}$$

This formulation allows us to consider that values close to 0 and 2π, respectively, are at a short distance from each other. The distance values can also be normalized, using for instance $\frac{d(\theta,\theta')}{\pi}$ or $\sin \frac{|\theta-\theta'|}{2}$. These formulas extend to higher dimensions.

3 Definition of some dilations of distributions

3.1 Morphological dilation of a sup-normalized distribution

We assume in this section that the distributions are normalized by the sup (and we restrict this work to distributions with bounded sup), or at least that they all have the same maximum value. To simplify the presentation, we consider binary structuring elements, defined as subsets of M.

If the distributions are defined on the real line ($M = \mathbb{R}$ or $M = \mathbb{R}^+$), classical mathematical morphology applies and the dilation of f by a structuring element B is expressed by

$$\forall x \in M, \delta_B(f)(x) = \sup_{y \in B_x} f(y),$$

where B_x denotes as usual the translation of B at x ($B_x = x + B$).

If the distributions are periodic, this periodicity should be taken into account in the dilation and the structuring element. The following definition details the case where $\rho = 2\pi$ but can be easily extended to any periodic function.

Definition 1. *Let f be a distribution on the unit circle. Its dilation by a structuring element of size α is defined by:*

$$\forall \theta \in M = [0, 2\pi], \delta_{B^\alpha}(f)(\theta) = \sup_{\theta' \in B^\alpha_\theta} f(\theta') \tag{2}$$

where B^α is a structuring element of aperture α, defined as:

- *if $\alpha \leq \pi$:*
  - $B^\alpha_\theta = [\theta - \alpha, \theta + \alpha]$ *if* $\theta - \alpha \geq 0$ *and* $\theta + \alpha \leq 2\pi$,
  - $B^\alpha_\theta = [0, \theta + \alpha] \cup [\theta - \alpha + 2\pi, 2\pi]$ *if* $\theta - \alpha \leq 0$ *and* $\theta + \alpha \leq 2\pi$,
  - $B^\alpha_\theta = [\theta - \alpha, 2\pi] \cup [0, \theta + \alpha - 2\pi]$ *if* $\theta - \alpha \geq 0$ *and* $\theta + \alpha \geq 2\pi$,
- *if $\alpha \geq \pi$: $B^\alpha_\theta = [0, 2\pi]$. (The case $\theta - \alpha \leq 0$ and $\theta + \alpha \geq 2\pi$ implies $\alpha \geq \pi$.)*

Figure 1 illustrates the definition of B^α.

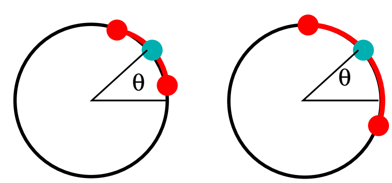

Figure 1: Two examples of B^α_θ, i.e. a ball of radius α of the ground distance, centered at θ (in red). In the first example, $B^\alpha_\theta = [\theta - \alpha, \theta + \alpha]$, and in the second one $B^\alpha_\theta = [0, \theta + \alpha] \cup [\theta - \alpha + 2\pi, 2\pi]$.

Figure 2 illustrates the dilation of a distribution on $[0, 2\pi]$. Note that Definition 1 extends directly to any periodic function.

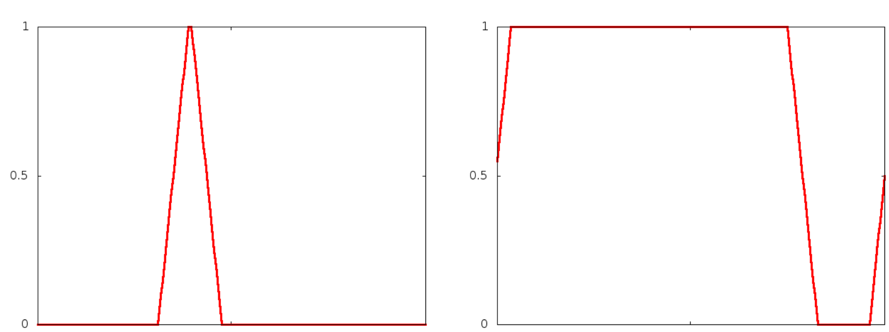

Figure 2: Distribution on $[0, 2\pi]$ and example of a dilation accounting for the periodicity.

The normalization ensures that the core of the distribution (set of points with maximum value) is extended according to the size of the structuring element. In particular, it is always possible to find a size of dilation such that a given point of the support of the distribution belongs to the core of the dilated distribution. This property will be used for Hausdorff distances defined from such dilations. The following proposition is easy to show (by a direct computation):

Proposition 1. *For all α, B^α is a ball of radius α of the ground distance d (Equation 1), and for all f and α, we have* $\forall \theta \in [0, 2\pi]$, $\delta_{B^\alpha}(f)(\theta) = \sup\{f(\theta') \mid \theta' \in [0, 2\pi], d(\theta, \theta') \leq \alpha\}$.

3.2 Dilations of cumulative distributions in the non-periodic case

In this section we consider a cumulative distribution either as a function F from M into $[0, 1]$, or as a subset SF of $M \times [0, 1]$.

Let us consider as a structuring element a segment of length 2ε, with $\varepsilon \geq 0$. We denote by $B^\varepsilon_x = [x - \varepsilon, x + \varepsilon] \cap M$ the translation of this structuring element at x, restricted to the support.

Proposition 2. *The dilation of F by B^ε is expressed as:*

$$\forall x \in M, \delta_{B^\varepsilon}(F)(x) = \sup_{y \in B^\varepsilon_x} F(y) = \begin{cases} F(x+\varepsilon) & \textit{if } x + \varepsilon \in M \\ 1 & \textit{otherwise} \end{cases}$$

Proof. This result follows directly from the fact that F is increasing. □

Let us now consider the dilation of SF, using different structuring elements, that will prove useful in the following. Let us first consider a ball of radius ε of the L_∞ distance, with a positive proportionality factor λ on M ($\lambda > 0$) to account for the different scales of the two dimensions (i.e. the structuring element is a rectangle). It is expressed, when translated at (x, y), as:

$$(B^{\varepsilon,\lambda}_1)_{(x,y)} = (\check{B}^{\varepsilon,\lambda}_1)_{(x,y)} = [x - \lambda\varepsilon, x + \lambda\varepsilon] \times [y - \varepsilon, y + \varepsilon].$$

where $\check{B}$ denotes the symmetrical of B with respect to the origin.

Proposition 3. *The dilation of any SF by $B^{\varepsilon,\lambda}_1$ is expressed as:*

$$\delta^{\varepsilon,\lambda}_1(SF) = \{(x, y) \in M \times [0, 1] \mid \exists x' \in M, \max(\frac{|x - x'|}{\lambda}, |y - F(x')|) \leq \varepsilon\}. \quad (3)$$

Proof. It follows directly from the development of $\delta^{\varepsilon,\lambda}_1(SF) = \{(x, y) \in M \times [0, 1] \mid (\check{B}^{\varepsilon,\lambda}_1)_{(x,y)} \cap SF \neq \emptyset\}$. □

This dilation is illustrated in Figure 3, for $\lambda = 1$.

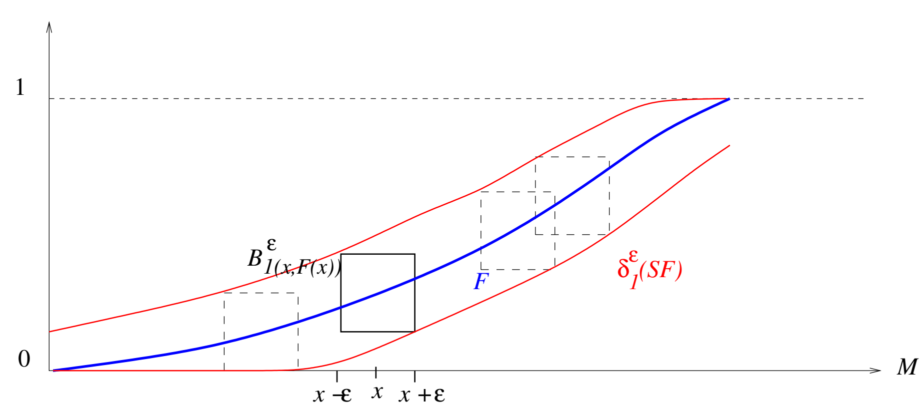

Figure 3: Dilation with a symmetrical structuring element.

Let us now consider an asymmetric dilation, with the following structuring element centered at (x, y) and of size ε (still with the factor λ on M): $(B_2^{\varepsilon,\lambda})_{(x,y)} = [x - \lambda\varepsilon, x + \lambda\varepsilon] \times [y - \varepsilon, 1]$. Its symmetrical with respect to (x, y) is then: $(\check{B}_2^{\varepsilon,\lambda})_{(x,y)} = [x - \lambda\varepsilon, x + \lambda\varepsilon] \times [0, y + \varepsilon]$.

Proposition 4. *The asymmetric dilation of SF by $B_2^{\varepsilon,\lambda}$ is expressed as:*

$$\delta_2^{\varepsilon,\lambda}(SF) = \{(x, y) \in M \times [0, 1] \mid \exists x' \in M, \max(\frac{|x - x'|}{\lambda}, F(x') - y) \leq \varepsilon\}.$$

Proof. The proof is similar to the one of Proposition 3. □

This asymmetric dilation is illustrated in Figure 4.

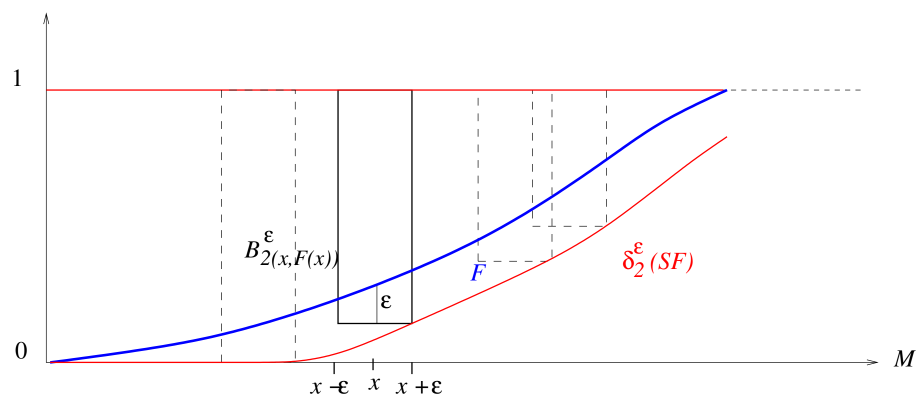

Figure 4: Dilation with a non-symmetrical structuring element.

Finally, let us consider another asymmetric dilation, but without saturation along the ordinate axis, with $(B_3^{\varepsilon,\lambda})_{(x,y)} = [x, x + \lambda\varepsilon] \times [y - \varepsilon, y]$, and $(\check{B}_3^{\varepsilon,\lambda})_{(x,y)} = [x - \lambda\varepsilon, x] \times [y, y + \varepsilon]$.

Proposition 5. *The dilation of SF by $B_3^{\varepsilon,\lambda}$ is expressed as:*

$$\delta_3^{\varepsilon,\lambda}(SF) = \{(x, y) \mid \exists x' \in M, x - \lambda\varepsilon \leq x' \leq x, y \leq F(x') \leq y + \varepsilon\}. \quad (4)$$

Proof. Again the proof is direct from the development of $\delta_3^{\varepsilon,\lambda}(SF) = \{(x, y) \mid \exists x' \in M, (x', F(x')) \in (\check{B}_3^{\varepsilon,\lambda})_{(x,y)}\}$. □

This dilation is illustrated in Figure 5.

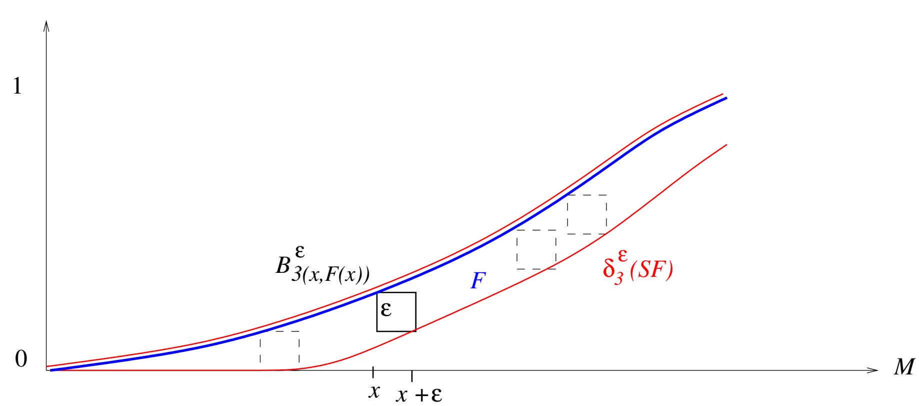

Figure 5: Asymmetric dilation, in the sub-graph of F only.

Proposition 6. *We have the following relationship between the two asymmetrical definitions:*

$$\delta_2^{\varepsilon,\lambda}(SF) = \delta_3^{\varepsilon,\lambda}(SF) \cup \{(x, y) \mid x \in M, F(x) \le y \le 1\}$$

(i.e. δ_2 is obtained from δ_3 by adding the sup-graph of F bounded by 1).

Proof. The proof is immediate. □

In all these definitions, we could also assume that M and $[0, 1]$ are co-normalized and then restrict the 2D space $\mathbb{R}^2$ to $[0, 1] \times [0, 1]$. Then λ can be set to 1 in all the above equations. This applies also for the periodic case considered next.

3.3 Dilations of cumulative distributions in the periodic case

All the definitions introduced above apply also to the periodic case, using the following embedding of F into $\mathbb{R}$:

$$\forall x \in \mathbb{R}, F(x + \rho) = F(x) + 1. \tag{5}$$

However, the computation does not need to be performed on the whole real line. For instance if $\rho = 2\pi$, it is sufficient to consider an embedding in $]-\pi, 3\pi[\times[-1, 2]$ since for $\lambda\varepsilon \ge \pi$, the dilation would provide the whole space $M \times [0, 1]$. The extension of SF then writes:

$$SF^E = SF \cup \{(\theta, F(\theta + 2\pi) - 1), \theta \in]-\pi, 0]\} \cup \{(\theta, F(\theta - 2\pi) + 1), \theta \in [2\pi, 3\pi[\}. \tag{6}$$

Dilations can be expressed directly from this set, and we have the following simple form.

Proposition 7. *The dilation of SF with a symmetrical structuring element and $\lambda\varepsilon \le \pi$ is expressed as:*

$$\delta_{c1}^{\varepsilon,\lambda}(SF) = \{(\theta, y) \in [0, 2\pi] \times [0, 1] \mid \exists\theta' \in [0, 2\pi], |\theta - \theta'| \le \lambda\varepsilon \text{ and } |F(\theta') - y| \le \varepsilon\}. \tag{7}$$

For $\lambda\varepsilon > \pi$, then $\delta_{c1}^{\varepsilon,\lambda}(SF) = [0, 2\pi] \times [0, 1]$.

Proof. The sketch of the proof is as follows: we first develop the expression of dilation, considering each part of the disjunction in SF^E. Then we analyze the cases where $\theta \le \lambda\varepsilon$ and $\theta \ge 2\pi - \lambda\varepsilon$ and show that, due to the fact that F is increasing, $F(0) = 0$ and $F(2\pi) = 1$, these cases can be simplified, and that neighbors outside $[0, 2\pi]$ do not need to be considered, thus providing the simple result in the proposition.

Let us detail these steps. The dilation of SF with a symmetrical structuring element and $\lambda\varepsilon \le \pi$ is defined as:

$$\begin{aligned}
\delta_{c1}^{\varepsilon,\lambda}(SF) =& \delta_1^{\varepsilon,\lambda}(SF^E) \cap ([0, 2\pi] \times [0, 1]) \\
=& \{(\theta, y) \in [0, 2\pi] \times [0, 1] \mid (\exists\theta' \in [0, 2\pi], |\theta - \theta'| \le \lambda\varepsilon \text{ and } |F(\theta') - y| \le \varepsilon) \\
& \text{or } (\exists\theta' \in [-\pi, 0[, |\theta - \theta'| \le \lambda\varepsilon \text{ and } |F(\theta' + 2\pi) - 1 - y| \le \varepsilon) \\
& \text{or } (\exists\theta' \in]2\pi, 3\pi], |\theta - \theta'| \le \lambda\varepsilon \text{ and } |F(\theta' - 2\pi) + 1 - y| \le \varepsilon)\} \qquad (8)\\
=& \{(\theta, y) \in [0, 2\pi] \times [0, 1] \mid \exists\theta' \in [0, 2\pi], (\theta', F(\theta')) \in [\theta - \lambda\varepsilon, \theta + \lambda\varepsilon] \times [y - \varepsilon, y + \varepsilon] \\
& \text{or } (\theta', F(\theta')) \in [0, \theta + \lambda\varepsilon] \times [y - \varepsilon, y + \varepsilon] \cup [\theta - \lambda\varepsilon + 2\pi, 2\pi] \times [y + 1 - \varepsilon, y + 1 + \varepsilon] \\
& \text{or } (\theta', F(\theta')) \in [\theta - \lambda\varepsilon, 2\pi] \times [y - \varepsilon, y + \varepsilon] \cup [0, \theta + \lambda\varepsilon - 2\pi] \times [y - 1 - \varepsilon, y - 1 + \varepsilon]\} \qquad (9)
\end{aligned}$$

This derivation (Equation 9) shows that it is sufficient to look for θ' in $[0, 2\pi]$ (which is interesting in practice to reduce the computation time), and to use a circular neighborhood, defined as (when centered at (θ, y)):

$$\begin{aligned}
B_{\theta,y}^{\varepsilon,\lambda} =& [\theta - \lambda\varepsilon, \theta + \lambda\varepsilon] \times [y - \varepsilon, y + \varepsilon] \text{ if } \theta - \lambda\varepsilon \ge 0 \text{ and } \theta + \lambda\varepsilon \le 2\pi \\
& [0, \theta + \lambda\varepsilon] \times [y - \varepsilon, y + \varepsilon] \cup [\theta - \lambda\varepsilon + 2\pi, 2\pi] \times [y + 1 - \varepsilon, y + 1 + \varepsilon] \text{ if } \theta - \lambda\varepsilon \le 0 \text{ and } \theta + \lambda\varepsilon \le 2\pi \\
& [\theta - \lambda\varepsilon, 2\pi] \times [y - \varepsilon, y + \varepsilon] \cup [0, \theta + \lambda\varepsilon - 2\pi] \times [y - 1 - \varepsilon, y - 1 + \varepsilon] \text{ if } \theta - \lambda\varepsilon \ge 0 \text{ and } \theta + \lambda\varepsilon \ge 2\pi
\end{aligned}$$

Note that the θ part, i.e. the projection of $B^{\varepsilon,\lambda}_{\theta,y}$ on $M = [0, 2\pi]$, is equal to $B^{\lambda\varepsilon}_{\theta}$ as defined in Section 3.1 for the circular dilation. This also means that the condition on θ' is always equivalent to $d(\theta, \theta') \leq \lambda\varepsilon$ for d being the circular distance (Section 2), since $B^{\lambda\varepsilon}$ is a ball of this distance. The case where $\theta - \lambda\varepsilon \leq 0$ and $\theta + \lambda\varepsilon \geq 2\pi$ is not considered here, since it implies $\lambda\varepsilon \geq \pi$ and then the dilation yields the whole space.

Let us now further develop the expression of the dilation in Equation 8, taking into account that F is an increasing function taking values 0 at 0 and 1 at 2π, in particular for θ being close to the bounds of the support of F:

- If $0 \leq \theta \leq \lambda\varepsilon$ (and $y \geq 0$):
  - if $0 \leq y \leq \varepsilon$, then $(0, F(0)) = (0, 0)$ belongs to the structuring element centered at (θ, y), and (θ, y) belongs to the dilation (it is therefore not necessary to look for another $\theta' < 0$);
  - if $y > \varepsilon$:
    - ⋆ if $\exists \theta' \geq 0$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $|F(\theta') - y| \leq \varepsilon$, then (θ, y) belongs to the dilation;
    - ⋆ otherwise, we look for $\theta' \in [-\pi, 0[$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $|F(\theta' + 2\pi) - 1 - y| \leq \varepsilon$. Since $F(\theta' + 2\pi) - 1 \leq 0$, we have $|F(\theta' + 2\pi) - 1 - y| = y + 1 - F(\theta' + 2\pi)$ which is greater than y. We cannot have both $y > \varepsilon$ and $|F(\theta' + 2\pi) - 1 - y| \leq \varepsilon$. Therefore the values of θ' in $[-\pi, 0[$ are not involved in the result of the dilation.
- If $2\pi - \lambda\varepsilon \leq \theta \leq 2\pi$:
  - if $1 - y \leq \varepsilon$, then the point $(2\pi, F(2\pi)) = (2\pi, 1)$ belongs to the structuring element centered at (θ, y) and (θ, y) belongs to the dilation;
  - if $1 - y > \varepsilon$:
    - ⋆ if $\exists \theta' \leq 2\pi$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $|F(\theta') - y| \leq \varepsilon$, then (θ, y) belongs to the dilation;
    - ⋆ otherwise, we look for a $\theta' \in]2\pi, 3\pi]$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $|F(\theta' - 2\pi) + 1 - y| \leq \varepsilon$. We have $|F(\theta' - 2\pi) + 1 - y| = F(\theta' - 2\pi) + 1 - y$ which is greater than $1 - y$ and hence cannot be less than ε.

Finally, only the first case for θ' needs to be considered, i.e. $(\theta, y) \in \delta^{\varepsilon,\lambda}_{c1}(SF)$ iff $\exists \theta' \in [0, 2\pi]$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $|F(\theta') - y| \leq \varepsilon$, hence the result. □

Note that the simple expression obtained in Proposition 7 corresponds to a geodesic way to process the boundaries of the domain, by truncating the translated structuring element to limit it to the part included in $[0, 2\pi] \times [0, 1]$. This dilation is illustrated in Figure 6.

Considering now the structuring element $B^{\varepsilon,\lambda}_2$ to dilate only the subgraph (and saturating its complement to 1) leads also to a simple expression:

Proposition 8. *The dilation of SF with an asymmetrical structuring element and $\lambda\varepsilon \leq \pi$ is expressed as:*

$$\delta^{\varepsilon,\lambda}_{c2}(SF) = \{(\theta, y) \in [0, 2\pi] \times [0, 1] \mid \exists \theta' \in [0, 2\pi], |\theta - \theta'| \leq \lambda\varepsilon \text{ and } F(\theta') - y \leq \varepsilon\}. \tag{10}$$

For $\lambda\varepsilon > \pi$, we have $\delta^{\varepsilon,\lambda}_{c2}(SF) = [0, 2\pi] \times [0, 1]$.

Proof. The proof follows the same reasoning as for Proposition 7: we have $(\theta, y) \in \delta^{\varepsilon,\lambda}_{c2}(SF)$ iff

1. $\exists \theta' \in [0, 2\pi]$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $F(\theta') \in [0, y + \varepsilon]$,
2. or $\exists \theta' \in [-\pi, 0[$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $F(\theta' + 2\pi) - 1 \in [0, y + \varepsilon]$,
3. or $\exists \theta' \in]2\pi, 3\pi]$ such that $|\theta - \theta'| \leq \lambda\varepsilon$ and $F(\theta' - 2\pi) + 1 \in [0, y + \varepsilon]$.

Cases 2 and 3 are not possible since $F(\theta' + 2\pi) - 1 \leq 0$ and $F(\theta' - 2\pi) + 1 \geq 1$, so only the first case remains. □

In all these definitions, the classical properties of dilations hold (commutativity with the supremum, monotony with respect to ε, iterativity property, etc.). The envelop of the dilation of SF is delimited by two functions which are cumulative distributions of distributions with a jump at 0 for the upper envelop and at 1 for the lower envelop.

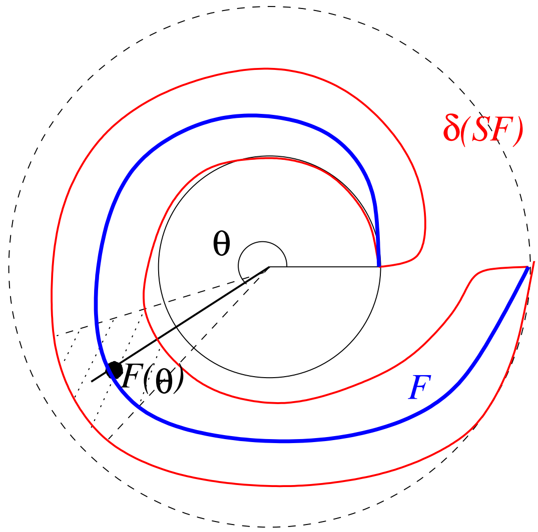

Figure 6: Dilation in the periodic case, for a symmetrical structuring element. The central circle corresponds to 0 and the larger one to 1. The dashed area is an example of structuring element centered at $(\theta, F(\theta))$.

4 Distances between distributions on the real line

In this section, we define distances between distributions on the real line using two approaches. The first one exploits the link between Hausdorff distances and morphological dilations to derive definitions of Hausdorff distances from the dilations introduced in the previous section, either for distributions or for cumulative distributions. The second approach is inspired by optimal transport. We consider the Lévy-Prokhorov distance, and exploit its Hausdorff like expression on cumulative distributions. We show that it is equivalent to one of the morphological expressions. This link, which constitutes an original contribution, will be then further exploited in the next section for periodic distributions.

4.1 Morphological approach

Let us first recall the general link between Hausdorff distance and dilation. In the classical set theoretical setting, it writes as:

$$d_H(F, G) = \inf\{\varepsilon > 0 \mid SG \subseteq \delta^\varepsilon(SF) \text{ and } SF \subseteq \delta^\varepsilon(SG)\},$$

and in the functional setting as:

$$d_H(F, G) = \inf\{\varepsilon > 0 \mid G \leq \delta^\varepsilon(F) \text{ and } F \leq \delta^\varepsilon(G)\},$$

where δ^ε denotes the dilation by a structuring element of size ε (a ball of radius ε of the ground distance). The same notation is used for dilations of sets and of functions, since no ambiguity can arise.

4.1.1 Hausdorff distance from dilations of cumulative distributions

Let us first consider $\delta_1^{\varepsilon,\lambda}$ introduced in Section 3.2, and let us derive a Hausdorff distance from it (see Figure 7, for $\lambda = 1$).

Proposition 9. *The Hausdorff distance associated with δ_1 is:*

$$d_{H1}(F, G) = \max(\sup_{x\in M}\inf_{y\in M}\max(\frac{|x-y|}{\lambda}, |G(x) - F(y)|), \sup_{y\in M}\inf_{x\in M}\max(\frac{|x-y|}{\lambda}, |F(y) - G(x)|)). \tag{11}$$

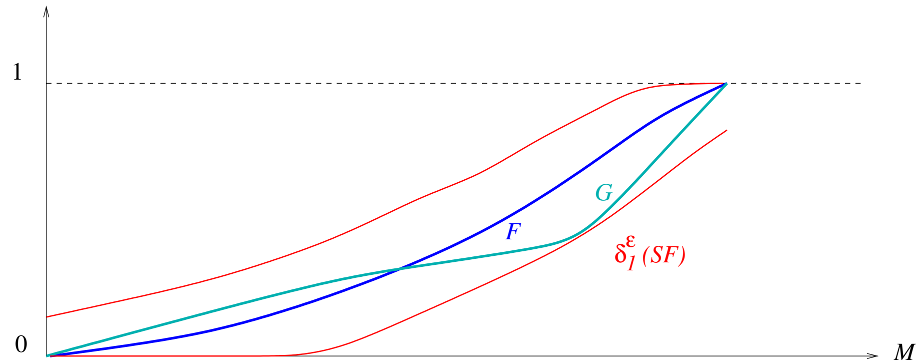

Figure 7: Minimal size of the dilation of SF by δ_1 such that it contains SG.

Proof. We have:

$$\begin{aligned} SG \subseteq \delta_1^{\varepsilon,\lambda}(SF) \quad &\Leftrightarrow \quad \forall x \in M, \exists y \in M, \max(\frac{|x-y|}{\lambda}, |G(x) - F(y)|) \leq \varepsilon \\ &\Leftrightarrow \quad \sup_{x \in M} \inf_{y \in M} \max(\frac{|x-y|}{\lambda}, |G(x) - F(y)|) \leq \varepsilon, \end{aligned}$$

which is illustrated in Figure 7 for $\lambda = 1$. A similar expression is obtained for $SF \subseteq \delta_1^{\varepsilon,\lambda}(SG)$. The result then follows from $d_{H1}(F, G) = \inf\{\varepsilon > 0 \mid SF \subseteq \delta_1^{\varepsilon,\lambda}(SG) \text{ and } SG \subseteq \delta_1^{\varepsilon,\lambda}(SF)\}$. □

Let us now consider the asymmetric dilation δ_2.

Proposition 10. *The Hausdorff distance derived from δ_2 is:*

$$d_{H2}(F, G) = \max(\sup_{x \in M} \inf_{y \in M} \max(\frac{|x-y|}{\lambda}, G(y) - F(x)), \sup_{y \in M} \inf_{x \in M} \max(\frac{|x-y|}{\lambda}, F(x) - G(y))). \tag{12}$$

Proof. We have:

$$d_{H2}(F, G) = \inf\{\varepsilon > 0 \mid SF \subseteq \delta_2^{\varepsilon,\lambda}(SG) \text{ and } SG \subseteq \delta_2^{\varepsilon,\lambda}(SF)\},$$

$$SG \subseteq \delta_2^{\varepsilon,\lambda}(SF) \Leftrightarrow \sup_{y \in M} \inf_{x \in M} \max(\frac{|x-y|}{\lambda}, F(x) - G(y)) \leq \varepsilon,$$

$$SF \subseteq \delta_2^{\varepsilon,\lambda}(SG) \Leftrightarrow \sup_{x \in M} \inf_{y \in M} \max(\frac{|x-y|}{\lambda}, G(y) - F(x)) \leq \varepsilon.$$

Hence the result. The computation from the dilation is illustrated in Figure 8 (only the lower envelop of the dilation is shown). □

Finally, let us derive the Hausdorff distance from cumulative distributions considered as functions.

Proposition 11. *We have:*

$$\begin{aligned} d_H(F, G) \quad &= \quad \inf\{\varepsilon > 0 \mid \forall x \in M, G(x) \leq F(x+\varepsilon) \textit{ and } F(x) \leq G(x+\varepsilon)\} \\ &= \quad \inf\{\varepsilon > 0 \mid \forall x \in M, G(x-\varepsilon) \leq F(x) \leq G(x+\varepsilon)\}. \end{aligned} \tag{13}$$

Proof. It follows directly from $G \leq \delta^\varepsilon(F) \Leftrightarrow \forall x \in M, G(x) \leq F(x+\varepsilon)$. □

This is illustrated in Figure 9.

Proposition 12. *All distances defined in this section are metrics (i.e. positive, separable, symmetrical and satisfy the triangular inequality). If the distributions are Dirac functions (with a unique non zero value at f_0 and g_0), the proposed distances are all equal to $d(f_0, g_0)$, where d is the ground distance.*

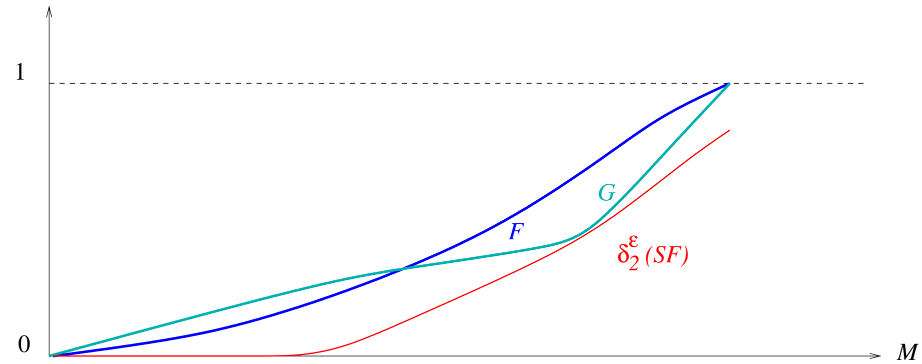

Figure 8: Minimal size of the dilation of SF by δ_2 such that it contains SG.

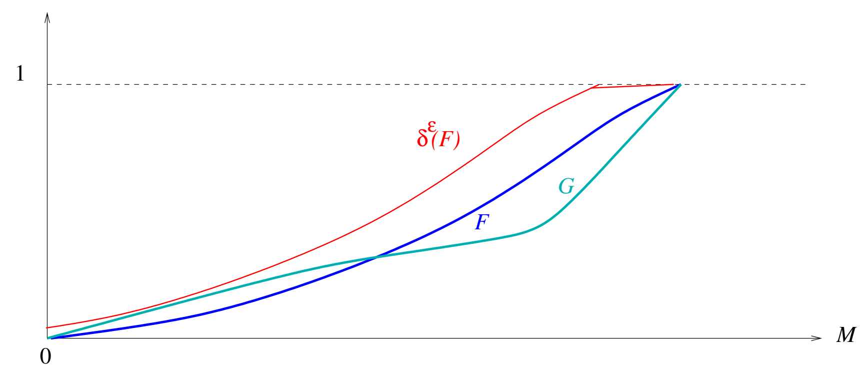

Figure 9: Computation of the Hausdorff distance by dilating the cumulative distributions considered as functions.

Proof. Since all distances are Hausdorff distances (for which we proved explicit expressions based on dilations), they are metrics. If distributions are Dirac functions, then we have $F(x) = 0$ for $x < f_0$ and $F(x) = 1$ otherwise, and a similar expression for G. Then the minimal size of dilation, such that SF is included in the dilation of SG and SG is included in the dilation of SF, is $|f_0 - g_0|$, which is the ground distance between f_0 and g_0 for one-dimensional distributions. □

4.1.2 Fuzzy Hausdorff distance from dilations of distributions

The idea here is to exploit the link between morphological dilation and some distances, such as minimum and Hausdorff distances, in the case of sets [3, 23]. Indeed, the Hausdorff distance between two sets is equal to the minimal size of the ball of the ground distance, such that the dilation of each set by this ball contains the other set. We propose to use the same principle on distributions.

Definition 2. *[3] The fuzzy Hausdorff distance is defined from the dilation of the distributions, considered as fuzzy sets, and from an inclusion operator $\Delta_{\subseteq}(f,g)$, expressing the degree to which f is included in g:*

$$\forall \ell \in \mathbb{R}^{+*}, d_H(f,g)(\ell) = t(d_H'(f,g)(\ell), d_H'(g,f)(\ell)) \tag{14}$$

with

$$d_H'(f,g)(\ell) = t(\Delta_{\subseteq}(f,\delta_{B^{\ell}}(g)), \inf_{0 \le \ell' < \ell} c(\Delta_{\subseteq}(f,\delta_{B^{\ell'}}(g)))),$$

and $d_H'(f,g)(0) = \Delta_{\subseteq}(f,g)$, with t a t-norm.

Note that when values of ℓ are quantified, which is the case in practice, it is sufficient to consider for ℓ' only the largest value less than ℓ.

The value $d_H(f,g)(\ell)$ expresses the degree to which the Hausdorff distance between f and g is equal to ℓ, and $d_H(f,g)$ is then a distance density, in the sense of [21]. A common definition of an inclusion degree in the fuzzy set framework is

$$\Delta_{\subseteq}(f,g) = \inf_{x\in M} I(f(x), g(x))$$

where I is a fuzzy implication. If a crisp number is needed, the center of gravity of this fuzzy number can be used:

$$\frac{\int_0^\infty d_H(f,g)(\ell)\ell d\ell}{\int_0^\infty d_H(f,g)(\ell)d\ell},$$

or the following definition:

$$d_H(f,g) = \inf\{\ell \in \mathbb{R}^+ \mid \forall x \in M, \delta_{B^\ell}(f)(x) \geq g(x) \text{ and } \delta_{B^\ell}(g)(x) \geq f(x)\}, \tag{15}$$

which corresponds to a crisp version of the inclusion. This simplified expression corresponds to the definitions in [9, 16] for flat structuring elements.

Proposition 13. *[3] The fuzzy distances introduced in Equations 14 and 15 are positive and symmetrical. The morphological Hausdorff distance between the distributions and computed with a crisp version of the inclusion degree (Equation 15) is separable and satisfies the triangular inequality, while the fuzzy version of the inclusion degree yields a distance (Equation 14) which is a fuzzy number, and separable for Lukasiewicz implication* ($I(a,b) = \min(1, 1-a+b)$)*, but does not satisfy the triangular inequality.*

4.2 Lévy and Prokhorov distances

An interesting distance between probability distributions, related to optimal transport problems [24] and which involves dilations, is the Lévy-Prokhorov metric d_{Pr} [17], defined for two distributions f and g as:

$$d_{Pr}(f,g) = \inf\{\varepsilon > 0 \mid \forall Z \in \mathcal{B}(M), f(Z) \leq g(\delta^{\lambda\varepsilon}(Z)) + \varepsilon \text{ and } g(Z) \leq f(\delta^{\lambda\varepsilon}(Z)) + \varepsilon\} \tag{16}$$

where $\delta^{\lambda\varepsilon}(Z)$ is the dilation of size $\lambda\varepsilon$ of Z (see Section 3.1, restricting functions to sets), and $\mathcal{B}(M)$ denotes the set of all Borel sets on M. The definition has been adapted here to introduce λ and thus to account for the potential different scales of M and $[0,1]$, as in [20].

This distance generalizes the Lévy distance (also a metric), defined in 1D between two cumulative distributions F and G as:

$$d_L(F,G) = \inf\{\varepsilon > 0 \mid \forall x \in \mathbb{R}, G(x-\lambda\varepsilon) - \varepsilon \leq F(x) \leq G(x+\lambda\varepsilon) + \varepsilon\}. \tag{17}$$

By restricting the Borel sets of $\mathbb{R}$ to the intervals of the form $Z =]-\infty, x[$ (or equivalently $Z =]x, +\infty[$), which generate $\mathcal{B}(M)$, d_{Pr} is indeed equivalent to d_L in 1D. Note that if all Borel sets are considered, then we only have $d_L \leq d_{Pr}$.

Let us provide the Hausdorffian expression of d_L from [20].

Proposition 14. *[20] The Lévy distance can be expressed in a similar way as the Hausdorff distance and we have:*

$$d_L(F,G) = \max(\sup_{x\in M}\inf_{y\in M}\max(\frac{|x-y|}{\lambda}, G(y)-F(x)), \sup_{y\in M}\inf_{x\in M}\max(\frac{|x-y|}{\lambda}, F(x)-G(y))). \tag{18}$$

Note that this expression involves explicitly the ground distance on M. Figure 10 illustrates Equation 18, providing a geometrical interpretation.

We now exhibit links with Hausdorff distances derived from the dilations proposed in Section 3.2. Note that d_{Pr} already involves a dilation and that the links between d_{Pr}, d_L and its Hausdorff-like expression already suggest that all these notions are closely related.

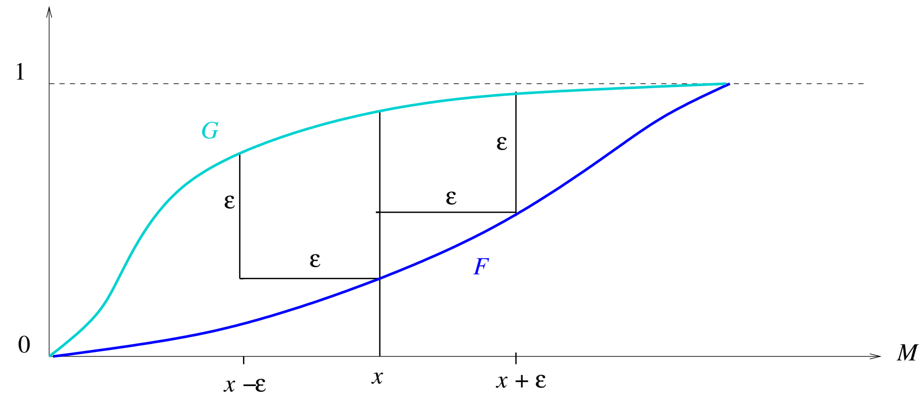

Figure 10: Computation of the Lévy distance between F and G.

Proposition 15. *Let F and G be any two cumulative distributions. We have the following equivalences between their distances:*

- *the Lévy distance can be formulated as a Hausdorff-like expression (Equation 18);*
- *Equation 11 is similar to Equation 18, but with absolute values on $G(x) - F(y)$, providing one of the definitions in [20];*
- *Equation 12 is equivalent to Equation 18;*
- *Equation 13 is equivalent to Equation 17;*
- *Equation 15 is similar to d_{Pr} expressed on points.*

Proof. These results are straightfoward from the expressions of distances. For the two last ones, they can be proved by replacing ε by $\frac{\varepsilon}{\lambda}$ and taking the limit for $\lambda \to +\infty$. □

All these links make it easier to extend the definitions to the periodic case (next section).

Proposition 16. *d_L is a probability metric [20]. Similarly, the Hausdorff distances defined in Equations 11 and 13 are probability metrics.*

Proof. The distance d_L (for $\lambda = 1$) can be expressed as:

$$d_L(F, G) = \inf\{t > 0 \mid v(f, G, t) < t\} \tag{19}$$

for v defined as:

$$v(F, G, t) = \max\left(\sup_{x\in M}\inf_{y\in M, |x-y|<t}(G(x) - F(y)), \sup_{y\in M}\inf_{x\in M, |x-y|<t}(F(x) - G(y))\right) \tag{20}$$

This extends directly to $\lambda \neq 1$.

If v verifies the following four properties, then the distance expressed in the form of Equation 19 is a probability metric [20]. Let us prove that these four properties hold for v in Equation 20.

1. If the probability that the two random variables underlying F and G are equal is equal to 1, then $v(F, G, t) = 0$ for all t. This property holds since in this case $F = G$ and the infimum in v is 0 (it is obtained for $x = y$).
2. $v(F, G, t) = v(G, F, t)$: this symmetry property holds by construction.
3. If $t < t'$ then $v(F, G, t) \geq v(F, G, t')$: indeed, if $t < t'$, then $|x-y| < t \Rightarrow |x-y| < t'$, and $\inf_{y\in M, |x-y|<t}(G(x) - F(y)) \geq \inf_{y\in M, |x-y|<t'}(G(x) - F(y))$. The same inequality holds for the second part of v, and finally $v(F, G, t) \geq v(F, G, t')$.

4. $\nu(F, G, t + t') \le \nu(F, H, t) + \nu(H, G, t')$: indeed, starting from $G(x) - F(y) = G(x) - H(z) + H(z) - F(y)$, and $(|x - z| < t'$ and $|y - z| < t) \Rightarrow |x - y| < t + t'$, we have (for all x, y, z):

$$\begin{aligned}\inf_{|x-y|<t+t'}(G(x) - F(y)) &\le G(x) - H(z) + \inf_{|y-z|<t}(H(z) - F(y)) \\ &\le G(x) - H(z) + \sup_z \inf_{|y-z|<t}(H(z) - F(y)) \\ &\le \inf_{|x-z|<t'}(G(x) - H(z)) + \nu(H, F, t)\end{aligned}$$

$$\Rightarrow \sup_x \inf_{|x-y|<t+t'}(G(x) - F(y)) \le \sup_x \inf_{|x-z|<t'}(G(x) - H(z)) + \nu(H, F, t)$$

Hence $\nu(F, G, t + t') \le \nu(G, H, t') + \nu(H, F, t)$.

A similar proof applies for the version with the absolute values in $|G(x)-F(y)|$, and with the scale factor λ.

□

5 Distances between periodic distributions

In this section we now assume periodic distributions. To fix the ideas, we set, without loss of generality, $\rho = 2\pi$ and $M = [0, 2\pi]$. The proposed approach relies on the link previously established between Lévy-Prokhorov distance and Hausdorff distance derived from a dilation. We show that this approach leads to a simple and elegant way to deal with the more complex case of periodic distributions.

5.1 Lévy and Prokhorov distances

Let us start again from d_{Pr}. We propose to express this distance from a circular dilation and by restricting the Borelian sets to $Z = [0, \theta]$ (which are generating all Borelian sets on $[0, 2\pi]$), taking 0 as origin, arbitrarily. If the origin is taken at θ_0, then the cumulative distribution is $\int_{\theta_0}^{\theta} f(t)dt = \int_0^{\theta} f(t)dt - \int_0^{\theta_0} f(t)dt = F(\theta) - F(\theta_0)$ if $\theta_0 \le \theta \le 2\pi$, and $\int_{\theta_0}^{2\pi} f(t)dt + \int_0^{\theta} f(t)dt = 1 - F_0(\theta_0) + F_0(\theta)$ if $0 \le \theta \le \theta_0$. If we want a distance which is independent of the choice of the origin, then $\inf_{\theta_0} d_L^c(F_{\theta_0}, G_{\theta_0})$ could be considered.

Let us define a dilation of size ε, in the positive direction, as: $\delta^{\varepsilon}(Z) = [0, \theta + \varepsilon]$ if $\theta + \varepsilon \le 2\pi$ and $[0, 2\pi]$ otherwise. This morphological expression allows us to derive easily the following result.

Proposition 17. *The Lévy distance, derived from the Prokhorov distance in 1D in the periodic case, is expressed as:*

$$d_L^c(F, G) = \inf\{\varepsilon > 0 \mid \forall\theta \in [0, 2\pi], F(\theta) \le G(\theta + \lambda\varepsilon) + \varepsilon \text{ and } G(\theta) \le F(\theta + \lambda\varepsilon) + \varepsilon\}. \tag{21}$$

by setting $G(\theta + \lambda\varepsilon) = F(\theta + \lambda\varepsilon) = 1$ if $\theta + \lambda\varepsilon \ge 2\pi$.

Proof. For $\lambda = 1$ (without loss of generality), we have for $\theta + \varepsilon \le 2\pi$:

$$\begin{aligned}f(Z) \le g(\delta^{\varepsilon}(Z)) + \varepsilon \Leftrightarrow \int_0^{\theta} f(t)dt &\le \int_0^{\theta+\varepsilon} g(t)dt + \varepsilon \\ &\Leftrightarrow F(\theta) - F(0) \le G(\theta + \varepsilon) - G(0) + \varepsilon \\ &\Leftrightarrow F(\theta) \le G(\theta + \varepsilon) + \varepsilon\end{aligned}$$

For $\theta + \varepsilon \ge 2\pi$, we have $f(Z) \le g(\delta^{\varepsilon}(Z)) + \varepsilon \Leftrightarrow F(\theta) \le 1 + \varepsilon$ which is always true. □

Note that it could be proved that any transport metric d_T can be extended to compare distributions with non equal masses, as suggested in [12] (Chapter 3.1/2.B), as follows:

$$D(f, g) = d_T\left(\frac{f}{f(X)}, \frac{g}{g(Y)}\right) + |f(X) - g(Y)|, \tag{22}$$

where X and Y are the supports of f and g, respectively. This expression wil be used for computing the distance between distributions normalized by the sup instead of the sum.

5.2 Morphological approach

5.2.1 Hausdorff distance from dilations of cumulative distributions

Proposition 18. *The Hausdorff distance derived from δ_{c1} computed with a symmetrical structuring element is:*

$$d_{Hc1}(F,G) = \max\Big(\sup_{\theta\in[0,2\pi]}\inf_{\theta'\in[0,2\pi]}\max\Big(\frac{|\theta-\theta'|}{\lambda}, |F(\theta')-G(\theta)|\Big), \sup_{\theta\in[0,2\pi]}\inf_{\theta'\in[0,2\pi]}\max\Big(\frac{|\theta-\theta'|}{\lambda}, |G(\theta')-F(\theta)|\Big)\Big).$$

The asymmetrical dilation δ_{c2} leads to similar results, and the derived Hausdorff distance has a similar expression, without the absolute values:

$$d_{Hc2}(F,G) = \max\Big(\sup_{\theta\in[0,2\pi]}\inf_{\theta'\in[0,2\pi]}\max\Big(\frac{|\theta-\theta'|}{\lambda}, G(\theta)-F(\theta')\Big), \sup_{\theta\in[0,2\pi]}\inf_{\theta'\in[0,2\pi]}\max\Big(\frac{|\theta-\theta'|}{\lambda}, F(\theta)-G(\theta')\Big)\Big).$$

Proof. As in Section 4.1 the proof is direct, by developing $SG \subseteq \delta^{\varepsilon,\lambda}(SF)$ and $SF \subseteq \delta^{\varepsilon,\lambda}(SG)$, for $\delta = \delta_{c1}$ and $\delta = \delta_{c2}$. □

The computation of $d_{Hc1}(F,G)$ is illustrated in Figure 11, where the minimal size of dilation of SF, such that it includes SG, is shown.

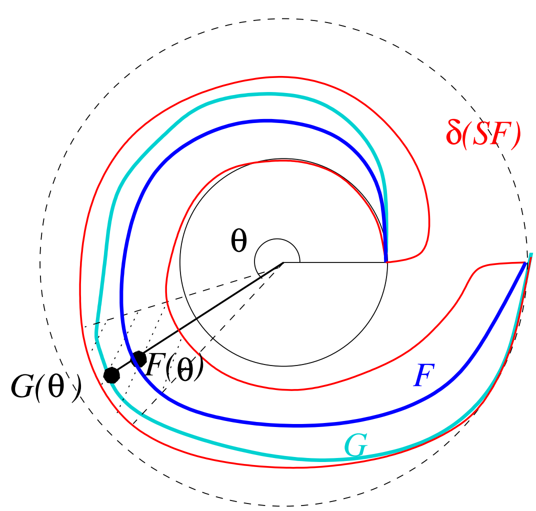

Figure 11: Dilation in the periodic case, for a symmetrical structuring element, such that the dilation of SF includes SG, illustrating the computation of the Hausdorff distance.

Proposition 19. *As in the non-periodic case, the Hausdorff distance derived from asymmetrical dilation and the Lévy distance are equal:*

$$d_{Hc2}(F,G) = d_L^c(F,G). \tag{23}$$

Proof. We detail the proof for $\lambda = 1$ to simplify the equations. Its extension to any λ is straightforward. We prove that $d^c_L(F, G) \leq \varepsilon \Leftrightarrow d_{Hc2}(F, G) \leq \varepsilon$.

$$\begin{aligned}
d^c_L(F, G) \leq \varepsilon \quad & \Rightarrow \quad \forall \theta \in [0, 2\pi], F(\theta) \leq G(\theta + \varepsilon) + \varepsilon \\
& \Rightarrow \quad \forall \theta \in [0, 2\pi], \exists \theta' \in [0, 2\pi] \text{ (e.g. } \theta' = \theta + \varepsilon), |\theta - \theta'| \leq \varepsilon, F(\theta) - G(\theta') \leq \varepsilon \\
& \Rightarrow \quad \forall \theta \in [0, 2\pi], \exists \theta' \in [0, 2\pi], \max(|\theta - \theta'|, F(\theta) - G(\theta')) \leq \varepsilon \\
& \Rightarrow \quad \sup_{\theta \in [0, 2\pi]} \inf_{\theta' \in [0, 2\pi]} \max(|\theta - \theta'|, F(\theta) - G(\theta')) \leq \varepsilon
\end{aligned}$$

and, from a similar derivation for the second term, we get $d_{Hc2}(F, G) \leq \varepsilon$. Conversely:

$$\begin{aligned}
d_{Hc2}(F, G) \leq \varepsilon \quad & \Rightarrow \quad \forall \theta \in [0, 2\pi], \exists \theta' \in [0, 2\pi], |\theta - \theta'| \leq \varepsilon, F(\theta) - G(\theta') \leq \varepsilon \\
& \Rightarrow \quad \forall \theta \in [0, 2\pi], \exists \theta' \in [0, 2\pi], \theta - \varepsilon \leq \theta' \leq \theta + \varepsilon, F(\theta) \leq G(\theta') + \varepsilon \leq G(\theta + \varepsilon) + \varepsilon
\end{aligned}$$

because G is increasing. The same reasoning applies to the second term, and finally $d^c_L(F, G) \leq \varepsilon$. □

5.2.2 Hausdorff distance from dilations of distributions

The definitions proposed in Equations 14 and 15 apply directly to periodic distributions, by considering appropriate dilations, taking the periodicity into account, as defined in Section 3.1.

An example of distribution on $[0, 2\pi]$ is given in Figure 12, with three translations. The Hausdorff distances values, computed using morphological dilations of the distributions (using Equation 15), between the first distribution of Figure 12 and the others, correspond to the distance between the cores of the distributions, as expected in this simple case.

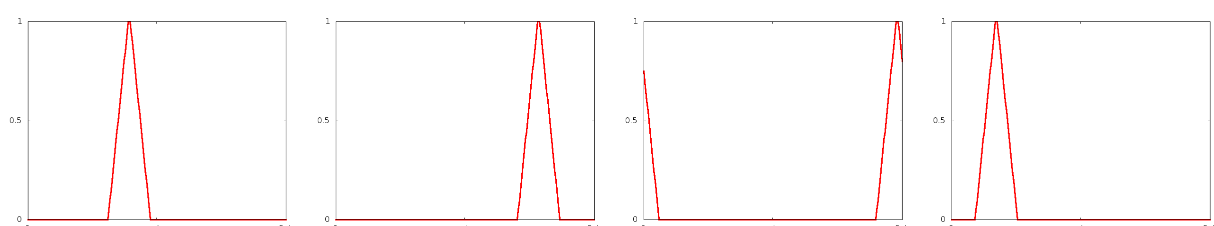

Figure 12: Example of distribution on $[0, 2\pi]$ and three translations ($T = 2.45$, $T = 3.68$, $T = 4.9$). The distances values (in radians) are 0 for $T = 0$, 2.45 for $T = 2.45$, 2.60 for $T = 3.68$, and 1.37 for $T = 4.9$.

6 Comparison between directional spatial relations

Observing the evolution of a pathology in medical images, or of soil occupation in remote sensing, detecting changes in video sequences, updating a spatial information system are examples that can all benefit from quantification and comparison of spatial relations between objects in the observed scenes. In this paper, to illustrate the proposed approaches, we consider spatial relations represented as distributions or fuzzy numbers, with the typical example of directional relations, represented as a periodic function on $[0, 2\pi]$ via the angle histogram [14]. The normalized angle histogram $ha_{A,B}$ between two 2D objects A and B is defined as:

$$\forall \theta \in [0, 2\pi], ha_{A,B}(\theta) = \frac{h'_{A,B}(\theta)}{\sup_{\theta' \in [0, 2\pi]} h'_{A,B}(\theta')},$$

with

$$h'_{A,B}(\theta) = |\{(a,b), a \in A, b \in B \mid \angle(a,b) = \theta\}|$$

and $\angle(a,b)$ the angle modulo 2π between the vector $\vec{ab}$ and the horizontal axis. This sum is further weighted by the membership values of a to A and of b to B if the objects are fuzzy.

Let us consider, as an example, the application of the proposed approach to quantify the evolution of directional relations between objects in a simulated video sequence (Figure 13). The grey object gets close to the white one in a constant direction, and then changes direction and goes away. The angle histograms ha between these two objects are also illustrated in this figure.

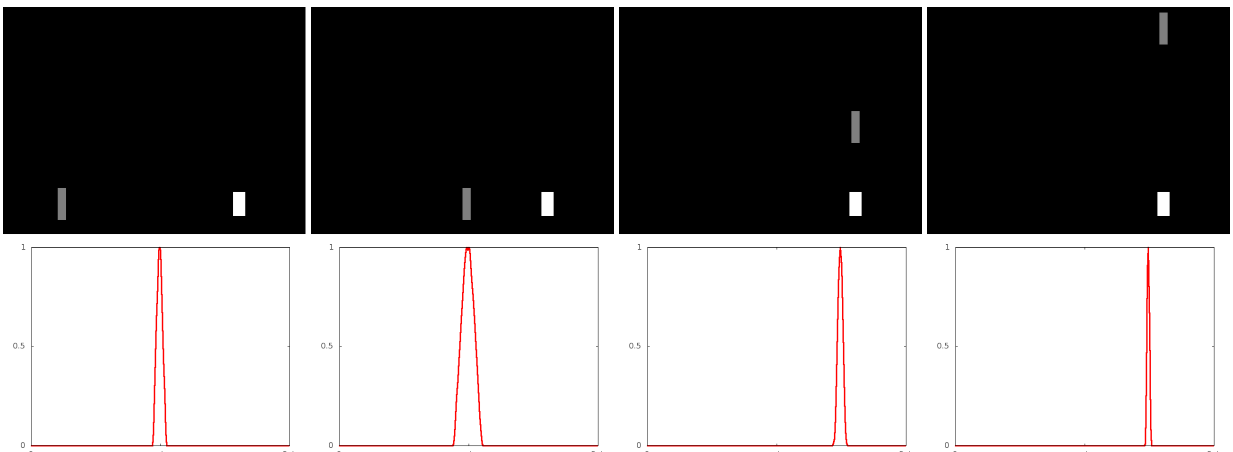

Figure 13: Simulated video sequence (top, some frames) and angle histograms (bottom). The grey object gets close to the white one in a constant direction, and then changes direction and goes away in this second direction.

These histograms have been compared using the different proposed measures, by computing the distance between the histogram at time t and the histogram in the first frame. The curves showing the evolution of this distance along time are displayed in Figure 14 for the morphological Hausdorff distance (Equation 15, using periodic fuzzy dilations) and for the Lévy-Prokhorov distance (Equation 21 when distributions are normalized by the sum, hence of equal masses, and Equation 22 if they are normalized by the sup). In all these curves a jump is observed at the instant where the change in direction occurs, which was expected. We can also notice the strong similarity between these curves.

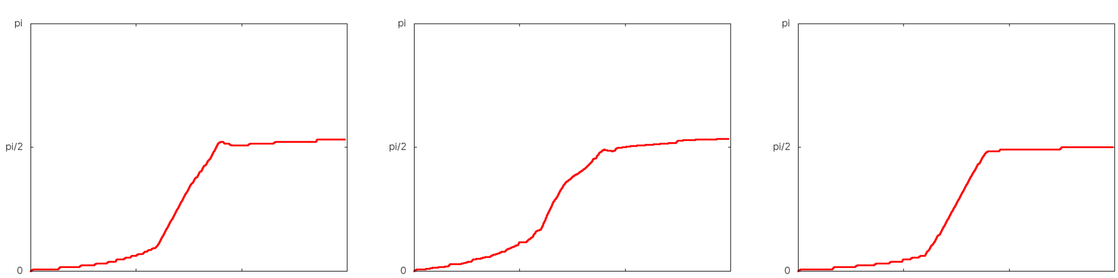

Figure 14: Morphological Hausdorff distances between the histogram in each frame and the one in the first frame (left). Lévy-Prokhorov distance between the histogram in each frame and the one in the first frame, for histograms normalized by the sup (middle) and by the sum (right).

Let us now consider a second example, where an object turns around another one and then moves away. A few frames are displayed in Figure 15, and the corresponding angle histograms in Figure 16. The Hausdorff distance between the histogram of each frame and the one in the first frame is shown in Figure 17, and again is consistent with what was intuitively expected. The Lévy-Prokhorov distance is illustrated in the same figure. As for the first example, the curves are very similar.

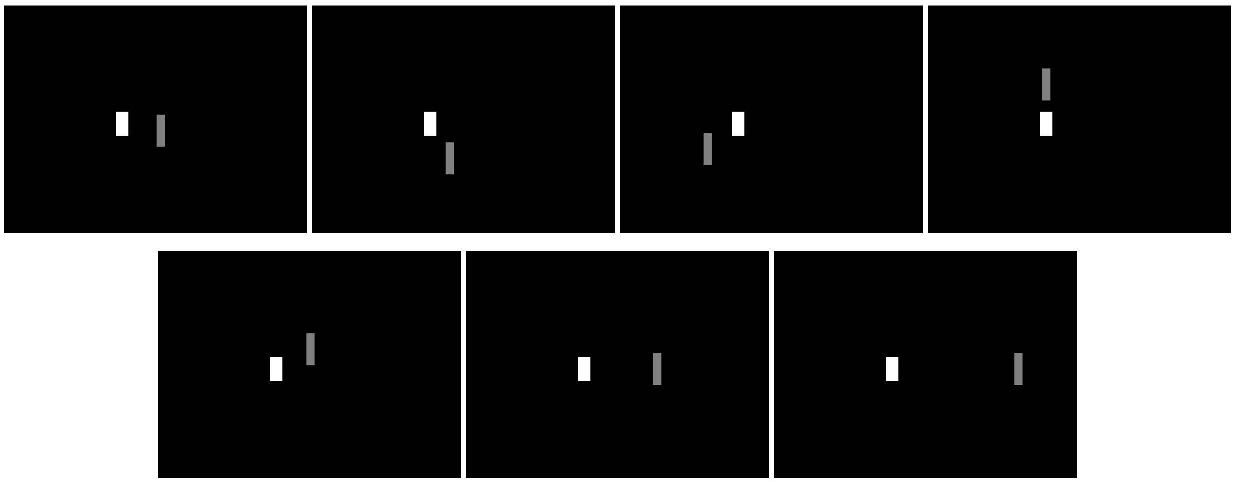

Figure 15: A few frames from a synthetic video sequence. The gray object turns around the white one and then moves away in a constant direction.

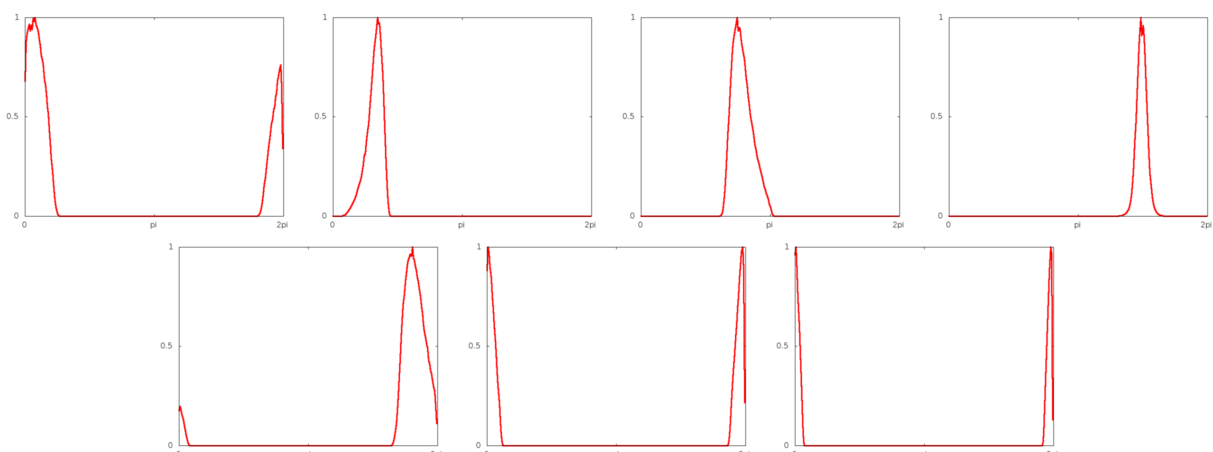

Figure 16: Angle histograms between the two objects in Figure 15.

In the third example, an object is getting closer to another one, turns above it, goes away in the opposite direction, completes the turn below and moves until the initial position. Some frames are displayed in Figure 18, with the corresponding histograms in Figure 19. The Hausdorff distance between the histogram of each frame and the one in the first frame is shown in Figure 20, as well as the Lévy-Prokhorov distance. As for the previous examples, the curves are very similar, and well fit the intuition. This was also observed on other video examples.

All this extends to other spatial relations, such as distances.

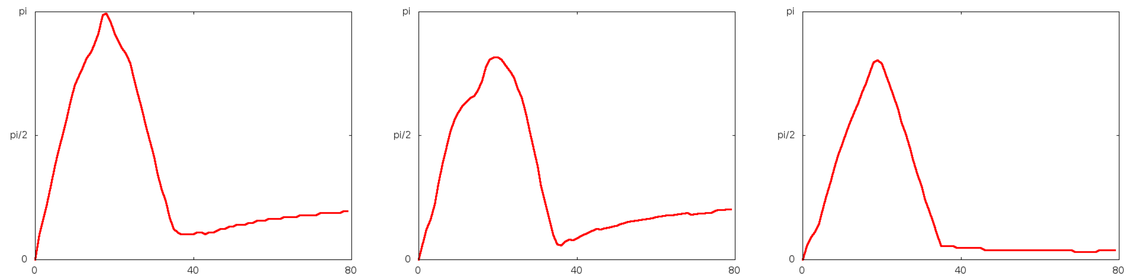

Figure 17: Left: Hausdorff distance between the angle histogram in each frame and the one in the first frame, for the sequence in Figure 15. Middle: Lévy-Prokhorov distance when histograms are normalized by the sup. Right: Lévy-Prokhorov distance when histograms are normalized by the sum.

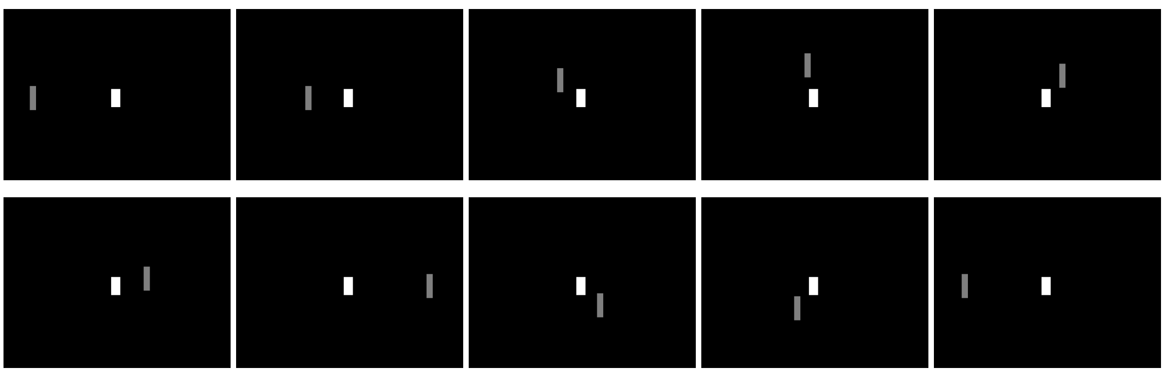

Figure 18: A few frames from a synthetic video sequence. The gray object gets close to the white one, turns above it, moves away in a constant direction, turns below it and moves to its initial position.

7 Conclusion

In this paper we have investigated several forms of Hausdorff distances for comparing distributions or cumulative distributions. A first contribution relies in the definition of several dilations of distributions. This allows deriving Hausdorff distances from their expression in terms of dilations. Then, based on existing definitions and new ones proposed in this paper, we have exhibited interesting links between optimal transport metrics, in particular Lévy-Prokhorov distance, and morphological ones. In particular, these links have allowed adaptations and extensions to the case of periodic distributions, which would have been more difficult to address otherwise. This is another important contribution of this work.

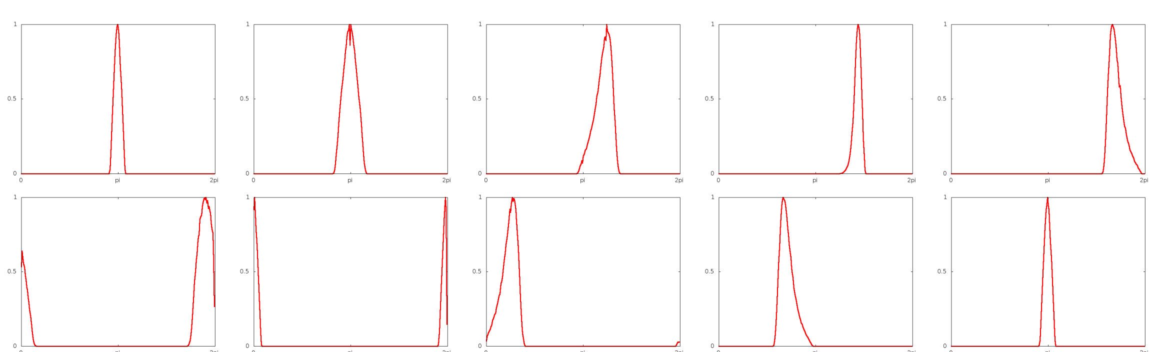

Figure 19: Angle histograms between the two objects in Figure 18.

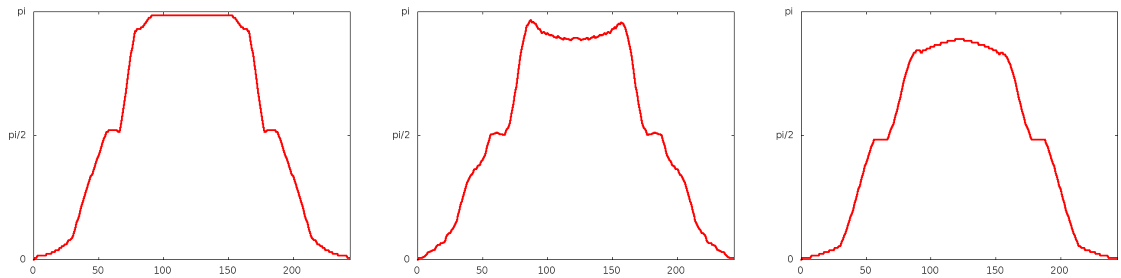

Figure 20: Left: Hausdorff distance between the angle histogram in each frame and the one in the first frame, for the sequence in Figure 18. Middle: Lévy-Prokhorov distance when histograms are normalized by the sup. Right: Lévy-Prokhorov distance when histograms are normalized by the sum.

As an illustration, we have shown that the proposed distances allow comparing spatial relations between objects in images or videos, represented as distributions. This could lead to future applications for detection of ruptures in temporal sequences [1], for comparing different spatial configurations of objects, as a guide for structural recognition and scene understanding, and more generally for spatial reasoning.

In our future work we will investigate extensions to higher dimensions, and we anticipate that the increased complexity of the transport approach will be overcome by the equivalence with morphological expressions.

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