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Modal Logics Based on Mathematical Morphology for Qualitative Spatial Reasoning

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Modal Logics Based on Mathematical Morphology for Qualitative Spatial Reasoning

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ABSTRACT. *We propose in this paper to construct modal logics based on mathematical morphology. The contribution of this paper is twofold. First we show that mathematical morphology can be used to define modal operators in the context of normal modal logics. We propose definitions of modal operators as algebraic dilations and erosions, based on the notion of adjunction. We detail the particular case of morphological dilations and erosions, and of there compositions, as opening and closing. An extension to the fuzzy case is also proposed. Then we show how this can be interpreted for spatial reasoning by using qualitative symbolic representations of spatial relationships (topological and metric ones) derived from mathematical morphology. This allows to establish some links between numerical and symbolic representations of spatial knowledge.*

KEYWORDS: *Modal Logics, Mathematical Morphology, Adjunction, Dilation, Erosion, Spatial Relations, Qualitative Spatial Reasoning.*

1. Introduction

When looking at the algebraic properties of mathematical morphology operators on the one hand, and of modal logic operators on the other hand, several similarities can be shown, and suggest that links between both theories are worth to be investigated. We propose in this paper to define a pair of modal operators $(\Box, \Diamond)$ as morphological erosion and dilation. Extending the work presented in [BLO 00c], we address the more general case of algebraic dilations and erosions and define $(\Box, \Diamond)$ as an adjunction, which is a fundamental notion in mathematical morphology [HEI 90].

Mathematical morphology provides tools for spatial reasoning at several levels. The notion of structuring element captures the local spatial context and leads to anal-

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ysis of a scene using operators involving the neighborhood of each point. At a more global level, several spatial relations between spatial entities can be expressed as morphological operations, in particular using dilations. Therefore mathematical morphology leads also to structural analysis of a scene.

The importance of relationships between objects has been highlighted in very different types of works: in vision, for identifying shapes and objects, in database system management, for supporting spatial data and queries, in artificial intelligence, for planning and reasoning about spatial properties of objects, in cognitive and perceptual psychology, in geography, for geographic information systems, etc.

Usually vision and image processing make use of quantitative representations of spatial relationships. In artificial intelligence, mainly symbolic representations are developed (see [VIE 97] for a survey). Limitations of purely qualitative reasoning have already been stressed in [DUT 91], as well as the interest of adding semiquantitative extension to qualitative value (as done in the fuzzy set theory for linguistic variables [ZAD 75, DUB 80]) for deriving useful and practical conclusions (as for recognition). An example can be found in [GUE 96] based on Allen's intervals. Purely quantitative representations are limited in the case of imprecise statements, and of knowledge expressed in linguistic terms. On the other hand, communication about spatial knowledge is often simpler in a linguistic way, as stressed in [DEN 96]. For instance reasoning with words about geographical information is becoming an important field [GUE 98]. In [BLO 00b] we proposed to integrate both quantitative and qualitative knowledge, using semiquantitative interpretation of fuzzy sets. As already mentioned in [FRE 75], this allows to provide a computational representation and interpretation of imprecise spatial constraints, expressed in a linguistic way, possibly including quantitative knowledge. In this paper, spatial relationships have been derived from morphological operations applied to reference objects and represented as spatial fuzzy sets.

Until now mathematical morphology has been used mainly for quantitative representations of spatial relations. For qualitative spatial reasoning, several symbolic approaches have been developed, but mathematical morphology has not been used in this context to our knowledge. In this paper we show how modal operators based on morphological operators can be used for symbolic representations of spatial relations.

The lattice structure of formulas is briefly recalled in Section 2. In Section 3, we show that modal operators can be constructed from morphological dilations and erosions, as introduced in [BLO 00c]. In Section 4, we introduce a new way to build modal operators, from the notion of adjunction and from algebraic dilations and erosions. Morphological dilation and erosion constitute a particular case. We show that conversely, any modal logic which satisfies a number of axioms can be characterized in terms of algebraic dilations and erosions. In Section 5 we define modal operators from morphological opening and closing. An extension to the fuzzy case is proposed in Section 6. Then we show how this can be interpreted for spatial reasoning by using qualitative representations of spatial relationships derived from mathematical morphology (Section 7). This allows to establish some links between numerical and symbolic representations of spatial knowledge. The interest of such links has been for instance highlighted in [DEN 96] in the context of spatial cognition.

In a similar way as in [JEA 94], the modal operators are used here for representing spatial relationships, while classical predicates represent the semantic part of the information. In [JEA 94], inclusion and adjacency are considered. Here we consider more spatial relationships, including metric ones, and model all of them using mathematical morphology. This is an original aspect of our work, which has not been used in a logical setting until now, although the modal flavor of mathematical morphology has been briefly mentioned in [AIE 99], but without further development.

2. Notations and lattice structure

Let PS be a finite set of propositional symbols. The language is generated by PS, the usual connectives, and modal operators that will be defined in the following. The set of formulas is denoted by Φ. We will use standard Kripke's semantics and denote by $\mathcal{M}$ a model composed of a set of worlds Ω, a binary relation R between worlds and a truth valuation. For any φ in Φ, $Mod(\varphi) = \{\omega \in \Omega \mid \omega \models \varphi\}$ is the set of worlds in which φ is satisfied in the model $\mathcal{M}$. For any subset B of Ω, we define $B \models \varphi$ as $\forall \omega \in B, \omega \models \varphi$ (i.e. $B \subseteq Mod(\varphi)$).

Morphological operations on logical formulas have been proposed in [BLO 00d], by exploiting equivalences between logical and set theoretical notions and by identifying a formula φ (and all equivalent formulas) with $Mod(\varphi)$.

Considering the inclusion relation on 2^Ω, $(2^\Omega, \subseteq)$ is a complete lattice. Similarly a lattice is defined on $\Phi_\equiv$, where $\Phi_\equiv$ denotes the quotient space of Φ by the equivalence relation between formulas (with the equivalence defined as $\varphi \equiv \psi$ iff $Mod(\varphi) = Mod(\psi)$). In the following, this will be implicit assumed, and we will simply use the notation Φ. Any subset $\{\varphi_i\}$ of Φ has a supremum $\vee_i \varphi_i$, and an infimum $\wedge_i \varphi_i$ (corresponding respectively to union and intersection in 2^Ω). The greatest element is $\top$ and the smallest one is $\bot$ (corresponding respectively to 2^Ω and $\emptyset$). This lattice structure is important for the algebraic point of view of mathematical morphology, as will be seen in Section 4. Indeed, it is the fundamental structure on which adjunctions can then be defined.

We define a canonical formula φ_ω associated with a world ω by:

$$Mod(\varphi_\omega) = \{\omega\}. \tag{1}$$

Let $\mathcal{C}$ be the subset of Φ containing all canonical formulas. The canonical formulas are sup-generating, i.e:

$$\forall \varphi \in \Phi,\ \exists \{\varphi_i\} \subseteq \mathcal{C},\ \varphi \equiv \vee_i \varphi_i. \tag{2}$$

The formulas φ_i are associated with the worlds ω_i which satisfy φ: for all ω_i such that $\omega_i \models \varphi$, $\varphi_i \equiv \varphi_{\omega_i}$. This decomposition will be used in some proofs, in particular in Section 4.

3. Modal operators from morphological dilations and erosions

In this Section, we show that morphological erosions and dilations can be used for defining modal operators $\Box$ and $\Diamond$ having several interesting properties.

3.1. *Morphological dilation and erosion of formulas*

Let us first recall the definitions of dilation and erosion of a set X (typically $X \subseteq \mathbb{R}^n$) by a structuring element B in $\mathbb{R}^n$, denoted respectively by $D_B(X)$ and $E_B(X)$ [SER 82]:

$$D_B(X) = \{x \in \mathbb{R}^n \mid B_x \cap X \neq \emptyset\}, \tag{3}$$

$$E_B(X) = \{x \in \mathbb{R}^n \mid B_x \subset X\}, \tag{4}$$

where B_x denotes the translation of B at x,

In these equations, B defines a neighborhood that is considered at each point. It can also be seen as a relationship between points.

The most important properties of dilation and erosion are the following ones [SER 82]:

– monotonicity: if $X \subseteq Y$, then $D_B(X) \subseteq D_B(Y)$ and $E_B(X) \subseteq E_B(Y)$; if $B \subseteq B'$, then $D_B(X) \subseteq D_{B'}(X)$ and $E_{B'}(X) \subseteq E_B(X)$;

– extensivity of dilation and anti-extensivity of erosion if the origin belongs to B: $X \subseteq D_B(X)$, $E_B(X) \subseteq X$;

– iteration property: dilating (eroding) a set successively by two structuring elements is equivalent to perform one dilation (erosion) by the sum of the structuring elements;

– dilation commutes with union and erosion with intersection;

– duality with respect to complementation: $E_B(X) = [D_B(X^C)]^C$.

Using the previous equivalences, and based on set definitions of morphological operators [SER 82], dilation and erosion of a formula φ have been defined in [BLO 00d] as follows:

$$Mod(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap Mod(\varphi) \neq \emptyset\}, \tag{5}$$

$$Mod(E_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \models \varphi\}. \tag{6}$$

In these equations, the structuring element B represents a relationship between worlds, i.e. $\omega' \in B(\omega)$ iff ω' satisfies some relationship to ω. The condition in Equation 5 expresses that the set of worlds in relation to ω should be consistent with φ, i.e.:

$$\exists \omega' \in B(\omega),\ \omega' \models \varphi.$$

The condition in Equation 6 is stronger and expresses that φ should be satisfied in all worlds in relation to ω.

The main properties of dilation and erosion also hold in the logical setting proposed here. They are detailed in [BLO 00d].

These definitions are particular cases of the more general definition of algebraic dilations and erosions that will be presented in Section 4.

3.2. *Structuring element as accessibility relation*

The structuring element B representing a relationship between worlds defines a "neighborhood" of worlds. If it is symmetrical, it leads to symmetrical structuring elements. If it is reflexive, it leads to structuring elements such that $\omega \in B(\omega)$, which leads to interesting properties, as will be seen later. Here we define this relationship as an accessibility relation as in normal modal logics [HUG 68, CHE 80].

An interesting way to choose the relationship is to base it on distances between worlds, which is an important information in spatial reasoning. This allows to define sequences of increasing structuring elements defined as the balls of a distance. For any distance δ between worlds, a structuring element of size n centered at ω takes the following form:

$$B^n(\omega) = \{\omega' \in \Omega \mid \delta(\omega, \omega') \leq n\}. \tag{7}$$

For instance a distance equal to 1 can represent a connectivity relation between worlds, defined for instance as a difference of one literal (i.e. one propositional symbol taking different truth values in both worlds).

To illustrate this, we make use of a graph representation of worlds, where each node represents a world and a link represents an elementary connection between two worlds, i.e. being at distance 1 from each other. A ball of radius 1 centered at ω is constituted by ω and the ends of the arcs originating in ω. This allows for an easy visualization of the effects of transformations. Let us consider an example with three propositional symbols a, b, c. The possible worlds are represented in Figure 1. Two successive dilations of the formula $\varphi = \neg a \wedge b \wedge c$ are shown as well.

Another way to choose the relationship is to rely on an indistinguishability relation between worlds [ORL 93, BAL 99], for instance based on spatial attributes of spatial entities represented by these worlds. Interestingly enough, as shown in [ORL 93], modal logics based on such relationships show some links with Pawlak's work on rough sets and rough logic [PAW 82, PAW 87], while rough sets can be constructed from morphological operators as shown in [BLO 00a]. This consideration suggests some further links to be exploited between these different approaches.

We define an accessibility relation from any structuring element B as follows:

$$R(\omega, \omega') \text{ iff } \omega' \in B(\omega). \tag{8}$$

Conversely, a structuring element can be defined from an accessibility relation using this equivalence.

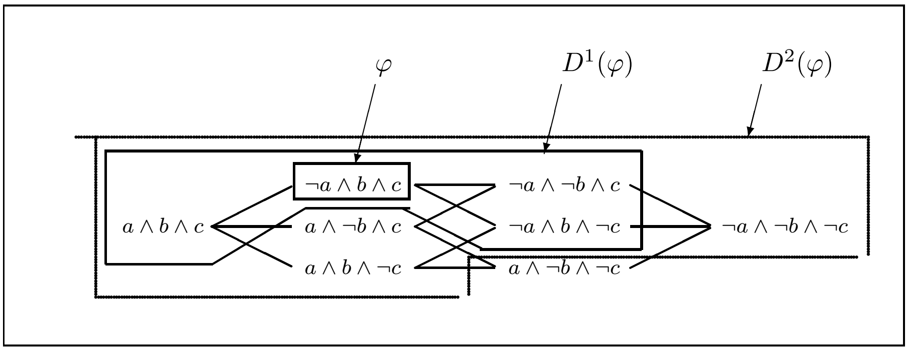

Figure 1. *Graph representation of possible worlds with 3 symbols and an example of φ and two successive dilations. An arc between two nodes means that the corresponding nodes are at distance 1.*

The accessibility relation R is reflexive iff $\forall \omega \in \Omega,\ \omega \in B(\omega)$. It is symmetrical iff $\forall (\omega, \omega') \in \Omega^2,\ \omega \in B(\omega')$ iff $\omega' \in B(\omega)$ (this is the case in the example of Figure 1). In general, accessibility relations derived from a structuring element are not transitive. Indeed in general if $\omega' \in B(\omega)$ and $\omega'' \in B(\omega')$, we do not necessarily have $\omega'' \in B(\omega)$.

3.3. *Modal logic from morphological dilations and erosions*

Modal operators $\Box$ and $\Diamond$ are usually defined from an accessibility relation as [CHE 80]:

$$\mathcal{M}, \omega \models \Box\varphi \text{ iff } \forall \omega' \in \Omega \text{ such that } R(\omega, \omega'),\ \text{then } \mathcal{M}, \omega' \models \varphi, \tag{9}$$

$$\mathcal{M}, \omega \models \Diamond\varphi \text{ iff } \exists \omega' \in \Omega,\ R(\omega, \omega') \text{ and } \mathcal{M}, \omega' \models \varphi, \tag{10}$$

where $\mathcal{M}$ is a standard model related to R, that we will omit in the following in order to simplify notations (it will be always implicitly related to the considered accessibility relation).

Equation 9 can be rewritten as:

$$\begin{aligned} \omega \models \Box\varphi &\Leftrightarrow \{\omega' \in \Omega \mid R(\omega, \omega')\} \models \varphi \\ &\Leftrightarrow \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \models \varphi \\ &\Leftrightarrow B(\omega) \models \varphi, \end{aligned}$$

which corresponds exactly to the definition of the erosion of a formula as defined in Equation 6.

Similarly, Equation 10 can be rewritten as:

$$\begin{aligned} \omega \models \Diamond\varphi &\Leftrightarrow \{\omega' \in \Omega \mid R(\omega, \omega')\} \cap Mod(\varphi) \neq \emptyset \\ &\Leftrightarrow \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \cap Mod(\varphi) \neq \emptyset \\ &\Leftrightarrow B(\omega) \cap Mod(\varphi) \neq \emptyset, \end{aligned}$$

which exactly corresponds to a dilation according to Equation 5.

This shows that we can define modal operators based on an accessibility relation as erosion and dilation with a structuring element:

$$\Box\varphi \equiv E_B(\varphi), \tag{11}$$

$$\Diamond\varphi \equiv D_B(\varphi). \tag{12}$$

3.4. *Properties*

Theorem 1 of [BLO 00c] summarizes the axioms and inference rules that are satisfied by these modal operators, based on properties of morphological operators and on equivalences between set theoretical and logical concepts. They are detailed below.

THEOREM 1. — *The modal logic built from morphological erosions and dilations has the following theorems and rules of inference (we use similar notations as in [CHE 80]):*

– **T***:* $\Box\varphi \to \varphi$ *and* $\varphi \to \Diamond\varphi$ *if* $\forall\omega \in \Omega, \omega \in B(\omega)$ *(reflexive accessibility relation).*
– **Df***:* $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ *and* $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$.
– **D***:* $\Box\varphi \to \Diamond\varphi$ *iff* R *is serial (or in other words,* $\forall\omega \in \Omega, B(\omega) \neq \emptyset$*).*
– **B***:* $\Diamond\Box\varphi \to \varphi$ *and* $\varphi \to \Box\Diamond\varphi$.
– **5c***:* $\Box\Diamond\varphi \to \Diamond\varphi$ *and* $\Box\varphi \to \Diamond\Box\varphi$ *if* $\forall\omega \in \Omega, \omega \in B(\omega)$.
– **4c***:* $\Box\Box\varphi \to \Box\varphi$ *and* $\Diamond\varphi \to \Diamond\Diamond\varphi$ *if* $\forall\omega \in \Omega, \omega \in B(\omega)$.
– **N***:* $\Box\top$ *and* $\neg\Diamond\perp$.
– **M***:* $\Box(\varphi \wedge \psi) \to (\Box\varphi \wedge \Box\psi)$ *and* $(\Diamond\varphi \vee \Diamond\psi) \to \Diamond(\varphi \vee \psi)$.
– **M'***:* $\Diamond(\varphi \wedge \psi) \to (\Diamond\varphi \wedge \Diamond\psi)$ *and* $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$.
– **C***:* $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$ *and* $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$.
– **R***:* $(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi)$ *and* $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$.
– **RN***:*

$$\frac{\varphi}{\Box\varphi}.$$

– **RM***:*

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \; and \; \frac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}.$$

– **RR***:*

$$\frac{(\varphi \wedge \varphi') \rightarrow \psi}{(\Box\varphi \wedge \Box\varphi') \rightarrow \Box\psi} \; and \; \frac{(\varphi \vee \varphi') \rightarrow \psi}{(\Diamond\varphi \vee \Diamond\varphi') \rightarrow \Diamond\psi}.$$

– **RE***:*

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi} \; and \; \frac{\varphi \leftrightarrow \psi}{\Diamond\varphi \leftrightarrow \Diamond\psi}.$$

– **K***:* $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ *and by duality* $(\neg\Diamond\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\neg\varphi \wedge \psi)$.

PROOF. — These properties are deduced from the algebraic properties of morphological operators, using the equivalences between set theoretical concepts and logical ones:

– **T** comes from the anti-extensivity of erosion and from the extensivity of dilation for structuring elements derived from a reflexive relation.

– **Df** corresponds to the duality between erosion and dilation with respect to complementation (negation of formulas).

– **D** holds iff the accessibility relation is serial, i.e. $\forall\omega \in \Omega, \exists\omega' \in \Omega,\; R(\omega, \omega')$, which is equivalent to $\forall\omega \in \Omega,\; B(\omega) \neq \emptyset$. This is in particular true if R is reflexive (and then **D** can be simply derived from **T**).

– **B** comes from the extensivity of closing (dilation followed by an erosion) and from the anti-extensivity of opening (erosion followed by a dilation).

– **5c** is **T** applied to $\Diamond\varphi$.

– **4c** is **T** applied to $\Box\varphi$. It corresponds to the fact that accessibility relations constructed from structuring elements are weakly dense, i.e. we have $\forall(\omega, \omega') \in \Omega^2,\; R(\omega, \omega') \rightarrow \exists\omega'' \in \Omega,\; R(\omega, \omega'') \wedge R(\omega'', \omega')$. Indeed, at least if the relation is reflexive, if we have $\omega' \in B(\omega)$ then $\exists\omega'' \in B(\omega) \cap B(\omega')$. Dual expressions hold for $\Diamond$.

– **M**, **C** and **R** come from the fundamental property of dilation (respectively erosion) which commutes with union or disjunction (respectively with intersection or conjunction).

– Increasingness of both operators leads to **RM** (monotonicity).

– **RR** is deduced from **RM** and **R**, and **RE** from **RM** applied to $\varphi \rightarrow \psi$ and to $\psi \rightarrow \varphi$.

– Dilatation does not commute with intersection and only an inclusion holds, leading to **M'**. Similarly, erosion does not commute with union.

– **RN** is derived from **N** and **RR** (see e.g. [CHE 80]).

– Since **K** is not directly derived from a property that is usually found in textbooks about mathematical morphology, we give here the main lines of the proof. From $((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \psi$ we deduce, using monotonicity, $\Box((\varphi \rightarrow \psi) \wedge \varphi) \rightarrow \Box\psi$, and using **M** and **C**, $(\Box(\varphi \rightarrow \psi) \wedge \Box\varphi) \rightarrow \Box\psi$. Using classical rules of propositional logic, we deduce $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$. ■

Since the proposed system contains **Df**, **N**, **C** and is closed by **RM**, it is a normal modal logic [CHE 80].

THEOREM 2. — *On the contrary, the following expressions are not satisfied in general:*

– **5**: $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ *(since the dilation followed by an erosion is a closing which does not necessarily contains the dilation).*

– **4**: $\Box\varphi \rightarrow \Box\Box\varphi$ *(since eroding a region twice produces a smaller region).*

PROOF. — **5** is not satisfied because accessibility relations derived from structuring elements are in general not Euclidean, i.e. the following property does not hold: $\forall(\omega,\omega',\omega'') \in \Omega^3,\ R(\omega,\omega') \wedge R(\omega,\omega'') \rightarrow R(\omega',\omega'')$. Let us consider the example of Figure 1 and show a counter-example: let

$$\varphi \equiv (a \wedge b \wedge c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c).$$

Then we have:

$$\Diamond\varphi \equiv a \vee b \vee c$$

and

$$\Box\Diamond\varphi \equiv (a \wedge b) \vee (b \wedge c) \vee (a \wedge c)$$

and $\Diamond\varphi \rightarrow \Box\Diamond\varphi$ does not hold.

Similarly, **4** is not satisfied since in general R is not transitive and we can have $R(\omega,\omega') \wedge R(\omega',\omega'')$ but $\neg R(\omega,\omega'')$. ■

Let us now denote by $\Box^n$ the iteration of n times $\Box$ (i.e. n erosions by the same structuring element). Since the succession of n erosions by a structuring element is equivalent to one erosion by a larger structuring element, of size n (iterativity property of erosion), $\Box^n$ is a new modal operator, constructed as in Equation 11. In a similar way, we denote by $\Diamond^n$ the iteration of n times $\Diamond$, which is again a new modal operator, due to iterativity property of dilation, constructed as in Equation 12 with a structuring element of size n. We set $\Box^1 = \Box$ and $\Diamond^1 = \Diamond$.

We also have the following theorems:

– $\Box^n\Box^{n'}\varphi \leftrightarrow \Box^{n+n'}\varphi$, and $\Diamond^n\Diamond^{n'}\varphi \leftrightarrow \Diamond^{n+n'}\varphi$ (iterativity properties of dilation and erosion).

– $\Diamond\Box\Diamond\Box\varphi \leftrightarrow \Diamond\Box\varphi$, and $\Box\Diamond\Box\Diamond\varphi \leftrightarrow \Box\Diamond\varphi$ (idempotence of opening and closing). This is actually a theorem from any KB logic: $\Diamond\Box\Diamond\Box\varphi \rightarrow \Diamond\Box\varphi$ is **B** applied to $\Diamond\Box\varphi$ and $\Diamond\Box\varphi \rightarrow \Diamond\Box\Diamond\Box\varphi$ comes from **B** applied to $\Box\varphi$ and from **RM**.

– More generally, we derive from properties of opening and closing the following theorems:

$$\Diamond^n\Box^n\Diamond^{n'}\Box^{n'}\varphi \leftrightarrow \Diamond^{\max(n,n')}\Box^{\max(n,n')}\varphi,$$

and

$$\Box^n\Diamond^n\Box^{n'}\Diamond^{n'}\varphi \leftrightarrow \Box^{\max(n,n')}\Diamond^{\max(n,n')}\varphi.$$

– For $n < n'$, the following expressions are theorems: $\Diamond^n\varphi \rightarrow \Diamond^{n'}\varphi$, $\Box^{n'}\varphi \rightarrow \Box^n\varphi$, $\Box^n\Diamond^n\varphi \rightarrow \Box^{n'}\Diamond^{n'}\varphi$, $\Diamond^{n'}\Box^{n'}\varphi \rightarrow \Diamond^n\Box^n\varphi$.

4. Modal operators from adjunction

In this Section, we consider the more general framework of algebraic erosions and dilations and the fundamental property of adjunction [HEI 90].

4.1. *Adjunction, algebraic erosions and dilations*

Algebraic erosions and dilations are defined as operations in a lattice that commute with infimum and supremum respectively. We do not make any additional assumption, and in particular we do not refer to any structuring element.

Generalizing the definitions of [BLO 00d], we define here an algebraic dilation δ on Φ as an operation which commutes with disjunction, and an algebraic erosion ε as an operation which commutes with conjunction, i.e. we have the two following expressions for any family $\{\varphi_i\}$:

$$\delta(\vee_i\varphi_i) \equiv \vee_i\delta(\varphi_i), \quad (13)$$

$$\varepsilon(\wedge_i\varphi_i) \equiv \wedge_i\varepsilon(\varphi_i). \quad (14)$$

One of the fundamental properties in the algebraic framework is the one of adjunction [HEI 90]. A pair of operators (ε, δ) on sets is an adjunction iff $\forall(X, Y), \delta(X) \subseteq Y$ iff $X \subseteq \varepsilon(Y)$. It can be proved that if (ε, δ) is an adjunction, then ε is an algebraic erosion and δ is an algebraic dilation.

In this Section, we use similar concepts on Φ for defining modal operators. A pair of modal operators $(\Box, \Diamond')$ is an adjunction on Φ iff:

$$\forall(\varphi, \psi) \in \Phi^2,\ \models (\Diamond'\varphi \rightarrow \psi\ \equiv\ \varphi \rightarrow \Box\psi), \quad (15)$$

or in other words:

$$\frac{\varphi \rightarrow \Box\psi}{\Diamond'\varphi \rightarrow \psi} \text{ and } \frac{\Diamond'\varphi \rightarrow \psi}{\varphi \rightarrow \Box\psi}.$$

In terms of worlds, this can also be expressed as:

$$\forall(\varphi, \psi) \in \Phi^2,\ Mod(\Diamond'\varphi) \subseteq Mod(\psi) \text{ iff } Mod(\varphi) \subseteq Mod(\Box\psi). \quad (16)$$

At this point, we use the notation $(\Box, \Diamond')$ instead of the classical notation $(\Box, \Diamond)$ because, as will be seen later, the two operators are not necessarily dual. In general they are two different modal operators.

THEOREM 3. — *If* $(\Box, \Diamond')$ *is an adjunction on* Φ*, then* $\Box$ *is an algebraic erosion, and* $\Diamond'$ *is an algebraic dilation, i.e. for any family* $\{\varphi_i\}$*, we have:*

$$\Box \wedge_i \varphi_i \equiv \wedge_i \Box \varphi_i, \tag{17}$$

$$\Diamond' \vee_i \varphi_i \equiv \vee_i \Diamond' \varphi_i. \tag{18}$$

These equivalences are also true for empty families, since we have $\Diamond' \bot \equiv \bot$.

The proof is similar as the one for adjunctions on sets (see e.g. [HEI 90]).

4.2. *Properties*

THEOREM 4. — *Let* $(\Box, \Diamond')$ *be an adjunction on the lattice of logical formulas. The modal logic based on these operators has the following theorems and rules of inference (we use similar notations as in Theorem 1 but* $\Diamond$ *has to be replaced by* $\Diamond'$*):* **B**, **N**, **M**, **M'**, **C**, **R**, **RN**, **RM**, **RR**, **RE**, **K**.

The proof is derived mainly from Theorem 3, from Equations 2 and 15-18 and from the following result:

THEOREM 5. — *We can write* $\Box$ *and* $\Diamond'$ *as:*

$$\Box\varphi \equiv \vee\{\psi \in \Phi,\ \Diamond'\psi \to \varphi\}, \tag{19}$$

$$\Diamond'\varphi \equiv \wedge\{\psi \in \Phi,\ \varphi \to \Box\psi\}. \tag{20}$$

Again formulas are considered up to the equivalence relation, and therefore $\vee$ and $\wedge$ are taken over a finite family.

PROOF (OF THEOREM 4). — More precisely, **M**, **C** and **R** are deduced from Theorem 3. This proposition implies **RM**, which leads to **RR** and **RE**. **RM** can also be deduced from Theorem 5. **K** is deduced from monotonicity, **M**, **C** and classical rules of propositional logics, as for Theorem 1. **M'** is deduced from increasingness. **B** is deduced from Equations 19 and 20. **N** is obtained by applying the commutativity of $\Box$ with conjunction and of $\Diamond'$ with disjunction on empty families. **RN** can be deduced from **N** and **RR** (see e.g. [CHE 80]). ∎

THEOREM 6. — **T**, **5c** *and* **4c** *are not always satisfied, and we have the following results:*

– **T** *if* $\forall\omega \in \Omega,\ \omega \models \Diamond'\varphi_\omega$,
– **5c** *if* $\forall\omega \in \Omega,\ \omega \models \Diamond'\varphi_\omega$,
– **4c** *if* $\forall\omega \in \Omega,\ \omega \models \Diamond'\varphi_\omega$.

PROOF. — **T** is simply obtained by reasoning on canonical decomposition and by using the commutativity of $\Diamond'$ with disjunction. By applying **T** to $\Box\varphi$ and $\Diamond'\varphi$ we deduce **5c** and **4c**. ∎

Note that the condition on B for **T** in Theorem 1 corresponds to the one above, and we have $B(\omega) = Mod(\Diamond\varphi_\omega)$.

THEOREM 7. — *We have the two following additional theorems:*

– $\Box\Diamond'\Box\varphi \leftrightarrow \Box\varphi$ *and* $\Diamond'\Box\Diamond'\varphi \leftrightarrow \Diamond'\varphi$.

– $\Diamond'\Box\Diamond'\Box\varphi \leftrightarrow \Diamond'\Box\varphi$ *and* $\Box\Diamond'\Box\Diamond'\varphi \leftrightarrow \Box\Diamond'\varphi$.

PROOF. — The proof is straightforward by using **B** and monotonicity. ■

THEOREM 8. — *Let* $(\Box, \Diamond')$ *be an adjunction on* Φ*. Let* $\Box_*\varphi \equiv \neg\Box\neg\varphi$ *and* $\Diamond'_*\varphi \equiv \neg\Diamond'\neg\varphi$*. Then* $(\Diamond'_*, \Box_*)$ *is an adjunction.*

This property expresses a kind of duality between both operators.

Note that we do not always have:

– **Df**: $\Diamond'\varphi \leftrightarrow \neg\Box\neg\varphi$ and $\Box\varphi \leftrightarrow \neg\Diamond'\neg\varphi$.

– **D**: $\Box\varphi \rightarrow \Diamond'\varphi$.

THEOREM 9. — **Df** *is satisfied by an adjunction* $(\Box, \Diamond')$ *if and only if* $\Diamond'$ *satisfies the following property:*

$$\forall(\omega, \omega') \in \Omega^2,\ \omega \models \Diamond'\varphi_{\omega'} \textit{ iff } \omega' \models \Diamond'\varphi_\omega. \tag{21}$$

D *is satisfied by an adjunction* $(\Box, \Diamond')$ *if and only if* $\Diamond'$ *satisfies one of the two following properties:*

$$\forall\omega \in \Omega,\ \omega \models \Diamond'\varphi_\omega \tag{22}$$

or

$$\forall(\omega, \omega') \in \Omega^2,\ \omega \models \Diamond'\varphi_{\omega'} \textit{ iff } \omega' \models \Diamond'\varphi_\omega \textit{ and } \{\omega', \omega' \models \Diamond'\varphi_\omega\} \neq \emptyset. \tag{23}$$

The last result means in particular that we can have **D** without having **T**.

In cases where **Df** is satisfied, then we note simply $\Diamond$ instead of $\Diamond'$.

THEOREM 10. — *The operators* $(\Box, \Diamond)$ *defined by Equations 11 and 12 build an adjunction.*

This shows that modal operators derived from morphological erosions and dilations are particular cases of modal operators derived from algebraic erosions and dilations.

The results obtained in this section show that the use of general algebraic dilations and erosions defined from the adjunction property lead to the properties of normal modal logics. This justifies the use of Kripke's semantics in Section 3, introduced for the particular case of morphological dilations and erosions. This also guarantees a completeness result.

4.3. *Characterizing modal logics in terms of morphological operators*

Conversely, the following result shows that modal operators satisfying some axioms can be expressed in morphological terms.

THEOREM 11. — *If two modal operators* $\Box$ *and* $\Diamond$ *satisfy* **B** *and* **RM**, *then* $(\Box, \Diamond)$ *is an adjunction on* Φ, $\Box$ *is an algebraic erosion and* $\Diamond$ *is an algebraic dilation.*

PROOF. — Let us assume that $\Diamond\varphi \rightarrow \psi$. Monotonicity (**RM**) implies $\Box\Diamond\varphi \rightarrow \Box\psi$. From **B** we derive $\varphi \rightarrow \Box\Diamond\varphi$ and thus $\varphi \rightarrow \Box\psi$. Similarly if $\varphi \rightarrow \Box\psi$ we derive from **B** and **RM** $\Diamond\varphi \rightarrow \psi$. $(\Box, \Diamond)$ is therefore an adjunction on Φ, and Theorem 3 leads to the conclusion that $\Box$ is an algebraic erosion and $\Diamond$ is an algebraic dilation. ■

THEOREM 12. — *Moreover, if we define a relation* R *between worlds by* $R(\omega, \omega')$ *iff* $\omega \models \Diamond\varphi_{\omega'}$, *where* φ_ω *is a canonical formula associated with* ω *(*$Mod(\varphi_\omega) = \{\omega\}$*), then* $\Box$ *and* $\Diamond$ *are exactly given by:*

$$Mod(\Box\varphi) = \{\omega \in \Omega \mid \forall\omega', R(\omega', \omega) \Rightarrow \omega' \models \varphi\}, \tag{24}$$

$$Mod(\Diamond\varphi) = \{\omega \in \Omega \mid \exists\omega', R(\omega, \omega'), \omega' \models \varphi\}. \tag{25}$$

PROOF. — The proof is directly derived from the decomposition of a formula in canonical formulas (Equation 2). ■

These equations are similar to the ones used in Section 3 for defining modal operators from an accessibility relation and a structuring element, except that here we consider $R(\omega, \omega')$ for one operator, and $R(\omega', \omega)$ for the other. If R is symmetrical, both are equivalent. In cases where the structuring element (and the accessibility relation) is not symmetrical, we consider its symmetrical in one of the operations (note that this is not the standard convention used in mathematical morphology in [SER 82] and in Section 3, but the one used in the algebraic framework of adjunctions).

The operators proposed in Section 3 (and [BLO 00c]) are therefore particular cases of the more general expressions proposed here based on adjunctions. This corresponds to the same levels as in the case of set operations: the most general dilations and erosions are the operations that commute with union and intersection respectively (as used in this section). If they are moreover invariant by translation (in the spatial domain), then there exists a structuring element B such that these operations are expressed under their morphological form as in Section 3 [SER 82, SER 88]. This property of invariance by translation is often a requirement in spatial information processing.

5. Modal operators from morphological opening and closing

Morphological opening and closing of a formula are defined in [BLO 00d] similarly as for sets [SER 82]:

$$O(\varphi) \equiv D_B(E_B(\varphi)),\ C(\varphi) \equiv E_B(D_B(\varphi)). \tag{26}$$

These operators are dual from each other, as dilation and erosion are, increasing and idempotent. Moreover, opening is anti-extensive and closing is extensive.

We can define modal operators from them as:

$$\Box\varphi \equiv O(\varphi), \tag{27}$$

$$\Diamond\varphi \equiv C(\varphi). \tag{28}$$

Unfortunately, this leads to weaker properties than operators derived from erosion and dilation. This comes partly from the fact that no accessibility relation can be derived from opening and closing as easily as from erosion and dilation.

However, it would be interesting to link this approach with the topological interpretation of modal logic as proposed in [AIE 99], since opening and closing are related to the notions of topological interior and closure. Note that considering erosion and dilation only leads to a pre-topology (where closure is not idempotent).

Another interesting direction could be to consider the neighborhood semantics [AIE 99], where here the neighborhoods of ω would be all elements of the set $N(\omega) = \{B(\omega') \mid \omega' \in \Omega \text{ and } \omega \in B(\omega')\}$. With this semantics, we can prove:

$$\omega \models \Box\varphi \Leftrightarrow \exists\omega' \in \Omega \mid B(\omega') \in N(\omega) \text{ and } B(\omega') \models \varphi. \tag{29}$$

The proof of this expression comes from the following rewriting of opening:

$$Mod(\Box\varphi) = Mod(O(\varphi)) = \{\omega \in \Omega \mid \exists\omega' \in \Omega,\ \omega \in B(\omega') \text{ and } B(\omega') \models \varphi\}. \tag{30}$$

Kripke's semantics can be seen as a particular case, where the neighborhood of ω is reduced to the singleton $\{B(\omega)\}$.

THEOREM 13. — *The modal logic constructed from opening and closing satisfies* **T**, **Df**, **D**, **4**, **4c**, **5c**, **N**, **M**, **M'**, **RM**, **RE**, *but not* **5**, **B**, **K**, **C**, **R**, **RR**.

PROOF. — **T** is guaranteed by extensivity of closing and anti-extensivity of opening (whatever the structuring element). **Df** is the expression of duality of both operations. **D** is deduced from **T**. **4** and **4c** are deduced from idempotence of opening and closing. **4** corresponds to a fundamental property of interior operator, which goes with the topological interpretation of some modal logics [BEN 95, AIE 99]. **5c** is again guaranteed from anti-extensivity of opening. **M** and **M'** are deduced from **M**, **M'** and **C** for dilation and erosion. **RM** expresses monotonicity, derived from the composition of two monotonic operations. **B** is satisfied only for formulas that are open with respect to the considered structuring element, i.e. in very particular cases. ∎

The fact that **K** is not satisfied goes with the interpretation in terms of neighborhood semantics, which leads to a weaker logic, where **RM** (monotonicity) is satisfied, but not **K** in general [AIE 99].

6. Extension to the fuzzy case

In this Section, we consider fuzzy formulas, i.e. formulas φ for which $Mod(\varphi)$ is a fuzzy subset of Ω. Several definitions of mathematical morphology on fuzzy sets with fuzzy structuring elements have been proposed in the literature (see e.g. [BLO 93, BLO 95, SIN 92, BAE 93, BAE 95, SIN 97]). Here we use the approach of [BLO 95] using t-norms and t-conorms as fuzzy intersection and fuzzy union [1]. However, what follows applies as well if other definitions are used. Erosion of a fuzzy set μ by a fuzzy structuring element ν, both defined in a space $\mathcal{S}$ (for instance $\mathcal{S} = \mathbb{R}^n$), is defined as:

$$\forall x \in \mathcal{S},\ E_\nu(\mu)(x) = \inf_{y \in \mathcal{S}} T[c(\nu(y-x)), \mu(y)], \tag{31}$$

where T is a t-conorm and c a fuzzy complementation [2]. By duality with respect to the complementation c, fuzzy dilation is then defined as:

$$\forall x \in \mathcal{S},\ D_\nu(\mu)(x) = \sup_{y \in \mathcal{S}} t[\nu(y-x), \mu(y)], \tag{32}$$

where t is the t-norm associated with the t-conorm T with respect to the complementation c .

These definitions guarantee that most properties of dilation and erosion are preserved when extended to fuzzy sets. Extensivity of closing, anti-extensivity of opening, and idempotence of these operations are satisfied only for specific t-norms and t-conorms, as Lukasiewicz operators (see [BLO 95] for further details).

Modal operators in the fuzzy case can then be constructed from fuzzy erosion and dilation in a similar way as in the crisp case using Equations 11 and 12. The fuzzy structuring element can be interpreted as a fuzzy relation between worlds. The properties of this fuzzy modal logic are the same as in the crisp case, since fuzzy dilations and erosions have the same properties as the binary ones.

This extension can also be considered from the algebraic point of view of adjunction, based on the results of [DEN 00] and on a definition of fuzzy erosion in terms of residual implication.

The use of fuzzy structuring elements will appear as particularly useful for expressing intrinsically vague spatial relationships such as directional relative position.

It could be also interesting to relate this approach to the possibilistic logic proposed for belief fusion in [BOL 95], and to similarity-based reasoning [EST 97, DUB 97].

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1. A triangular norm (or t-norm) is a function from $[0,1] \times [0,1]$ into $[0,1]$ which is commutative, associative, increasing, and for which 1 is unit element and 0 is null element [MEN 42, SCH 63]. Examples of t-norms are min, product, etc. [DUB 80].
2. A fuzzy complementation is a function c from $[0,1]$ into $[0,1]$ such that $c(0) = 1$, $c(1) = 0$, c is involutive: $\forall x \in [0,1], c(c(x)) = x$, and c is strictly decreasing.

7. Qualitative representation of spatial relationships

For qualitative spatial reasoning, worlds can represent spatial entities, like regions of the space. Formulas then represent combinations of such entities, and define regions, objects, etc., which may be not connected. For instance, if a formula φ is a symbolic representation of a region X of the space, it can be interpreted for instance as "the object we are looking at is in X". In an epistemic interpretation, it could represent the belief of an agent that the object is in X. The interest of such representations could be also to deal in a qualitative way with any kind of spatial entities, without referring to points.

Using these interpretations, if φ represents some knowledge or belief about a region X of the space, then $\Box\varphi$ represents a restriction of X. If we are looking at an object in X, then $\Box\varphi$ is a necessary region for this object. Similarly, $\Diamond\varphi$ represents an extension of X, and a possible region for the object. In an epistemic interpretation, $\Box\varphi$ can represent the belief of an agent that the object is necessarily in the erosion of X while $\Diamond\varphi$ is the belief that it is possibly in the dilation of X. Interpretations in terms of rough regions are also possible.

In this Section, we address the problem of qualitative representation of spatial relationships between regions or objects represented by logical formulas. According to the semantical hierarchy of Kuipers [KUI 88], we consider topological and metric relationships (corresponding to levels 3 and 4 of this hierarchy). Many authors have stressed the importance of topological relationships, e.g. [ALL 83, VAR 96, RAN 92, COH 97, ASH 95, CLE 97, KUI 78, PUL 88]. But distances and directional relative position (constituting the metric relationships) are also important, e.g. [PEU 88, DUT 91, KUI 88, GAP 94, KRI 93, WAN 99, LIU 98].

In previous works, we have shown that several spatial relationships can be expressed using morphological dilation or fuzzy dilation, as distances [BLO 99c], adjacency [BLO 97], directional relative position [BLO 99a, BLO 99b]. Spatial representations of these relationships have been proposed in [BLO 00b]. Now, we propose to use the modal operators introduced in this paper to provide symbolic and qualitative representations of such spatial knowledge.

7.1. *Topological relationships*

Let us first consider topological relationships. Let φ and ψ be two formulas representing two regions X and Y of the space. Note that all what follows holds in both crisp and fuzzy cases. Simple topological relations such as inclusion, exclusion, intersection do not call for more operators than the standard ones of propositional logic (see e.g. [BEN 95]). But other relations such that X *is a tangential part of* Y can benefit from the morphological modal operators. Such a relationship can be expressed as:

$$\varphi \rightarrow \psi \text{ and } \Diamond\varphi \wedge \neg\psi \text{ consistent,} \tag{33}$$

or, equivalently,

$$\varphi \rightarrow \psi \text{ and } \varphi \wedge \neg\Box\psi \text{ consistent.} \tag{34}$$

Indeed, if X is a tangential part of Y, it is included in Y but its dilation is not, and equivalently it is not included in the erosion of Y, as illustrated in Figure 2.

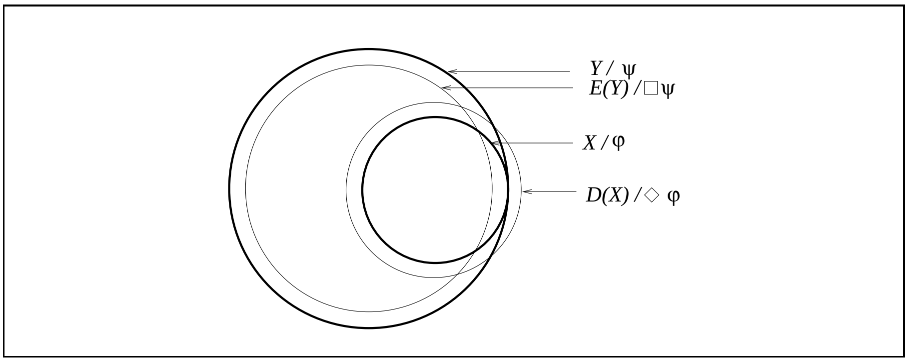

Figure 2. *Illustration of tangential part relationship, and its expression in terms of dilation and erosion.*

In a similar way, a relation such that X is a non tangential part of Y is expressed as:

$$\varphi \rightarrow \psi \text{ and } \Diamond\varphi \rightarrow \psi, \tag{35}$$

or, equivalently,

$$\varphi \rightarrow \psi \text{ and } \varphi \rightarrow \Box\psi, \tag{36}$$

(i.e. in order to verify that X is a non tangential part of Y, we have to prove these relations).

If we also want X to be a proper part, we have to add the following condition:

$$\neg\varphi \wedge \psi \text{ consistent.} \tag{37}$$

Let us now consider adjacency (or external connection). Saying that X is adjacent to Y means that they do not intersect and as soon as one region is dilated, it has a non empty intersection with the other. In symbolic terms, this relation can be expressed as:

$$\varphi \wedge \phi \text{ inconsistent and } \Diamond\varphi \wedge \psi \text{ consistent and } \varphi \wedge \Diamond\psi \text{ consistent.} \tag{38}$$

Actually, this expression holds in a discrete domain. If φ and ψ represent spatial entities in a continuous spatial domain, some problems may occur if these entities are closed sets and have parts of local dimension less than the dimension of the space (see [BLO 97] for a complete discussion). Such problems can be avoided if the entities are reduced to regular ones, i.e. that are equal to the closure of their interior. Using the topological interpretation of modal logic, this amounts to deal with formulas for which we can prove $\varphi \leftrightarrow \Diamond\Box\varphi$.

It could be interesting to link these types of representations with the ones developed in the community of mereology and mereotopology, where such relations are defined respectively from parthood and connection predicates [ASH 95, RAN 92, COH 97, VAR 96, REN 01]. Interestingly enough, erosion is defined from inclusion (i.e. a parthood relationship) and dilation from intersection (i.e. a connection relationship). Some axioms of these domains could be expressed in terms of dilation. For instance from a parthood postulate $P(X, Y)$ between two spatial entities X and Y and from dilation D, tangential proper part could be defined as $TPP(X, Y) = P(X, Y) \wedge \neg P(Y, X) \wedge \neg P(D(X), Y)$. Further links certainly deserve to be investigated, in particular with the work presented in [COH 97, CRI 00, GAL 00], etc.

7.2. *Metric relationships*

Distances and directional position are important relationships in order to describe a scene by means of the spatial arrangement of the objects, and to account for the structural information of the scene in spatial reasoning.

7.2.1. *Distances*

Distances between objects X and Y can be expressed in different forms, as *the distance between X and Y is equal to n, the distance between X and Y is less (respectively greater) than n, the distance between X and Y is between n_1 and n_2*. Several distances can be related to morphological dilation, as minimum distance and Hausdorff distance. We used these relations as a basis for defining distances between fuzzy sets in [BLO 99c]. For instance for the minimum distance, denoted by $d_{\min}$, the following equations hold, where D^n denotes the dilation of size n:

$$d_{\min}(X, Y) = \min\{n \in \mathbb{N},\ D^n(X) \cap Y \neq \emptyset \text{ and } D^n(Y) \cap X \neq \emptyset\}, \tag{39}$$

$$d_{\min}(X, Y) = n \Leftrightarrow$$
$$\begin{cases} \forall m < n, D^m(X) \cap Y = D^m(Y) \cap X = \emptyset \\ \text{and } D^n(X) \cap Y \neq \emptyset, D^n(Y) \cap X \neq \emptyset. \end{cases} \tag{40}$$

$$d_{\min}(X, Y) \leq n \Leftrightarrow D^n(X) \cap Y \neq \emptyset, D^n(Y) \cap X \neq \emptyset. \tag{41}$$

$$d_{\min}(X, Y) \geq n \Leftrightarrow \forall m < n, D^m(X) \cap Y = D^m(Y) \cap X = \emptyset. \tag{42}$$

$$n_1 \leq d_{\min}(X, Y) \leq n_2 \Leftrightarrow$$
$$\begin{cases} \forall m < n_1, D^m(X) \cap Y = D^m(Y) \cap X = \emptyset \\ \text{and } D^{n_2}(X) \cap Y \neq \emptyset, D^{n_2}(Y) \cap X \neq \emptyset. \end{cases} \tag{43}$$

The proof of these equations involves extensivity of dilation (for such structuring elements), and increasingness with respect to the structuring element. They can be extended to the fuzzy case by using fuzzy dilation [BLO 99c, BLO 00b].

Similarly for the Hausdorff distance d_{Haus}, we have:

$$d_{\text{Haus}}(X,Y) = \min\{n \in \mathbb{N},\ Y \subseteq D^n(X) \text{ and } X \subseteq D^n(Y)\}, \tag{44}$$

and similar equations for the other types of distance information.

Now, the translation into a logical formalism is straightforward. Expressing that $d_{\min}(X,Y) = n$ leads to:

$$\begin{cases} \forall m < n, \Diamond^m \varphi \wedge \psi \text{ inconsistent and } \Diamond^m \psi \wedge \varphi \text{ inconsistent} \\ \text{and } \Diamond^n \varphi \wedge \psi \text{ consistent and } \Diamond^n \psi \wedge \varphi \text{ consistent.} \end{cases} \tag{45}$$

Expressions like $d_{\min}(X,Y) \leq n$ translate into:

$$\Diamond^n \varphi \wedge \psi \text{ consistent and } \Diamond^n \psi \wedge \varphi \text{ consistent.} \tag{46}$$

Expressions like $d_{\min}(X,Y) \geq n$ translate into:

$$\forall m < n, \Diamond^m \varphi \wedge \psi \text{ inconsistent and } \Diamond^m \psi \wedge \varphi \text{ inconsistent.} \tag{47}$$

Expressions like $n_1 \leq d_{\min}(X,Y) \leq n_2$ translate into:

$$\begin{cases} \forall m < n_1, \Diamond^m \varphi \wedge \psi \text{ inconsistent and } \Diamond^m \psi \wedge \varphi \text{ inconsistent} \\ \text{and } \Diamond^{n_2} \varphi \wedge \psi \text{ consistent and } \Diamond^{n_2} \psi \wedge \varphi \text{ consistent.} \end{cases} \tag{48}$$

The proof of these equations involves mainly **T** and the results on $\Diamond^n$ at the end of Section 3.

Similarly for Hausdorff distance, we translate $d_{\text{Haus}}(X,Y) = n$ by:

$$\begin{cases} \forall m < n, \psi \wedge \neg\Diamond^m \varphi \text{ consistent or } \varphi \wedge \neg\Diamond^m \psi \text{ consistent} \\ \text{and } \psi \rightarrow \Diamond^n \varphi \text{ and } \varphi \rightarrow \Diamond^n \psi. \end{cases} \tag{49}$$

The first condition corresponds to $d_{\text{Haus}}(X,Y) \geq n$ and the second one to $d_{\text{Haus}}(X,Y) \leq n$.

Let us consider an example of possible use of these representations for spatial reasoning. If we are looking at an object represented by ψ in an area which is at a distance in an interval $[n_1, n_2]$ of a region represented by φ, this corresponds to a minimum distance greater than n_1 and to a Hausdorff distance less than n_2. This is illustrated in Figure 3.

Then we have to check the following relation:

$$\psi \rightarrow \neg\Diamond^{n_1}\varphi \wedge \Diamond^{n_2}\varphi, \tag{50}$$

or equivalently:

$$\psi \rightarrow \Box^{n_1}\neg\varphi \wedge \Diamond^{n_2}\varphi. \tag{51}$$

This expresses in a symbolic way an imprecise knowledge about distances represented as an interval. If we consider a fuzzy interval, this extends directly by means of fuzzy dilation (see [BLO 00b] for detailed expressions of these dilations).

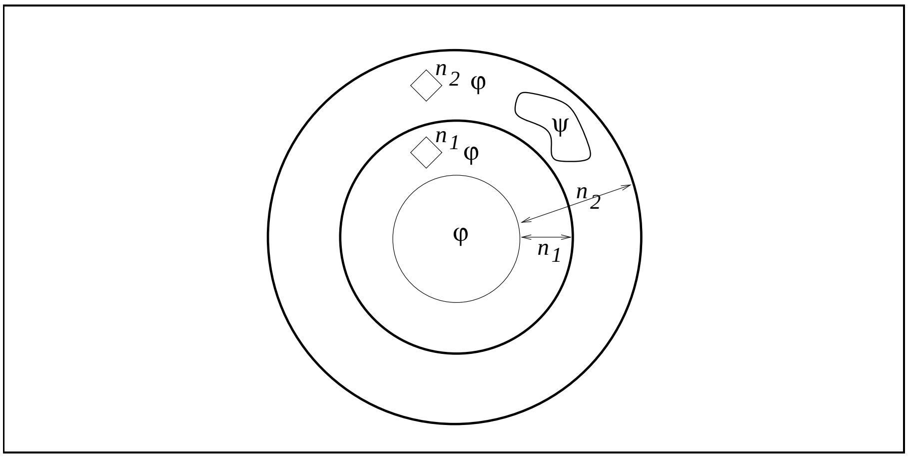

Figure 3. *Illustration of a distance relation expressed by an interval.*

These expressions show how we can convert distance information, which is usually defined in an analytical way, into algebraic expressions through mathematical morphology, and then into logical expressions through morphological expressions of modal operators.

7.2.2. *Directional relative position*

Contrary to the previous relationships, relative directional position (like *object X is on the right of object Y*) is an intrinsically vague information, for which the fuzzy set framework is appropriate for defining formally such relationships with good properties. To the best of our knowledge, almost all existing methods for defining fuzzy relative directional spatial position rely on angle measurements between points of the two objects of interest [KRI 93, MIY 94, KEL 95, MAT 99], and concern 2D objects (sometimes with possible extension to 3D). These approaches cannot easily be used for defining a fuzzy set in the space corresponding to the area where a directional relationship to an object is satisfied, nor to translate such information in a symbolic setting. Here we rely on the approach we proposed in [BLO 99a], which is completely different and more suitable to this task, since the relationship is defined directly in the considered space (spatial domain). It consists in dilating the reference object X with a particular structuring element, of radial form, having high membership values along lines in the desired direction, and decreasing membership values when going away from this direction. This dilation provides a fuzzy area of the space, based on which the relation of any other object to X can be assessed (for instance using pattern matching).

Let us denote by D^d the dilation corresponding to a directional information in the direction d, and by $\diamond^d$ the associated modal operator. Expressing that an object represented by ψ has to be in direction d with respect to a region represented by φ amounts to check the following relation:

$$\psi \rightarrow \Diamond^d \varphi. \tag{52}$$

In the fuzzy case, this relation can hold to some degree.

This formulation directly inherits the properties of directional relative position defined from dilation (see [BLO 99b] for details), such as invariance with respect to geometrical transformations. It also has a behavior that fits well the intuition if the distance to the reference object increases, and in case of concavities.

Usually for spatial reasoning several relationships have to be used together. This aspect can benefit from the developments in information fusion, both in a numerical and in a logical setting.

8. Conclusion

We proposed in this paper definitions of modal operators from mathematical morphology and the fundamental concept of adjunction. Conversely, we have shown that some modal logics can be characterized in terms of mathematical morphology. We discussed the properties of the proposed operators and their usefulness for deriving qualitative representations of spatial relationships, since several spatial relationships can be expressed in terms of mathematical morphology. Extensions to the fuzzy case are possible based on fuzzy mathematical morphology and need to be further exploited for dealing with imprecisely defined spatial entities and with vague relations. The proposed approach can be related to the possibilistic logic proposed for belief fusion in [BOL 99], and to similarity-based reasoning [EST 97]. Operators based on other morphological operators could also be investigated, based on algebraic opening and closing, or filters. The proposed modal logic, derived from links established between theories that were so far disconnected[3], merges the advantages of logical representations, of modal operators which allow to express in a common language different types of spatial knowledge, and of mathematical morphology which provides a unified framework for representing local spatial knowledge as well as relationships between spatial entities. We expect that this can be further exploited for spatial reasoning and for merging qualitative and quantitative aspects.

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3. These links between both theories are also currently studied by J. van Benthem et al. [BEN] in particular in the framework of linear logic and arrow logic.

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