Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Image and Vision Computing 23 (2005) 89–110
image AND vision COMPUTING
www.elsevier.com/locate/imavis

# Fuzzy spatial relationships for image processing and interpretation: a review

Isabelle Bloch*

*Ecole Nationale Supérieure des Télécommunications, Département TSI—CNRS UMR 5141 LTCI, 46 rue Barrault, 75013 Paris, France*

Received 16 January 2004; received in revised form 5 May 2004; accepted 29 June 2004

**Abstract**

In spatial reasoning, relationships between spatial entities play a major role. In image interpretation, computer vision and structural recognition, the management of imperfect information and of imprecision constitutes a key point. This calls for the framework of fuzzy sets, which exhibits nice features to represent spatial imprecision at different levels, imprecision in knowledge and knowledge representation, and which provides powerful tools for fusion, decision-making and reasoning. In this paper, we review the main fuzzy approaches for defining spatial relationships including topological (set relationships, adjacency) and metrical relations (distances, directional relative position).


*Keywords:* Fuzzy spatial relationships; Degree of intersection; Degree of inclusion; Degree of adjacency; Distances; Directional relative position; Structural pattern recognition; Image interpretation; Spatial reasoning

## 1. Introduction

Spatial reasoning can be defined as the domain of spatial knowledge representation, in particular spatial relations between spatial entities, and of reasoning on these entities and relations. This field has been largely developed in artificial intelligence, in particular using qualitative representations based on logical formalisms. In image interpretation and computer vision, it is much less developed and is mainly based on quantitative representations. A typical example in this domain concerns model-based structure recognition in images. The model constitutes a description of the scene where objects have to be recognized. This description can be of iconic type, as for instance a digital map or a digital anatomical atlas, or of symbolic type, as linguistic descriptions of the main structures. The model can be attached to a specific scene, the typical example being a digital map used for recognizing structures in an aerial or satellite image of a specific region. It can also be more generic, as an anatomical atlas, which is a schematic representation that can be used for recognizing structures in a medical image of any person. In both types of descriptions (iconic and symbolic), objects are usually described through some characteristics like shape, size, appearance in the images, etc. But this is generally not enough to discriminate all objects in the scene, in particular if they are embedded in a complex environment. For instance in a magnetic resonance image of the brain, several internal structures appear as smooth shapes with similar grey levels, making their individual recognition difficult. Similar examples can be found in other application domains. In such cases, spatial relationships play a crucial role, and it is important to include them in the model in order to guide the recognition [1]. The importance of spatial relationships has been similarly recognized in many different works. Many authors have stressed the importance of topological relationships, but distances and directional relative position are also important. Freeman [2] distinguishes the following primitive relationships: left of, right of, above, below, behind, in front of, near, far, inside, outside, surround. Kuipers [3,4] considers topological relations (set relations, but also adjacency which was not considered by Freeman)

* Tel.: +33-1-45-81-75-85; fax: +33-1-45-81-37-94.
*E-mail address:* isabelle.bloch@enst.fr.

0262-8856/$ - see front matter 
doi:10.1016/j.imavis.2004.06.013

and metrical relations (distances and directional relative position). In this paper, we will consider all these relations.

Moreover, imprecision has to be taken into account in such problems. Imprecision is often inherent to images, and its causes can be found at several levels: observed phenomenon (imprecise limits between structures or objects), acquisition process (limited resolution, numerical reconstruction methods), image processing steps (imprecision induced by a filtering procedure for instance). This may induce imprecision on the objects to be recognized (due to the absence of strong contours or to a rough segmentation). But imprecision can be found also in semantics of some relationships (such as 'left of', 'quite far', etc.), or in the type of knowledge available about the structures (for instance anatomical textbooks describe the caudate nucleus as 'an internal brain structure which is very close to the lateral ventricles') or even in the type of question we would like to answer (in mobile robotics for instance, we may want a robot 'go towards an object while remaining at some security distance of it').

In summary, the main ingredients in problems related to spatial reasoning include knowledge representation (including spatial relationships), imprecision representation and management, fusion of heterogeneous information and decision-making. Fuzzy set theory is of great interest to provide a consistent mathematical framework for all these aspects. It allows to represent imprecision of objects, relationships, knowledge and aims, it provides a flexible framework for information fusion as well as powerful tools for reasoning and decision-making.

The aim of this paper is to review the main approaches for modeling spatial relationships under imprecision in the fuzzy set framework. We distinguish between relationships that are mathematically well defined and relationships that are intrinsically vague. Topological relationships (such as set relationships and adjacency) and distances belong to the first class. If the objects are precisely defined, their relationships can be defined and computed in a numerical (purely quantitative) setting. But if the objects are imprecise, as is often the case if they are extracted from images, then the semi-quantitative framework of fuzzy sets proved to be useful for their representation, as spatial fuzzy sets. Definitions of relationships have then to be extended to be applicable on fuzzy objects. Results can also be semi-quantitative, and provided in the form of intervals or fuzzy numbers. Some metric relationships, like relative directional position, belong to the second class. Even for crisp objects, fuzzy definitions are then appropriate.

Section 2 contains some preliminaries about spatial fuzzy sets, some basic definitions, and general principles to extend a crisp relation to a fuzzy one. Set theoretical relationships (intersection and inclusion) are described in Section 3. Then, other topological relations (local such as neighborhood or more global such as adjacency) are addressed in Section 4. Distances are reviewed in Section 5 and finally directional relative position in Section 6.

## 2. Preliminaries

### 2.1. Spatial fuzzy sets

Let $\mathcal{S}$ be the image space, typically $\mathbb{Z}^2$ or $\mathbb{Z}^3$ for digital 2D or 3D images, or, in the continuous case, $\mathbb{R}^2$ or $\mathbb{R}^3$. A spatial fuzzy set (or fuzzy image object) is a fuzzy set defined on $\mathcal{S}$. Its membership function $\mu$ represents the imprecision in the spatial extent of the object. For any point $x$ of $\mathcal{S}$ (pixel or voxel), $\mu(x)$ is the degree to which $x$ belongs to the fuzzy object. As usual in the fuzzy set community, and for the sake of simplicity, $\mu$ will denote both the fuzzy set and its membership function. Using fuzzy sets may represent different types of imprecision, either on the boundary of the objects (due for instance to some partial volume effect or to the spatial resolution), or on their individual variability, etc. In the sequel, $\mathcal{F}$ denotes the set of all fuzzy sets defined on $\mathcal{S}$.

### 2.2. Notations and basic definitions

We recall here a few basic definitions for the sake of completeness and introduce some notations. A complete description of fuzzy set theory can be found in Ref. [5]. From $\mu \in \mathcal{F}$, some particular crisp (binary) sets can be derived, such as its core: $\mathrm{Core}(\mu) = \{x \in \mathcal{S}, \mu(x) = 1\}$, its support: $\mathrm{Supp}(\mu) = \{x \in \mathcal{S}, \mu(x) > 0\}$, its $\alpha$-cuts (for $\alpha \in ]0,1]$): $\mu_\alpha(x) = \{x \in \mathcal{S}, \mu(x) \geq \alpha\}$.

A fuzzy number is a convex upper semi-continuous (and unimodal) fuzzy set on $\mathbb{R}^+$ having a bounded support.

A few basic operators on membership values will be used in the following, such as t-norrns, t-conorms and complementation [5]. A t-norm is an operator $t$ from $[0,1] \times [0,1]$ into $[0,1]$ which is commutative, associative, increasing in both variables and that admits 1 as unit element. It represents a conjunction and generalizes intersection and logical 'and'. Typical examples are $\min(a, b)$, $ab$, $\max(a+b-1, 0)$, the last one being known as the Lukasiewicz t-norm. A t-conorm is an operator $T$ from $[0,1] \times [0,1]$ into $[0,1]$ which is commutative, associative, increasing in both variables and that admits 0 as unit element. It represents a disjunction and generalizes union and logical 'or'. Typical examples are $\max(a, b)$, $a+b-ab$, $\min(a+b, 1)$, the last one being the Lukasiewicz t-conorm. A complementation is an operator $c$ from $[0,1]$ into $[0,1]$ which is strictly decreasing, involutive, and such that $c(0)=1$, $c(1)=0$. The most used complementation is defined as $\forall a \in [0,1]$, $c(a) = 1-a$. From a t-norm $t$ and a complementation $c$ a dual t-conorm $T$ can be derived as $\forall (a,b) \in [0,1] \times [0,1]$, $T(a, b) = c(t(c(a), c(b)))$. A strictly monotonous Archimedian t-norm is a t-norm $t$ such that $\forall a \in [0,1]$, $t(a,a) < a$ and $\forall (a,b,b') \in [0,1]^3$, $b < b' \Rightarrow t(a,b) < t(a,b')$. Archimedian t-conorms are defined by duality. A typical example of Archimedian t-norm is the product.

We will denote by $R_B$ a relation between two binary (crisp) subsets of $\mathcal{S}$. This relation can provide a binary result: for $X \subseteq \mathcal{S}$, $Y \subseteq \mathcal{S}$, $R_B(X,Y) \in \{0,1\}$ (for instance if $R_B$ is an adjacency relation), or a numerical result: $R_B(X,Y) \in \mathbb{R}$, $R_B(X,Y) \in \mathbb{R}^+$ (for instance if $R_B$ is a distance), or $R_B(X,Y) \in [0,1]$. The aim of this paper is to extend spatial relationships to the fuzzy case, i.e. to relationships between two fuzzy sets. We will denote by $R$ the extension of $R_B$ to fuzzy sets.

Several definitions in this paper are based on mathematical morphology [6] and its extension to fuzzy sets, and more precisely on dilation and erosion. In the crisp case, the dilation $D_B(X)$ of a set $X$ by a structuring element $B$ (a subset of $\mathcal{S}$) is defined as

$$D_B(X) = \{x \in \mathcal{S}, X \cap B_x \neq \emptyset\}, \tag{1}$$

where $B_x$ denotes the translation of $B$ at $x$. The erosion $E_B(X)$ of a set $X$ by a structuring element $B$ is defined as:

$$E_B(X) = \{x \in \mathcal{S}, B_x \subseteq X\}. \tag{2}$$

Dilation and erosion are dual operators: $E_B(X) = D_B(X^C)^C$ where $X^C$ denotes the complement of $X$ in $\mathcal{S}$. Applying successively dilation (respectively erosion) by the same structuring element $n$ times will be denoted by $D_B^n(X)$ (respectively $E_B^n(X)$). If $B$ is a ball of radius $n$ of the Euclidean distance in $\mathcal{S}$ (or of a digital distance), then the dilation and the erosion by $B$ are simply denoted by $D^n(X)$ and $E^n(X)$.

Several definitions of fuzzy mathematical morphology have been proposed in the literature during the last decade. Here, we just give an example, chosen for its nice properties with respect to classical morphology, where dilation and erosion of a fuzzy set $\mu$ by a structuring element $\nu$ are, respectively, defined, for all $x \in \mathcal{S}$, by [7]

$$D_\nu(\mu)(x) = \sup\{t[\nu(y-x), \mu(y)], y \in \mathcal{S}\}, \tag{3}$$

$$E_\nu(\mu)(x) = \inf\{T[c(\nu(y-x)), \mu(y)], y \in \mathcal{S}\}, \tag{4}$$

where $t$ is a t-norm and $T$ the associated t-conorm with respect to the complementation $c$. Applying successively dilation (respectively erosion) $n$ times by the same structuring element $\nu$ will be denoted by $D_\nu^n(\mu)$ (respectively $E_n^\nu(\mu)$).

### 2.3. Constructing fuzzy relations from crisp relations

Common and generic methods that can be used for defining a fuzzy relationship from its equivalent binary (crisp) one can be categorized into three main classes. The first type relies on the 'extension principle', as introduced by Zadeh [8]. The second class relies on computation on $\alpha$-cuts. These two classes of definitions explicitly involve the relations on crisp sets. The third class of methods consists in providing directly fuzzy definitions of the relationships, by substituting all crisp expressions by their fuzzy equivalents.

Originally, the first generic method for extending crisp operators to fuzzy operators is due to Zadeh [8] and known as the extension principle. Let us first consider a function $f$ from $\mathcal{U}$ to $\mathcal{V}$. Let $\mu$ be a fuzzy set defined on $\mathcal{U}$. The extension of $f$ to a fuzzy set is a fuzzy set $\mu'$ defined on $\mathcal{V}$. It is constructed as follows:

$$\forall y \in \mathcal{V},\ \mu'(y) = \begin{cases} 0 & \text{if } f^{-1}(y) = \emptyset, \\ \sup_{x \in \mathcal{U} | y = f(x)} \mu(x) & \text{otherwise.} \end{cases} \tag{5}$$

This extends to the more general case where $f$ is defined on a product space.

A typical example of application of the extension principle is the compatibility between two fuzzy sets, as defined by Zadeh [9]. Let us consider a fuzzy set $\mu$ on some space $\mathcal{U}$. The value $\mu(x)$ may be interpreted as a degree of compatibility of $x$ with the fuzzy set $\mu$ [5] ($\mu$ being for instance a fuzzy number). The compatibility of a fuzzy set $\mu'$ of $\mathcal{U}$ with $\mu$ can be evaluated using the extension principle as a fuzzy set $\mu_{\mathrm{comp}}$ on [0,1]:

$$\forall t \in [0,1],\ \mu_{\mathrm{comp}}(t) = \begin{cases} 0 & \text{if } \mu^{-1}(t) = \emptyset, \\ \sup_{x \in \mathcal{U} | t = \mu(x)} \mu'(x) & \text{otherwise.} \end{cases} \tag{6}$$

This notion of compatibility has been used in Ref. [10] to define directional spatial relations (see Section 6.3).

One way to define crisp sets from a fuzzy set consists in taking the $\alpha$-cuts of this set. Conversely, a fuzzy set can be reconstructed from its $\alpha$-cuts. Therefore, a class of methods for defining fuzzy operations from crisp ones relies on the application of the crisp operation on each $\alpha$-cut and then combining the results to reconstruct a fuzzy operation by stacking up the $\alpha$-cuts. Let us denote by $\mu$ and $\nu$ the membership functions of two fuzzy sets defined on the space $\mathcal{S}$. Let us now consider a relation $R_B$ between crisp sets. The fuzzy equivalent $R$ of $R_B$, applied to $\mu$ and $\nu$, is defined as

$$R(\mu, \nu) = \int_0^1 R_B(\mu_\alpha, \nu_\alpha) d\alpha, \tag{7}$$

or by a double integration as:

$$R(\mu, \nu) = \int_0^1 \int_0^1 R_B(\mu_\alpha, \nu_\beta) \mathrm{d}\alpha\, \mathrm{d}\beta. \tag{8}$$

Other fuzzification equations are possible, like [7,11]

$$R(\mu, \nu) = \sup_{\alpha \in ]0,1]} \min(\alpha, R_B(\mu_\alpha, \nu_\alpha)), \tag{9}$$

if the relation $R_B$ takes values in [0,1], or:

$$R(\mu, \nu) = \sup_{\alpha \in ]0,1]} (\alpha R_B(\mu_\alpha, \nu_\alpha)). \tag{10}$$

Another method for combining the results on $\alpha$-cuts is similar to the extension principle [8]. In general it leads to a fuzzy set on $\mathcal{V}$. For instance if $\mathcal{V} = \mathbb{R}$, the crisp relation provides real values. While the corresponding fuzzy relation using previous equations also provides numbers, the following one provides fuzzy numbers

$$\forall n \in \mathcal{V}, \quad R(\mu,\nu)(n) = \sup_{R_B(\mu_\alpha,\nu_\alpha)=n} \alpha, \tag{11}$$

and represents the degree to which the relation between $\mu$ and $\nu$ is equal to $n$. If the relationship to be extended only takes binary values (0/1, or true/false), then this equation reduces to:

$$R(\mu,\nu) = \sup_{R_B(\mu_\alpha,\nu_\alpha)=1} \alpha. \tag{12}$$

A last class of methods consists in translating binary equations into their fuzzy equivalent. This approach completely differs from the two previous ones in the sense that it does not use explicitly the crisp relation or operation. Indeed, in the extension principle as well as in approaches based on $\alpha$-cuts, the definition of a fuzzy operation is a function of the corresponding crisp operation. Here, a fuzzy operation is given directly by an equation involving fuzzy terms, that just mimics the crisp equation. This translation is generally done term-by-term. For instance, intersection is replaced by a t-norm, sets by fuzzy set membership functions, etc. This translation is particularly straightforward if the binary relationship can be expressed in set theoretical and logical terms. Table 1 summarizes the main crisp concepts involved in set equations, and their fuzzy equivalent. The many possibilities to translate for instance set union using a t-conorm induce that many definitions are issued from this method, depending on the choice of the fuzzy operators used for translating the crisp corresponding ones.

Links between extension principle and combination of $\alpha$-cuts using Eq. (9) have been established in Ref. [11]. Other links exist between some definitions. For instance, if $R_B$ is a crisp relationship taking values in {0,1}, its extension using Eq. (12) is a value in [0,1] and is equivalent to the two fuzzification procedures given by Eqs. (9) and (10). The question of which extension should be used does not have a definite answer until now. However, it can be noted that the extension principle is well adapted for translating analytical expressions, while the formal translation method is well-adapted if the operators to be extended can be expressed using set theoretical and logical terms. The properties of the obtained extended operators have to play an important role in the choice of a method, since they may vary depending on it.

Table 1
Translation of crisp concepts in their fuzzy equivalents

| Crisp concept | Equivalent fuzzy concept |
|---|---|
| Set ($X$) | Fuzzy set/membership function $\mu$ |
| Complement of a set | Fuzzy complementation $c$ |
| Intersection ($\cap$) | t-norm $t$ |
| Union ($\cup$) | t-conorm $T$ |
| Existence ($\exists$) | Supremum |
| Universal symbol ($\forall$) | Infimum |

## 3. Set theoretical operations

Let us start our review of fuzzy relationships by set operations. We do not consider here the point of view of constructing a fuzzy set which is the result of the combination of two fuzzy sets by a set operation, but we take another point of view, which addresses the question: are two fuzzy sets satisfying some set relationships? For instance: is $\mu$ included in $\nu$? In the crisp case, such questions receive binary answers. When the objects are fuzzy, imprecisely defined, the answer to such questions becomes a matter of degree, and amounts to define a degree to which the relation is satisfied. We consider here degrees of intersection or non-intersection (corresponding to the 'outside' relation of Freeman) and degree of inclusion ('inside' relation of Freeman). They belong to the class of topological relations according to Kuipers.

### 3.1. Degree of intersection

Saying that two sets intersect translates in the fuzzy case as a degree $\mu_{\mathrm{int}}(\mu, \nu)$ to which two fuzzy sets $\mu$ and $\nu$ intersect. In the crisp case, the set equation expressing that two crisp subsets $X$ and $Y$ of $\mathcal{S}$ intersect is:

$$X \cap Y \neq \emptyset \text{ or equivalently } \exists x \in \mathcal{S},\ x \in X \cap Y. \tag{13}$$

On the other hand, the fact that $X$ and $Y$ do not intersect is expressed by the non-satisfaction of this equation. These two possible states in the crisp case correspond to a binary 'degree' of intersection, which is equal to 1 if the equation is satisfied, and to 0 if it is not.

#### 3.1.1. Direct extension

In the fuzzy domain, this binary degree becomes a degree in [0,1], which expresses the degree of satisfaction of this equation. It can be defined for instance using the formal translation method. The simplest fuzzy translation (see Table 1) provides

$$\mu_{\mathrm{int}}(\mu,\nu) = \sup_{x\in\mathcal{S}} t[\mu(x),\nu(x)], \tag{14}$$

where $t$ is any t-norm. This expression may vary from 0, which corresponds to no intersection at all (typically if $\mu$ and $\nu$ have disjoint supports) to 1 if at least one point $x$ belongs completely to both $\mu$ and $\nu$. Note that in this case, the other fuzzification methods (combination of $\alpha$-cuts, extension principle) provide the same result, or a particular case where $t=\min$. This form has been used for instance to define fuzzy morphological dilations (see Ref. [7] for more developments on this). Indeed, Eq. (3) corresponds to the degree of intersection of $\mu$ and the translation of $\nu$ at $x$.

From the degree of intersection, a degree of empty intersection (or of disjunctness) is then derived as

$$\mu_{\neg\mathrm{int}}(\mu,\nu) = c[\mu_{\mathrm{int}}(\mu,\nu)], \tag{15}$$

where $c$ is a fuzzy complementation (for instance defined as $\forall a \in [0,1]$, $c(a) = 1-a$). This form has already been widely used in the fuzzy set literature. In particular, it is often interpreted as a degree of conflict between two fuzzy sets or two possibility distributions [12].

#### 3.1.2. Introducing the volume of the overlapping domain

However, this form is not always adequate for image processing or vision purpose since it does not include any spatial information. This may even lead to counter-intuitive results: the expression $\sup_{x\in\mathcal{S}} t[\mu(x), \nu(x)]$ only represents the maximum height of the intersection. Although it is generally low for fuzzy sets that have almost disjoint supports, its value does not account for different overlapping situations. This will be illustrated in Sections 3.3 and 4.3. The degree of intersection and of non-intersection can therefore be reformulated in order to better represent the notion of spatial overlapping by considering the fuzzy hypervolume $V_n$ (in a space of dimension $n$) of the intersection. This also corresponds to a translation process, in the sense that disjunctness can be expressed in the crisp case as: $X\cap Y=\emptyset \Leftrightarrow V_n(X\cap Y)=0$. For defining the hypervolume of a fuzzy set, we simply use the classical fuzzy cardinality. This provides for a fuzzy set $\mu$ (having a bounded support) in the discrete case $V_n(\mu)=\sum_{x\in\mathcal{S}}\mu(x)$, and in the continuous case $V_n(\mu)=\int_{x\in\mathcal{S}}\mu(x)$.

From the hypervolume of $t(\mu, \nu)$, we can derive a degree of intersection in [0,1]. It should be equal to 0 if $\mu$ and $\nu$ have completely disjoint supports, be high if one set is included in the other, and increasing with respect to the hypervolume of the intersection. The following definition satisfies these requirements[1]:

$$\mu_{\text{int}}(\mu,\nu) = \frac{V_n[t(\mu,\nu)]}{\min[V_n(\mu), V_n(\nu)]}. \tag{16}$$

Again a degree of non-intersection can be derived from this expression using Eq. (15).

#### 3.1.3. Properties

An important property of these various definitions is that the intersection degrees defined by Eqs. (14) and (16) are both consistent with the binary definition. Moreover, they satisfy the following properties:

- symmetry: $\forall(\mu,\nu)\in\mathcal{F}^2$, $\mu_{\text{int}}(\mu,\nu)=\mu_{\text{int}}(\nu,\mu)$;
- reflexivity (Eq. (14)) if the fuzzy sets are normalized: $\exists x\in\mathcal{S}$, $\mu(x)=1\Rightarrow\mu_{\text{int}}(\mu,\mu)=1$; for Eq. (16), reflexivity holds if $t=\min$;
- if one of the sets is empty ($\forall x\in\mathcal{S}$, $\nu(x)=0$), then the degree of intersection is always 0;
- if one of the sets is equal to $\mathcal{S}(\forall x\in\mathcal{S}, \nu(x)=1)$, the degree of intersection is always equal to 1 using Eq. (16), and to 1 for normalized fuzzy sets using Eq. (14);
- invariance with respect to geometrical transformations (translation, rotation).

[1] Other definitions leading to similar properties are possible.

### 3.2. Degree of inclusion

We consider here the inclusion relation. It should be noted that a degree of equality can be derived from a degree of inclusion, by combining in a conjunctive way (using a t-norm) the degree of inclusion of one set in the other, and the degree of inclusion of the second set in the first one. Moreover, a fuzzy morphological erosion can be defined from a degree of inclusion [7] by computing the degree to which the structuring element translated at each point $x$ is included in the considered fuzzy set.

#### 3.2.1. Inclusion from other set operations

In the crisp case, we have for two sets $X$ and $Y$:

$$X\subseteq Y \Leftrightarrow X\cap Y^C=\emptyset \Leftrightarrow X^C\cup Y=\mathcal{S}. \tag{17}$$

Using the degree of intersection, we obtain for the degree of inclusion of $\mu$ in $\nu$:

$$\begin{aligned}\mathcal{I}(\mu,\nu) &= c[\mu_{\text{int}}(\mu, c(\nu))] = c\left[\sup_{x\in\mathcal{S}} t[\mu(x), c(\nu(x))]\right]\\ &= \inf_{x\in\mathcal{S}} T[c(\mu(x)), \nu(x)],\end{aligned} \tag{18}$$

where $t$ is a t-norm, $T$ is a t-conorm and $c$ is a fuzzy complementation. This definition leads to the erosion introduced in Eq. (4).

The properties of the degree of inclusion are directly derived from those of intersection since the degree of inclusion of $\mu$ in $\nu$ is equal to the degree of non-intersection of $\mu$ and $c(\nu)$:

- consistency with the binary definition;
- if $\mu$ is empty ($\forall x\in\mathcal{S}$, $\mu(x)=0$) then $\mathcal{I}(\mu,\nu)$ always equals 1;
- if $\nu$ is empty, then $\mathcal{I}(\mu,\nu)$ is equal to 0 for normalized fuzzy sets;
- if $\mu$ is equal to $\mathcal{S}(\forall x\in\mathcal{S}, \mu(x)=1)$, $\mathcal{I}(\mu,\nu)$ is equal to 0 for bounded support fuzzy sets;
- if $\nu$ is equal to $\mathcal{S}$, $\mathcal{I}(\mu,\nu)$ is always equal to 1;
- invariance with respect to geometrical transformations (translation, rotation).

A fuzzy inclusion based on integration over $\alpha$-cuts has been proposed in Ref. [13]. It consists in computing, for each $\alpha$-cut of $\mu$ and $\nu$ the ratio $r_\alpha$ between the surface of their intersection $\mu_\alpha\cap\nu_\alpha$ and the surface of $\nu_\alpha$, and then in integrating $r_\alpha$ over $\alpha$. The main drawback of this approach is its computational cost.

#### 3.2.2. Axiomatization of fuzzy inclusion

Other methods for defining a degree of inclusion rely on a set of axioms and on the determination of functions satisfying these axioms. This is the method adopted for instance in Refs. [14,15].

The axioms of Ref. [14] are as follows[2]:

(1) $\mathcal{I}(\mu,\nu)=1$ iff $\mu\subseteq\nu$ in Zadeh's sense, i.e. $\forall x\in\mathcal{S}$, $\mu(x)\leq\nu(x)$.
(2) $\mathcal{I}(\mu,\nu)=0$ iff $\mathrm{Core}(\mu)\cap[\mathrm{Supp}(\mu)]^C\neq\emptyset$.
(3) $\mathcal{I}$ is increasing in $\nu$: if $\nu_1\subseteq\nu_2$, then $\mathcal{I}(\mu,\nu_1)\leq\mathcal{I}(\mu,\nu_2)$.
(4) $\mathcal{I}$ is decreasing in $\mu$: if $\mu_1\subseteq\mu_2$, then $\mathcal{I}(\mu_1,\nu)\geq\mathcal{I}(\mu_2,\nu)$.
(5) $\mathcal{I}$ is invariant under geometric transformations such as translations, rotations.
(6) $\mathcal{I}(\mu,\nu)=\mathcal{I}(c(\nu),c(\mu))$.
(7) $\mathcal{I}(\mu\cup\mu',\nu)=\min[\mathcal{I}(\mu,\nu),\mathcal{I}(\mu',\nu)]$.
(8) $\mathcal{I}(\mu,\nu\cap\nu')=\min[\mathcal{I}(\mu,\nu),\mathcal{I}(\mu,\nu')]$.
(9) $\mathcal{I}(\mu,\nu\cup\nu')\geq\max[\mathcal{I}(\mu,\nu),\mathcal{I}(\mu,\nu')]$.

The degree of inclusion proposed in Ref. [14] according to these axioms is defined as

$$\forall\,(\mu,\nu)\in\mathcal{F}^2,\quad \mathcal{I}(\mu,\nu)=\inf_{x\in\mathcal{S}}\min[1,\lambda(\mu(x))+\lambda(1-\nu(x))] \tag{19}$$

where $\lambda$ is a function from [0,1] into [0,1] such that:

- $\lambda$ is non-increasing,
- $\lambda(0)=1$,
- $\lambda(1)=0$,
- the equation $\lambda(x)=0$ has a single solution,
- $\forall\,\alpha\in[0.5,1]$, the equation $\lambda(x)=\alpha$ has a single solution,
- $\forall\,a\in[0,1]$, $\lambda(a)+\lambda(1-a)\geq 1$.

A typical example for $\lambda$ is: $\lambda(a)=1-a^n$, with $n\geq 1$. In particular, for $n=1$, the degree of inclusion becomes:

$$\mathcal{I}(\mu,\nu)=\inf_{x\in\mathcal{S}}\min[1,1-\mu(x)+\nu(x)] \tag{20}$$

which is exactly the inclusion obtained from intersection or union (Eq. (18)) for the complementation $c(a)=1-a$ and for the Lukasiewicz t-norm and t-conorm.

Despite the apparent similarity between Eqs. (18) and (19), they are not equivalent. Indeed, the function defined as $\min[1,\lambda(1-a)+\lambda(1-b)]$, which plays in Eq. (19) the same role as the t-conorm $T$ in Eq. (18), is actually not a t-conorm, since it is not associative and it does not admit 0 as unit element, except for $\lambda(a)=1-a$ [7]. This induces a loss of properties of the inclusion degree in comparison to those of inclusion derived from a true t-conorm, as it can be seen for instance for fuzzy mathematical morphology [7].

Another axiomatization has been proposed in Ref. [15]. The axioms for degree of inclusion are the following:

(1) $\mathcal{I}(\mu,\nu)=1$ iff $\mu\subseteq\nu$ in Zadeh's sense, i.e. $\forall x\in\mathcal{S}$, $\mu(x)\leq\nu(x)$; this is the same as the first axiom of Ref. [14].
(2) Let $\nu$ be such that $\forall x\in\mathcal{S}$, $\nu(x)=0.5$. If $\nu\subseteq\mu$, then $\mathcal{I}(\mu,c(\mu))=0$ iff $\mu=\mathcal{S}$ (i.e. $\forall x\in\mathcal{S}$, $\mu(x)=1$); this contrasts with the second axiom of Ref. [14].
(3) if $\nu\subseteq\mu_1\subseteq\mu_2$, then $\mathcal{I}(\mu_1,\nu)\geq\mathcal{I}(\mu_2,\nu)$, which is weaker than axiom 4 of Ref. [14]; if $\nu_1\subseteq\nu_2$, then $\mathcal{I}(\mu,\nu_1)\leq\mathcal{I}(\mu,\nu_2)$, which is axiom 3 of Ref. [14].

These axioms are weaker than those of Ref. [14]. They lead to weaker properties of the degree of inclusion, and also to weaker properties than the degree of inclusion derived from t-norms and t-conorms.

[2] Some additional properties are proposed in Ref. [14], but not as mandatory. They are skipped in this presentation.

### 3.2.3. Inclusion and fuzzy entropy

Links between degree of inclusion and fuzzy entropy have been studied by several authors, including Refs. [15,16]. These links are expressed as

$$\forall\mu\in\mathcal{F},\quad \mathcal{I}(\mu\cup c(\mu),\mu\cap c(\mu))=E(\mu), \tag{21}$$

where $E(\mu)$ denotes the fuzzy entropy of $\mu$ [17], i.e.

$$E(\mu)=-\sum_{x\in\mathcal{S}}\mu(x)\log\mu(x)-\sum_{x\in\mathcal{S}}(1-\mu(x))\log(1-\mu(x)).$$

The definition of degree of inclusion of Ref. [16], defined for finite $\mathcal{S}$, is

$$\mathcal{I}(\mu,\nu)=\frac{|\mu\cap\nu|}{|\mu|}=\frac{\sum_{x\in\mathcal{S}}\min[\mu(x),\nu(x)]}{\sum_{x\in\mathcal{S}}\mu(x)}, \tag{22}$$

with the convention $\mathcal{I}(\mu,\nu)=1$ if $\forall x\in\mathcal{S}$, $\mu(x)=0$. This expression has been also used on each $\alpha$-cut in Ref. [13] as mentioned earlier and is related to Tversky's measures [18] (see Section 5.2).

The corresponding fuzzy entropy is then:

$$E(\mu)=\frac{\sum_{x\in\mathcal{S}}\min[\mu(x),1-\mu(x)]}{\sum_{x\in\mathcal{S}}\max[\mu(x),1-\mu(x)]}. \tag{23}$$

Another example is the degree of inclusion of Ref. [19], defined for finite $\mathcal{S}$, as:

$$\mathcal{I}(\mu,\nu)=\frac{1}{|\mathcal{S}|}\sum_{x\in\mathcal{S}}\min[1,1-\mu(x)+\nu(x)]. \tag{24}$$

The corresponding entropy measure is:

$$E(\mu)=\frac{1}{|\mathcal{S}|}\sum_{x\in\mathcal{S}}\min[\mu(x),1-\mu(x)], \tag{25}$$

which is the fuzzy entropy of Ref. [20].

This degree of inclusion is similar to the one of Ref. [21], that relies on the same combination operator but using a different type of normalization:

$$\mathcal{I}(\mu,\nu)=\frac{\inf_{x\in\mathcal{S}}\min[1,1-\mu(x)+\nu(x)]}{\sup_{x\in\mathcal{S}}\mu(x)}. \tag{26}$$

In [15], it is proved more generally that if $\mathcal{I}$ is an inclusion degree that satisfies the three axioms of Ref. [15], then the measure defined by

$$\forall\mu\in\mathcal{F},\quad E(\mu)=\mathcal{I}(\mu\cup c(\mu),\mu\cap c(\mu)) \tag{27}$$

is a fuzzy entropy measure on $\mathcal{S}$.

#### 3.2.4. Inclusion from fuzzy implication

Finally, inclusion can be defined from implication [15,22,23], as

$$\mathcal{I}(\mu,\nu) = \inf_{x\in\mathcal{S}} \mathrm{Imp}[\mu(x),\nu(x)], \tag{28}$$

where Imp[$\mu(x)$,$\nu(x)$] denotes the degree to which $\mu(x)$ implies $\nu(x)$. Fuzzy implication is often defined as [24]

$$\mathrm{Imp}(a,b) = T[c(a),b], \tag{29}$$

from which we recover exactly the same definition of degree of inclusion as the one obtained from a t-conorm (Eq. (18)). Another interesting approach is to use residual implications:

$$\mathrm{Imp}(a,b) = \sup\{\varepsilon\in[0,1], t(a,\varepsilon)\le b\}. \tag{30}$$

This provides the following expression for the degree of inclusion:

$$\mathcal{I}(\mu,\nu) = \inf_{x\in\mathcal{S}} \sup\{\varepsilon\in[0,1], t(\mu(x),\varepsilon)\le \nu(x)\}. \tag{31}$$

This definition coincides with the previous one if $t$ is an Archimedian t-norm with nilpotent elements, typically the Lukasiewicz t-norm.

### 3.3. Discussion

Set theoretical operations have a very large interest and constitute fundamental operations. Their extensions to fuzzy sets enable the management of imprecision in all applications of these operations. We already mentioned how degree of inclusion and degree of intersection can be used for defining morphological erosion and dilation. Other application domains include fusion (for instance fusion of two images providing imprecise spatial location of an object), registration (for instance based on criteria on the overlap area of two object images), measure of conflict, measure of information, etc. All these applications are important components of spatial reasoning.

The number of possible extensions of operations to fuzzy sets raises the problem of choosing the most appropriate definition. Although there is a large overlap between different classes of definitions and numerous common properties guaranteeing similar behaviors in practice, one problem remains concerning the consideration of the spatial overlapping. This problem has already been mentioned for intersection: the expression $\sup_{x\in\mathcal{S}} t[\mu(x),\nu(x)]$ represents the maximum height of the intersection and may depend on one point only while not accounting for different overlapping situations. This is shown in Fig. 1 (for sake of clarity, $\mathcal{S}$ is represented in 1D only).

Such situations are avoided by using the fuzzy volume of the overlapping area as in Eq. (16) which leads to different values for $\mu_{\mathrm{int}}(\mu,\nu)$ and $\mu_{\mathrm{int}}(\mu,\nu')$, in accordance with the fact that $\mu$ and $\nu'$ have a larger overlap than $\mu$ and $\nu$.

Similar questions occur for the degree of inclusion, since the definition in Eq. (18) has the same drawback as

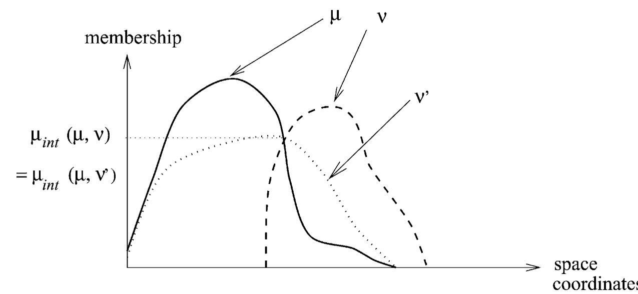



Fig. 1. Low discrimination power of the definition of degree of intersection between two fuzzy sets using the maximum of intersection: $\mu_{\mathrm{int}}(\mu,\nu')=\mu_{\mathrm{int}}(\mu,\nu)$, although $\mu$ and $\nu'$ strongly overlap and should be considered as more intersecting than $\mu$ and $\nu$.

the degree of intersection: it may depend on one point only. Here again, the overlap between both fuzzy sets could be taken into account.

## 4. Adjacency

In this section, we consider the important topological relation of adjacency, often used and important in vision and image processing, both locally in the neighborhood of a point and globally when spatial relationships between objects have to be assessed

### 4.1. Fuzzy neighborhood

Although we can use neighborhoods according to the digital topology defined on $\mathcal{S}$, fuzzy neighborhoods can also be defined, as well as a degree $n_{xy}$ to which two points $x$ and $y$ are neighbors Several definitions have been proposed [25,26], which are typically decreasing functions of the distance between both points

$$n_{xy} = \frac{1}{1+d_{\mathcal{S}}(x,y)}, \quad \text{or} \quad n_{xy} = \frac{1+\exp(-b)}{1+\exp b\left(\frac{d_{\mathcal{S}}(x,y)-1}{S}-1\right)}, \tag{32}$$

where $d_{\mathcal{S}}$ denotes the Euclidean distance in $\mathcal{S}$ and $b$ and $S$ are two positive parameters which control the shape of the curve. Other functions can be used, like S-functions for instance.

A lot of work related to this has led to the notion of degree of connectivity between two points in a fuzzy set and fuzzy connectedness, expressing to which extent points hold together to build an object [27,28]. This is merely a relation between points of a fuzzy set and not really between two fuzzy sets. It will therefore not be further considered here.

### 4.2. Adjacency between two fuzzy objects

Rosenfeld and Klette [29] define a degree of adjacency between two crisp sets, using a geometrical approach based on the notion of 'visibility' of a set from another one.

This definition is then extended to degree of adjacency between two fuzzy sets. However, this definition is not symmetrical, and probably not easy to transpose to higher dimensions. Another approach consists in using the notion of contours, frontiers, and neighborhood [25,26]. We present here this second approach. The space $\mathcal{S}$ is endowed with a digital connectivity $c$.

In the crisp digital case, two image regions $X$ and $Y$ are adjacent if

$$X \cap Y = \emptyset \quad \text{and} \quad \exists\, x \in X, \quad \exists\, y \in Y : n_c(x, y), \tag{33}$$

where $n_c(x, y)$ is the Boolean variable stating that $x$ and $y$ are neighbors in the sense of the digital $c$-connectivity.

The extension of this definition, as detailed in Ref. [26], involves a degree of intersection $\mu_{\text{int}}(\mu,\nu)$ between two fuzzy sets $\mu$ and $\nu$ defined on $\mathcal{S}$, as well as a degree of non-intersection $\mu_{\neg\text{int}}(\mu,\nu)$, and a degree of neighborhood $n_{xy}$ between two points $x$ and $y$ of $\mathcal{S}$. All these notions have been introduced previously. This leads to the following definition for fuzzy adjacency between $\mu$ and $\nu$

$$\mu_{\text{adj}}(\mu,\nu) = t[\mu_{\neg\text{int}}(\mu,\nu), \sup_{x\in\mathcal{S}} \sup_{y\in\mathcal{S}} t[\mu(x), \nu(y), n_{xy}]], \tag{34}$$

where $t$ is a t-norm realizing the conjunction between several conditions.[3]

This definition is symmetrical, consistent with the discrete binary definition (i.e. in the case where $\mu$ and $\nu$ are crisp and $n_{xy} = n_c(x, y)$), and decreasing with respect to the distance between the two fuzzy sets. It is invariant with respect to geometrical transformations (for scaling, only if $n_{xy}$ is itself invariant). It should be noted that the condition in Eq. (33) is achieved for $x$ and $y$ belonging to the boundary of $X$ and $Y$, respectively. This constraint could also be added in the fuzzy extension [26].

Definition (33) can also be expressed equivalently in terms of morphological dilation, as

$$X \cap Y = \emptyset \quad \text{and} \quad D_{B_c}(X) \cap Y \neq \emptyset, \quad D_{B_c}(Y) \cap X \neq \emptyset, \tag{35}$$

where $D_{B_c}(X)$ denotes the dilation of $X$ by the structuring element $B_c$.

The degree of adjacency between $\mu$ and $\nu$ involving fuzzy dilation is then defined as:

$$\mu_{\text{adj}}(\mu,\nu) = t[\mu_{\neg\text{int}}(\mu,\nu), \mu_{\text{int}}[D_{B_c}(\mu), \nu], \mu_{\text{int}}[D_{B_c}(\nu), \mu]]. \tag{36}$$

This definition represents a conjunctive combination of a degree of non-intersection between $\mu$ and $\nu$ and a degree of intersection between one fuzzy set and the dilation of the other. Again the same properties are satisfied. The structuring element $B_c$ can be taken as the elementary structuring element related to the considered connectivity (i.e. a central point and its neighbors as defined by the $c$-connectivity).

[3] Since any t-norm is associative, for the sake of simplicity we denote by $t(a, b, c)$ the expression $t(t(a,b),c)$.

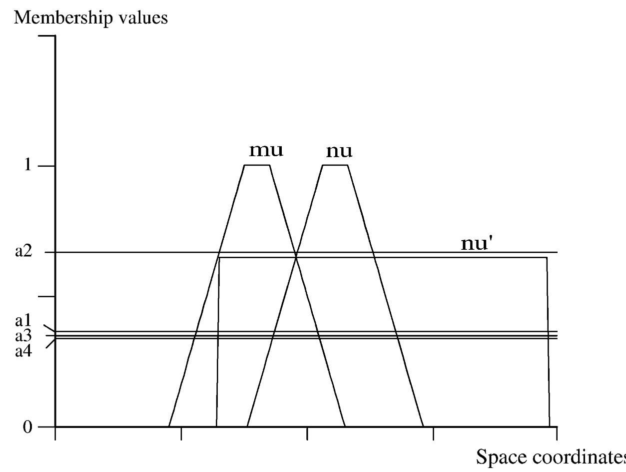



Fig. 2. Illustration of Eq. (34) when using different definitions for the degree of intersection. Using the maximum of the intersection we obtain $\mu_{\text{adj}}(\mu, \nu) = \alpha_1$ $(=0.36)$ and $\mu_{\text{adj}}(\mu, \nu') = \alpha_3$ $(=0.35)$, and using the fuzzy hypervolume $\mu_{\text{adj}}(\mu, \nu) = \alpha_2$ $(=0.67)$ and $\mu_{\text{adj}}(\mu, \nu') = \alpha_4$ $(=0.34)$.

It can also be a fuzzy structuring element, representing for instance spatial imprecision (i.e. the possibility distribution of the location of each point).

### 4.3. Discussion

The choice of the degree of intersection plays an important role in the definitions of adjacency between fuzzy sets. Fig. 2 shows the results obtained with Eq. (34) with the t-norm minimum. Using the maximum of the intersection for $\mu_{\text{int}}$ (Eq. (14)) we obtain $\mu_{\text{adj}}(\mu, \nu) = 0.36$ and $\mu_{\text{adj}}(\mu, \nu') = 0.35$, which are very similar values. On the contrary, using the fuzzy hypervolume to define the degree of intersection (Eq. (16)), Eq. (34) accounts for the differences in intersection and provides $\mu_{\text{adj}}(\mu, \nu) = 0.67$ and $\mu_{\text{adj}}(\mu, \nu') = 0.34$, which are this time very different.

Let us now compute this fuzzy adjacency on a few objects extracted from a real image. As an illustrative example, a slice of a brain image is shown in Fig. 3. It is obtained using a T1 weighted acquisition in magnetic resonance imaging. A few internal structures are

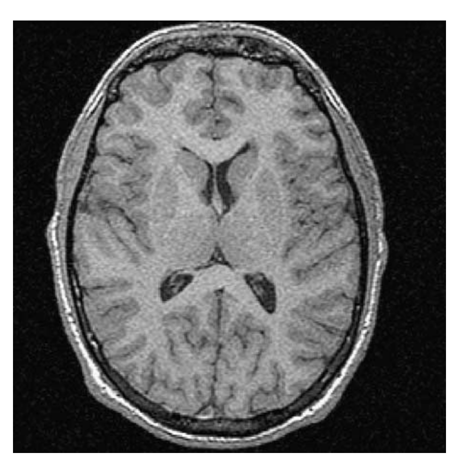


Fig. 3. Magnetic resonance image of a brain (one slice of a 3D volume).

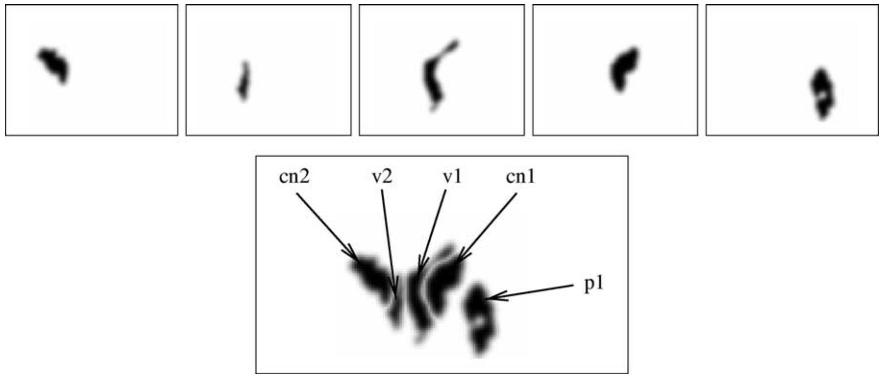



Fig. 4. Top: 5 fuzzy objects representing a few internal brain structures of the image shown in Fig. 3 (membership values rank between 0 and 1, from white to black). From left to right: right caudate nucleus (cn2), right lateral ventricle (v2), left lateral ventricle (v1), left caudate nucleus (cn1), left putamen (p1). Bottom: superimposition of these fuzzy objects (the maximum membership value is displayed at each point).

represented in Fig. 4 as spatial fuzzy sets, where membership degrees are represented using grey levels. The adjacency degrees between some of these fuzzy objects are given in Table 2. The results are in agreement with what can be expected from a brain model (an anatomical atlas for instance, where objects and adjacency are defined in a crisp way). Two classes of values are obtained: very low values which correspond to non-adjacency in the model and a set of higher values corresponding to adjacency in the model. In this case, crisp adjacency would provide completely different results in the model and in the image, preventing its use for recognition. This suggests that fuzzy adjacency degrees can indeed be used for pattern recognition purposes, as introduced in Section 1, of course combined with other spatial relationships.

## 5. Distances

The importance of distances in image processing and interpretation is well established. Their extensions to fuzzy sets can be useful in several problems in image processing under imprecision. Several definitions can be found in the literature for distances between fuzzy sets (which is the main addressed problem). They can be roughly divided in two classes: distances that take only membership functions into account and that compare them pointwise, and distances that additionally include spatial distances. The wide literature on fuzzy similarities, dissimilarities and distances is rather silent on methods dealing with spatial information, and, unfortunately, not all approaches are suitable to this purpose. The presentation given below is directly inspired by the classification proposed in Ref. [30], but adapted to image processing and vision purposes, by underlining for each definition its properties and the type of image information on which it relies. A complete review can be found in Ref. [31].

### 5.1. Representations

Before reviewing the main definitions, we address some representation issues.

The most used representation of a distance between two fuzzy sets is as a number $d$, taking values in $\mathbb{R}^+$ (or more specifically in [0,1] for some of them). However, since we consider fuzzy sets, i.e. objects that are imprecisely defined, we may expect that the distance between them is imprecise too [32,33]. Then, the distance is better represented as a fuzzy set, and more precisely as a fuzzy number.

In Ref. [33], Rosenfeld defines two concepts that will be used in the sequel. One is distance density, denoted by $\delta(\mu,\nu)$, and the other distance distribution, denoted by $\Delta(\mu,\nu)$, both being fuzzy sets on $\mathbb{R}^+$. They are linked together by the following relation:

$$\Delta(\mu,\nu)(n) = \int_0^n \delta(\mu,\nu)(n')\mathrm{d}n'. \tag{37}$$

While the distance distribution value $\Delta(\mu, \nu)$ $(n)$ represents the degree to which the distance between $\mu$ and $\nu$ is less than $n$, the distance density value $\delta(\mu, \nu)$ $(n)$ represents the degree to which the distance is equal to $n$.

Table 2
Results obtained for fuzzy adjacency

| Fuzzy object 1 | Fuzzy object 2 | Degree of adjacency | Adjacency in the model (crisp) |
|---|---|---|---|
| v1 | v2 | 0.368 | 1 |
| v1 | cn1 | 0.463 | 1 |
| v1 | p1 | 0.000 | 0 |
| v1 | cn2 | 0.035 | 0 |
| v2 | cn2 | 0.427 | 1 |
| cn1 | p1 | 0.035 | 0 |

Labels of structures are given in Fig. 4. High degrees are obtained between structures where adjacency is expected, while very low degrees are obtained in the opposite case.

Histograms of distances inspired from angle histograms (see Section 6.3), carrying a complete information about distance relationships but at the price of a heavier representation, have been introduced in Ref. [31]. They will not be detailed here.

The concept of distance can be represented as a linguistic variable. This assumes a granulation [9] of the set of possible distance values into symbolic classes such as 'near', 'far', etc. each of these classes being defined as a fuzzy set. This approach has been drawn, e.g. in Refs. [13, 34,35]. For instance in Ref. [13], the relation 'far' is defined as a decreasing function of the average distance between both sets.

Finally spatial representations are useful to define the regions of the space where some distance constraint to a reference object is satisfied [31,36]. Such constraints are often expressed as imprecise statements or in linguistic terms, which reinforces the usefulness of fuzzy modeling.

In the following sections, we will mainly present methods providing numbers or fuzzy numbers.

### 5.2. Comparison of membership functions

In this section, we review the main distances proposed in the literature that aim at comparing membership functions. They have generally been proposed in a general fuzzy set framework, and not specifically in the context of image processing. They do not really include information about spatial distances. The classification chosen here is inspired from the one in Ref. [30]. Similar classifications can be found in Refs. [37–39]. More details and properties are given in Ref. [31].

The *functional approach* is probably the most popular one. It relies on a $L_p$ norm between $\mu$ and $\nu$, leading to the following generic definition [32,40,41]:

$$d_p(\mu,\nu) = \left[\int_{x\in\mathcal{S}} |\mu(x)-\nu(x)|^p\right]^{1/p}, \tag{38}$$

$$d_\infty(\mu,\nu) = \sup_{x\in\mathcal{S}} |\mu(x)-\nu(x)|. \tag{39}$$

In the discrete finite case, these definitions use discrete sums and max, respectively.

A noticeable property of $d_p$ is that it takes a constant value if the supports of $\mu$ and $\nu$ are disjoint, irrespectively of how far the supports are from each other in $\mathcal{S}$.

Among the *information theoretic approaches*, definitions based on fuzzy entropy or fuzzy divergence have been proposed [17,42–44]. But one main drawback of most of these approaches is that the obtained distance is always equal to 0 for crisp sets.

In the *set theoretic approach*, distance between two fuzzy sets is seen as a dissimilarity function, based on fuzzy union and intersection. Examples are given in Ref. [30]. The basic idea is that the distance should be larger if the two fuzzy sets weakly intersect. Most of the proposed measures are inspired from the work by Tversky [18] that proposes two parametric similarity measures between two sets $A$ and $B$:

$$\theta f(A\cap B) - \alpha f(A-B) - \beta f(B-A), \tag{40}$$

and in a rational form

$$\frac{f(A\cap B)}{f(A\cap B)+\alpha f(A\cap B^C)+\beta f(B\cap A^C)}, \tag{41}$$

where $f(X)$ is typically the cardinality of $X$, and $\alpha$, $\beta$ and $\theta$ are parameters leading to different kinds of measures.

Let us mention a few examples (they are given in the finite discrete case). A measure being derived from the second Tversky measure by setting $\alpha=\beta=1$ has been used by several authors [5,30,37–39,45,46]:

$$d(\mu,\nu) = 1 - \frac{\sum_{x\in\mathcal{S}} \min[\mu(x),\nu(x)]}{\sum_{x\in\mathcal{S}} \max[\mu(x),\nu(x)]}. \tag{42}$$

It does not satisfy the triangular inequality, and always takes the constant value 1 as soon as the two fuzzy sets have disjoint supports. It also corresponds to the Jaccard index [45]. With respect to the typology presented in Ref. [47], this distance is a comparison measure, and more precisely a dissimilarity measure. Moreover, $1-d$ is a resemblance measure. Applications in image processing can be found, e.g. in Ref. [48], where it is used on fuzzy sets representing object features (and not directly spatial image objects) for structural pattern recognition on polygonal 2D objects.

A slightly different formula has been proposed in Ref. [49] and modified in Ref. [50] in order to achieve better properties:

$$d(\mu,\nu) = 1 - \frac{1}{|\mathrm{Supp}(\mu)\cup\mathrm{Supp}(\nu)|} \times \sum_{x\in\mathrm{Supp}(\mu)\cup\mathrm{Supp}(\nu)} \frac{t[\mu(x),\nu(x)]}{T[\mu(x),\nu(x)]}. \tag{43}$$

Another measure takes into account only the intersection of the two fuzzy sets [30,37,39]:

$$d(\mu,\nu) = 1 - \max_{x\in\mathcal{S}} \min[\mu(x),\nu(x)]. \tag{44}$$

Again it is a dissimilarity measure, and $1-d$ is a resemblance measure. It is always equal to 1 if the supports of $\mu$ and $\nu$ are disjoint.

If we set $(\mu\nu)(x)=\max[\min(\mu(x),1-\nu(x)),\min(1-\mu(x),\nu(x))]$, two other distances can be derived, as [30,39]:

$$d(\mu,\nu) = \sup_{x\in\mathcal{S}}(\mu\nu)(x), \tag{45}$$

$$d(\mu,\nu) = \sum_{x\in\mathcal{S}}(\mu\nu)(x). \tag{46}$$

These two distances are symmetrical measures. They are separable only for binary sets. Also we have $d(\mu,\mu)=0$ only for binary sets. They are dissimilarity measures. The first one is equal to 1 if $\mu$ and $\nu$ have disjoint supports and are normalized (if they are not normalized, then this constant value is equal to the maximum membership value of $\mu$ and $\nu$). The second measure is always equal to $|\mu|+|\nu|$ if $\mu$ and $\nu$ have disjoint supports.

These measures actually rely on measures of inclusion of each fuzzy set in the other (see Section 3.2). Indeed, the distance should be small if the two sets have a small degree of equality (the equality between $\mu$ and $\nu$ can be expressed by '$\mu$ included in $\nu$ and $\nu$ included in $\mu$', which leads to an easy transposition to fuzzy equality). Other inclusion indexes can be defined, e.g. from Tversky measure by setting $\alpha=1$ and $\beta=0$, leading to $f(A\cap B)/f(A)$ [45]. As an example, these measures have been applied in image processing for image database applications in Ref. [39].

The last definitions given by Eqs. (44) and (45) are, respectively, equivalent to $1-\Pi(\mu;\nu)$ and $1-\max[N(\mu;\nu), N(\nu;\mu)]$ (where $\Pi$ and $N$ are possibility and necessity functions) used in fuzzy pattern matching [51,52], which has a large application domain, including image processing and vision (see e.g. [53]).

Finally, the *pattern recognition approach* consists in first expressing each fuzzy set in a feature space (for instance cardinality, moments, skewness) and to compute the Euclidean distance between two feature vectors [30] or attribute vectors [54]. This approach may take advantage of some of the previous approaches, for instance by using entropy or similarity in the set of features. It has been applied for instance for database applications [54]. A similar approach, called signal detection theory, has been proposed in Ref. [39]. It is based on counting the number of similar and different features. These approaches are somewhat apart from our main focus since they do not use directly the fuzzy sets but features extracted from them.

### 5.3. Combination of spatial and membership comparisons

The second class of methods tries to include the spatial distance $d_{\mathcal{S}}$ (Euclidean distance in $\mathcal{S}$ for instance) in the distance between $\mu$ and $\nu$. In contrary to the definitions given above, in this second class the membership values at different points of $\mathcal{S}$ are linked using some formal computation, making the introduction of $d_{\mathcal{S}}$ possible. This leads to definitions that do not share the drawbacks of previous approaches, for instance when the supports of the two fuzzy sets are disjoint.

The *geometrical approach* consists in generalizing one of the distances between crisp sets. This has been done for instance for nearest point distance [32,33], mean distance [33], Hausdorff distance [32], and could easily be extended to other distances (see e.g. Ref. [55] for a review of crisp set distances). These generalizations can be obtained according to a fuzzification by integration over $\alpha$-cuts (see Section 2) [30,56].

Another method consists in weighting distances by membership values. For the mean distance this leads for instance to [33]:

$$d(\mu,\nu) = \frac{\sum_{x\in\mathcal{S}}\sum_{y\in\mathcal{S}} d_{\mathcal{S}}(x,y)\min[\mu(x),\nu(y)]}{\sum_{x\in\mathcal{S}}\sum_{y\in\mathcal{S}}\min[\mu(x),\nu(y)]}. \tag{47}$$

A third approach consists in defining a fuzzy distance as a fuzzy set on $\mathbb{R}^+$ instead of as a crisp number using the extension principle (see Section 2). For the nearest point distance this leads to [33]:

$$d(\mu,\nu)(r) = \sup_{x,y,d_{\mathcal{S}}(x,y)\le r} \min[\mu(x),\nu(y)]. \tag{48}$$

A similar approach has been used in Ref. [57], and the corresponding distance density is expressed as:

$$d(\mu,\nu)(r) = \sup_{x,y,d_{\mathcal{S}}(x,y)=r} \min[\mu(x),\nu(y)], \tag{49}$$

The fuzzy extension of the Hausdorff distance is probably the most widely studied of all the fuzzy extensions of distances between sets. One reason for this may be that it is a true metric in the crisp case, while other set distances like minimum or average distances have weaker properties. Another reason is that it has been used to determine a degree of similarity between two objects, or between an object and a model [58]. Extensions of this distance have been defined using fuzzification over the $\alpha$-cuts and using the extension principle [30,59,60–62]. One potential problem with these approaches occurs in the case of empty $\alpha$-cuts [63,64]. Boxer [62] proposed to add a crisp set to every set, but the result is highly dependent on this additional set, and does not reduce to the classical Hausdorff distance when applied on crisp sets. The solution proposed in Ref. [63] consists in clipping the distance at some maximum distance, but similar problems arise. Other authors use the Hausdorff distance between the endographs of the two membership functions [60] (but the additional dimension has not the same meaning as the spatial dimension). Several generalizations of Hausdorff distance have also been proposed under the form of fuzzy numbers [32]. Extensions of the Hausdorff distance based on fuzzy mathematical morphology has also been developed [65] and is presented below.

These distances share most of the advantages and drawbacks of the underlying crisp distance [55]: computation cost can be high (it is already high for several crisp distances); moreover, interpretation and robustness strongly depend on the chosen distance (for instance, Hausdorff distance may be noise sensitive, whereas average distance is less). Extensions of these definitions may be obtained by using other weighting functions, for instance by using t-norms instead of min.

A *morphological approach* has been proposed in Refs. [65,66]. We just give the examples of nearest point distance and Hausdorff distance. The main idea here is to use links between distances and morphological dilations to derive algebraic expressions of distances (instead of classical analytical ones), which are then easy to translate into fuzzy expressions.

In the binary case, and in a digital space $\mathcal{S}$, for $n>0$, the nearest point distance $d_N$ can be expressed in morphological terms as

$$d_N(X,Y) = n \Leftrightarrow D^n(X) \cap Y \neq \emptyset \text{ and } D^{n-1}(X) \cap Y = \emptyset \tag{50}$$

and the symmetrical expression. For $n=0$ we have: $d_N(X, Y)=0 \Leftrightarrow X \cap Y \neq \emptyset$. The translation of these equivalences provides, for $n>0$, the following distance density

$$\delta_N(\mu,\mu')(n) = t\left[\sup_{x\in\mathcal{S}} t[\mu'(x), D^n_\nu(\mu)(x)], c\left[\sup_{x\in\mathcal{S}} t[\mu'(x), D^{n-1}_\nu(\mu)(x)]\right]\right] \tag{51}$$

or a symmetrical expression derived from this one, and $\delta_N(\mu,\mu')(0)=\sup_{x\in\mathcal{S}} t[\mu(x),\mu'(x)]$ (i.e. a degree of intersection as in Eq. (14)).

Like for the nearest point distance, we can extend the Hausdorff distance by translating directly the binary equation defining the Hausdorff distance:

$$d_H(X,Y) = \max\left[\sup_{x\in X} d_{\mathcal{S}}(x,Y), \sup_{x\in Y} d_{\mathcal{S}}(y,Y)\right]. \tag{52}$$

This distance can be expressed in morphological terms as:

$$d_H(X,Y) = \inf\{n, X \subseteq D^n(Y) \quad \text{and} \quad Y \subseteq D^n(X)\}. \tag{53}$$

From Eq. (53), a distance distribution can be defined, involving fuzzy dilation

$$\Delta_H(\mu,\mu')(n) = t\left[\inf_{x\in\mathcal{S}} T[D^n_\nu(\mu)(x), c(\mu'(x))], \inf_{x\in\mathcal{S}} T[D^n_\nu(\mu')(x), c(\mu(x))]\right], \tag{54}$$

where $c$ is a complementation, $t$ a $t$-norm and $T$ a t-conorm. A distance density can be derived implicitly from this distance distribution.

A direct definition of a distance density can be obtained from: $d_H(X,Y)=0 \Leftrightarrow X=Y$, and for $n>0$:

$$d_H(X,Y) = n \Leftrightarrow X \subseteq D^n(Y) \text{ and } Y \subseteq D^n(X) \text{ and } (X \not\subset D^{n-1}(Y) \text{ or } Y \not\subset D^{n-1}(X)). \tag{55}$$

Translating these equations leads to a definition of the Hausdorff distance between two fuzzy sets $\mu$ and $\mu'$ as a fuzzy number:

$$\delta_H(\mu,\mu')(0) = t\left[\inf_{x\in\mathcal{S}} T[\mu(x), c(\mu'(x))], \inf_{x\in\mathcal{S}} T[\mu'(x), c(\mu(x))]\right], \tag{56}$$

$$\delta_H(\mu,\mu')(n) = t\left[\inf_{x\in\mathcal{S}} T[D^n_\nu(\mu)(x), c(\mu'(x))], \inf_{x\in\mathcal{S}} T[D^n_\nu(\mu')(x), c(\mu(x))], T\left(\sup_{x\in\mathcal{S}} t[\mu(x), c(D^{n-1}_\nu(\mu')(x))], \sup_{x\in\mathcal{S}} t[\mu'(x), c(D^{n-1}_\nu(\mu)(x))]\right)\right] \tag{57}$$

The above definitions of fuzzy nearest point and Hausdorff distances (defined as fuzzy numbers) between two fuzzy sets do not necessarily share the same properties as their crisp equivalent. All distances are positive, in the sense that the defined fuzzy numbers have always a support included in $\mathbb{R}^+$. By construction, all defined distances are symmetrical with respect to $\mu$ and $\mu'$. The separability property is not always satisfied. However, if $\mu$ is normalized, we have for the nearest point distance $\delta_N(\mu,\mu)$ $(0)=1$ and $\delta_N(\mu,\mu)$ $(n)=0$ for $n>1$. For the Hausdorff distance, $\delta_H(\mu,\mu')$ $(0)=1$ implies for $T$ being the bounded sum ($T(a, b)=\min(1, a+b)$), while it implies $\mu$ and $\mu'$ crisp and equal for $T=\max$. Also the triangular inequality is not satisfied in general.

Another morphological approach has been suggested in Ref. [67], based on links between the minimum distance and the Minkowski difference. In the crisp case, we have $d_N(X,Y)=\inf\{|z|, z \in Y \ominus X\}$, if $X$ and $Y$ are non-intersecting crisp sets. In order to account for possible intersection between the two sets, the authors introduce also the notion of penetration distance, defined along a direction $\sigma$ as the maximum translation of $X$ along $\sigma$ such that $X$ still meets $Y$ : $\delta(\sigma; X, Y) = \max\{k, (X+k\sigma) \cap Y \neq \emptyset\}$. The extension to fuzzy sets is done by assuming fuzzy numbers on each axis. This leads to nice reasonable computation times, but can unfortunately not be directly extended to any fuzzy objects because of this assumption.

A *tolerance-based approach* has been developed in Ref. [41]. The basic idea is to combine spatial information and membership values by assuming a tolerance value $\tau$, indicating the differences that can occur without saying that the objects are no more similar. Note that this approach has strong links with morphological approaches, since the neighborhood considered around each point can be considered as a structuring element.

According to a *graph theoretic approach*, a similarity function between fuzzy graphs may also induce a distance between fuzzy sets. This approach contrasts with the previous ones, since the objects are no more represented directly as fuzzy sets on $\mathcal{S}$ or as vectors of attributes, but as higher level structures. Fuzzy graphs in structural recognition can be used for representing objects, as in Ref. [68], or a scene, as in Ref. [69]. In the first case, nodes are parts of the objects and edges are links between these parts. In the second case, nodes are objects of the scene and edges are relationships between these objects. These two examples use different ways to consider distances (or similarity) between fuzzy graphs. In Ref. [68], the distance is defined from a similarity between nodes and between edges (both being fuzzy sets), given a correspondence between nodes (respectively between edges). The similarity used compares only membership functions, using a set theoretic approach (see Section 5.2) and corresponds to Eq. (42). Although it has not been considered in this reference, spatial distance can then be taken into account if we include it in the attribute set. This idea is probably worth to be further developed. In a similar way, several distances between graphs have been proposed as an objective function to find the correspondence between graphs. This function compares attributes of nodes of the two graphs to be matched, and attributes of arcs. One of the main difficulties is to deal with non-objective matching. This has been addressed for instance in Refs. [70–72], where a formalism for defining fuzzy morphisms between graphs is proposed, as well as optimization methods for finding the best morphism according to an objective function including spatial distance information as an edge attribute. Another way to consider distances between objects is in terms of cost of deformations to bring one set in correspondence with the other. Such approaches are particularly powerful in graph-based methods. The distance can then be expressed as the cost of the matching of two graphs, as done in Ref. [69] for image processing applications, or as the Levensthein distance accounting for the necessary transformations (insertions, substitutions, deletions) for going from the structural representation of one shape to the representation of the other [73]. In Ref. [69], the fuzzy aspect is taken into account as weighting factors, therefore the method is quite close to the weighted Levensthein distance of Ref. [73]. Spatial distances could also be introduced as one of the relationships between objects in these approaches. A distance between conceptual graphs is defined in Ref. [74], as an interval $[N, \Pi]$ where $N$ represents the necessity and $\Pi$ the possibility, obtained by a fuzzy pattern matching approach. Although the application is not related to image processing, the idea of expressing similarity as an interval is interesting and could certainly be exploited in other domains. A second interest of this approach is that the nodes of the graph are concepts, which could be (although not explicitly mentioned in this reference) represented as fuzzy sets (like linguistic variables). Although these examples are still far from the main concern of this paper, it is worth mentioning them, since they bring an interesting structural aspect that could be further developed.

### 5.4. Discussion

In the first class of methods (Section 5.2), the only way $\mu$ and $\nu$ are combined is by computation linking $\mu(x)$ and $\nu(x)$, i.e. only the memberships at the same point of $\mathcal{S}$. No spatial information is taken into account. A positive consequence is that the corresponding distances are easy to compute. The complexity is linear in the cardinality of $\mathcal{S}$. Considering image processing and vision applications, we suggest that the first class of methods (comparing membership functions only) be restricted when the two fuzzy sets to be compared represent the same structure or a structure and a model. Applications in model-based or case-based pattern recognition are foreseeable.

On the other hand, the definitions which combine spatial distance and fuzzy membership comparison (Section 5.3) allow for a more general analysis of structures in images, for applications where the topological and spatial arrangement of the structures of interest is important (segmentation, classification, scene interpretation). This is enabled by the fact that these distances combine membership values at different points in the space, therefore taking into account their proximity or farness in $\mathcal{S}$. The price to pay is an increased complexity, generally quadratic in the cardinality of $\mathcal{S}$.

When facing the problem of choosing a distance, several criteria can be used. First, the type of application at hand plays an important role. While both classes of methods can be used for comparing an object and a model object, only the second class can be used for evaluating distances between objects in the same image. Among the distances of the first class, the results we obtained show that entropy and divergence-based approaches are not satisfactory. Also normalized distances should be avoided in most cases. The choice among the remaining distances can be done by looking at the properties of the distances (for instance, do we need $d(\mu,\mu)=0$ for the application at hand?), and at the computation time. Among the distances of the second class, similar choice criteria can be used. Although we may speak about distances between image objects in a very general way, this expression does not make necessarily the assumption that we are dealing with true metrics. For several applications in image processing, it is not sure that all properties are needed. For instance the triangular inequality is not always a requirement.

In order to illustrate the differences between various definitions, we have computed distances from all structures shown in Fig. 4 to v2, using most of the definitions reviewed in this paper. The results obtained with the distances of the first class are summarized in Table 3. For definitions involving min and max as intersection and union, we computed the results obtained with extended definitions, using other t-norms and t-conorms. The results using distances of the second class are given in Table 4 for the geometrical approach, and in Table 5 for the morphological approach.

As can be observed from Table 3, the $d_p$ distances (lines 1–3) are not able to differentiate the structures with respect to v2: a value 0 is obtained for v2 (since $d_p(\mu,\mu)=0$) and for all other objects almost the same value is obtained. The problem of the constant value if the supports of $\mu$ and $\nu$ are disjoint can also be observed: in this example, v2 and nc1 have disjoint supports, as well as v2 and t1. Using these distances, nc1 and t1 have the same distance to v2, although t1 is farther from v2 than nc1 in $\mathcal{S}$.

Since the distance based on fuzzy entropy (line 4) does not combine points of $\mu$ with points of $\nu$, but only

Table 3
Distances between fuzzy sets of Fig. 4 using the definitions involving comparison of membership functions only

| | | Distance between v2 and: | | | | |
|---|---|---|---|---|---|---|
| | | nc2 | v2 | v1 | nc1 | t1 |
| 1 | $d_\infty$ | 1.000 | 0.000 | 0.996 | 1.000 | 1.000 |
| 2 | $d_1$ | 255.510 | 0.000 | 289.718 | 317.380 | 322.078 |
| 3 | $d_2$ | 13.348 | 0.000 | 13.532 | 15.054 | 14.849 |
| 4 | Fuzzy entropy | 40.831 | 0.000 | 127.679 | 55.575 | 87.030 |
| 5 | Fuzzy divergence | 164.720 | 0.000 | 125.834 | 0.000 | 0.000 |
| 6 | Pappis (diff/sum) | 0.945 | 0.000 | 0.957 | 1.000 | 1.000 |
| 7 | 1-sum of $t$ over sum of $T$ ($t$=min) | 0.971 | 0.000 | 0.978 | 1.000 | 1.000 |
| 8 | 1-sum of $t$ over sum of $T$ ($t$=prod) | 0.989 | 0.644 | 0.992 | 1.000 | 1.000 |
| 9 | 1-sum of $t$ over sum of $T$ (Lukasiewicz) | 1.000 | 0.821 | 1.000 | 1.000 | 1.000 |
| 10 | 1-norm. sum of $t$ over $T$ ($t$=min) | 0.126 | 0.000 | 0.165 | 0.163 | 0.168 |
| 11 | 1-norm. sum of $t$ over $T$ ($t$=prod) | 0.129 | 0.049 | 0.169 | 0.163 | 0.168 |
| 12 | 1-norm. sum of $t$ over $T$ (Lukasiewicz) | 0.131 | 0.055 | 0.170 | 0.163 | 0.168 |
| 13 | 1-max of inter. (min) | 0.667 | 0.110 | 0.749 | 1.000 | 1.000 |
| 14 | 1-max of inter. (product) | 0.890 | 0.208 | 0.910 | 1.000 | 1.000 |
| 15 | 1-max of inter. (Lukasiewicz) | 1.000 | 0.220 | 1.000 | 1.000 | 1.000 |
| 16 | max of non-inclusion (min) | 1.000 | 0.498 | 0.996 | 1.000 | 1.000 |
| 17 | max of non-inclusion (product) | 1.000 | 0.247 | 0.997 | 1.000 | 1.000 |
| 18 | max of non-inclusion (Lukasiewicz) | 1.000 | 0.000 | 0.997 | 1.000 | 1.000 |
| 19 | norm. sum of non-incl. (min) | 0.053 | 0.011 | 0.060 | 0.064 | 0.065 |
| 20 | norm. sum of non-incl. (product) | 0.053 | 0.007 | 0.060 | 0.064 | 0.065 |
| 21 | norm. sum of non-incl. (Lukasiewicz) | 0.052 | 0.000 | 0.059 | 0.064 | 0.065 |

a global measure of the fuzzy sets, made separately, the results can even be counter-intuitive. In this example, nc1 has a lower distance to v2 than v1, although v1 is closer to v2 than nc1 in $\mathcal{S}$ (spatially). For the fuzzy divergene (line 5), similar problems occur: some counter-intuitive results are obtained, nc1 and t1 have a null distance to v2.

The distances presented in lines 6–18 are not able to differentiate between nc1 and t1, and even v1 and nc2. Very similar values are obtained for all these structures, although they are spatially at very different distances from v2. The property $d(\mu, \mu)=0$ does not always hold (see lines 8, 9, 11–17). In lines 19–21, similar problems are observed. Additionally, the normalization leads to very low values for all structures. This is a general problem due to normalization for all these distances.

Using distances taking into account spatial information, more satisfactory results are obtained. Using the geometrical approach (Table 4), the lowest value is always obtained for v2. A null value is obtained only using the minimum and the Hausdorff distances, since they are the only ones which satisfy $d(\mu,\mu)=0$. Objects nc2 and v1 have similar distances to v2, as it appears in Fig. 4.

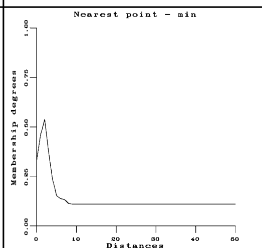
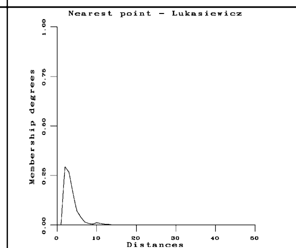
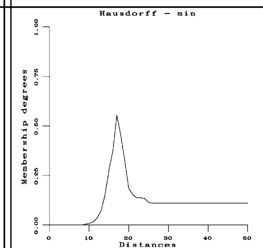
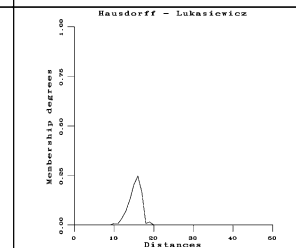
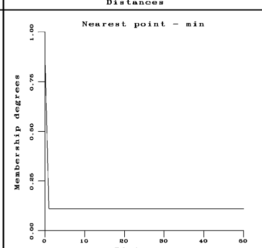
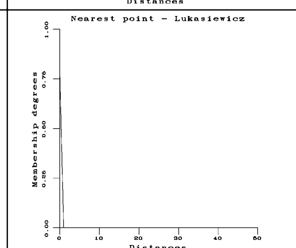
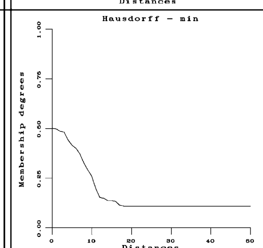
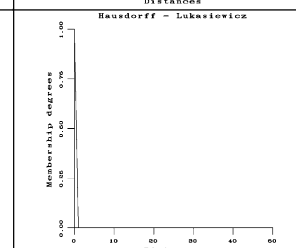
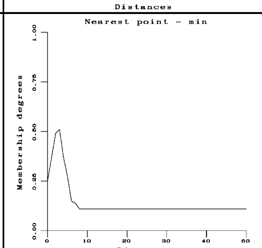
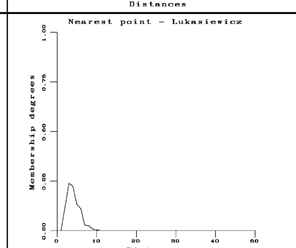
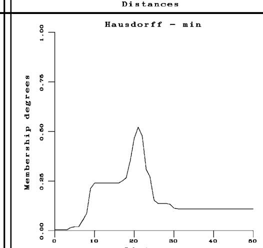
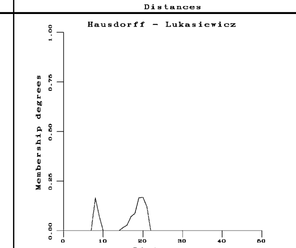
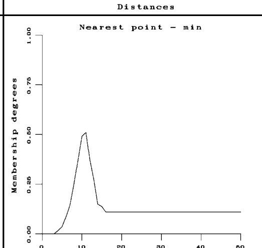
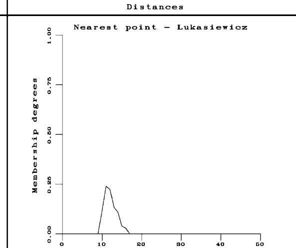
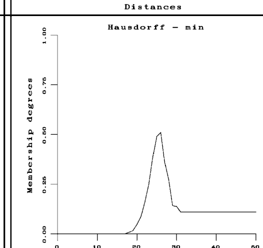
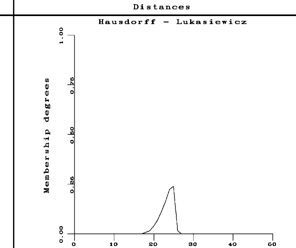
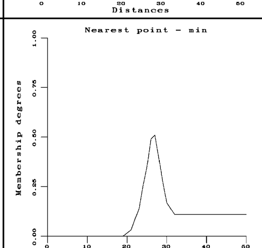
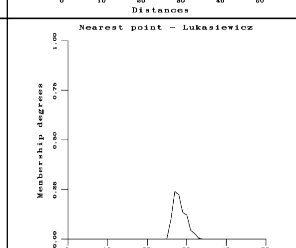
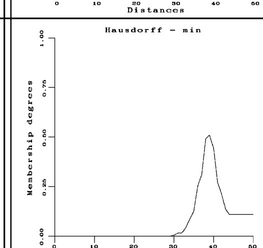
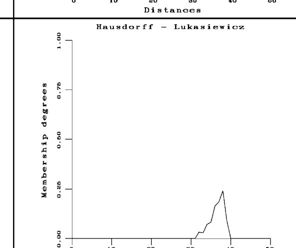
Then nc1 is found farther, and then t1. These results fit well the intuition. Using different t-norms in the weighted average distance changes the absolute values that are obtained, but not the ranking. Since the following inequalities hold: $\forall(a, b)\in[0,1]^2$, $\max(0, a+b-1)\leq ab\leq \min(a, b)$, similar inequalities between the derived distances are obtained. For this distance, the choice of the t-norm is not really important, since it does not change the properties of the distance, and for image processing purposes, the ranking between distance values is often more important that their absolute value.

All previous examples provide results as numbers. When using the morphological approach, the results take the form of fuzzy numbers as seen in Table 5. The curves in this table show the degrees to which the distance is equal to $n$ as a function of $n$. Again the results fit well the intuition. The distributions obtained for v2 are concentrated on the low distance values. Then, when the structures become farther

Table 4
Distances between fuzzy sets of Fig. 4 using the geometrical approach: weighted average distance using 3 different t-norms, fuzzification (using integral over $\alpha$-cuts) of mean, min and Hausdorff distances

| | Distance between v2 and: | | | | |
|---|---|---|---|---|---|
| | nc2 | v2 | v1 | nc1 | t1 |
| Weighted mean dist. (min) | 16.296 | 8.165 | 16.402 | 22.820 | 36.762 |
| Weighted mean dist. (prod) | 16.174 | 7.501 | 15.402 | 22.668 | 36.589 |
| Weighted mean dist. (Lukasiewicz) | 16.096 | 5.855 | 13.574 | 22.502 | 36.299 |
| Integral of mean dist. | 14.536 | 5.145 | 12.897 | 20.298 | 32.453 |
| Integral of min dist | 1.696 | 0.000 | 2.071 | 8.937 | 23.204 |
| Integral of Hausdorff dist | 19.068 | 0.000 | 21.944 | 25.373 | 35.952 |

Table 5
Distances between fuzzy sets of Fig. 4 using the morphological approach, for the nearest point distance and the Hausdorff distance, using two different t-norms

| Distance between v2 and: | | | | |
|---|---|---|---|---|
| Distance | $\delta_N$ (min) | $\delta_N$ (Lukasiewicz) | $\delta_H$ (min) | $\delta_H$ (Lukasiewicz) |
| nc2 |  |  |  |  |
| v2 |  |  |  |  |
| v1 |  |  |  |  |
| nc1 |  |  |  |  |
| t1 |  |  |  |  |

from v2, the curves are shifted towards higher distance values. Here again, the choice of a specific t-norm is not crucial as it changes mainly the absolute values. Lower membership degrees are obtained when using a smaller t-norm.

The fact that the Hausdorff distance provides higher values than the nearest point distance corresponds to the fact that the size of the dilation applied to one set needed to reach the other is less than the size of the dilation needed to completely include the other set. This is the case for crisp sets, and the same property holds in the fuzzy case.

## 6. Directional relative position between objects

Now we consider a completely different type of relationships, directional relative position, which is a typical example of relations that defy precise definitions.

Although such relations are very important and explicitly mentioned by Freeman and by Kuipers, crisp definitions are clearly not appropriate. Therefore, in this case, the general principles presented in Section 2 cannot be applied. The usual way to consider such relations in common language is as a matter of degree and in a non-exclusive manner (several relations between two given objects can be satisfied to some degree). Fuzzy set theory appears then as an appropriate tool for such modeling even for crisp sets. The main fuzzy approaches are reviewed here. More details can be found in Ref. [75].

### 6.1. Fuzzy relations describing relative position

In Refs. [10,76], the angle between the segment joining two points $a$ and $b$ and the $x$-axis of the coordinate frame (in 2D) is computed. This angle, denoted by $\theta(a, b)$, takes values in $[-\pi,\pi]$, which constitutes the domain on which primitive directional relations are defined.

The four such relations 'left', 'right', 'above' and 'below' are defined in Ref. [10] as $\cos^2\theta$ and $\sin^2\theta$ functions. Other functions are possible: in Ref. [76] trapezoidal shaped membership functions are used, for the same relations. Whatever the equations, the membership functions for these relations are denoted by $\mu_{\text{left}}$, $\mu_{\text{right}}$, $\mu_{\text{above}}$, and $\mu_{\text{below}}$, and are functions from $[-\pi,\pi]$ into [0,1]. The equations are chosen according to simplicity (e.g. cos or sin functions), to the fact that they define a fuzzy partition of $[-\pi,\pi]$, and to their invariance properties with respect to rotation (i.e. a rotation should correspond to a translation of the membership functions).

In the work relying on these definitions, only these four basic directions are used, other relations being expressed in terms of these. However, we can propose a straightforward extension to any direction. In 2D, a direction is defined by an angle $\alpha$ with the $x$-axis. Using this convention, the relationship 'right' corresponds to $\alpha=0$. From $\mu_{\text{right}}=\mu_0$, we derive $\mu_\alpha$, representing the relationship 'in direction $\alpha$', for any $\alpha$ as follows

$$\forall\theta,\quad \mu_\alpha(\theta) = \mu_0(\theta-\alpha) \tag{58}$$

with for instance:

$$\mu_0(\theta) = \begin{cases} \cos^2(\theta) & \text{if } \theta \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right] \\ 0 & \text{elsewhere} \end{cases} \tag{59}$$

This makes the definitions based on angle computation more general. Moreover, it guarantees geometric invariance.

Another solution for defining relations intermediate between the four basic ones can be based on logical combinations of these four basic relations. For instance, 'oblique right' is defined by '(above and right of) or (below and right of)'. The advantage of this approach is that only four membership functions have to be defined, which is consistent with the usual way of speaking about relative position. The drawback is that, contrary to the definition proposed in Eq. (58), we cannot achieve a great precision in direction using this approach. Also, the shape of the membership function will vary depending on the considered direction, leading to a high anisotropy and therefore a loss of rotation invariance, while it remains the same using Eq. (58).

### 6.2. Centroid method

A first simple solution to evaluate a fuzzy relationship between two objects consists in representing each object by a characteristic point. This point is chosen as the object centroid in Refs. [13,76]. Let $c_R$ and $c_A$ denote the centroids of objects $R$ and $A$. The degree of satisfaction of the proposition '$A$ is to the right of $R$' is then defined as

$$\mu^R_{\text{right}}(A) = \mu_{\text{right}}(\theta(c_R, c_A)), \tag{60}$$

where the membership function $\mu_{\text{right}}$ is defined as in Section 6.1. Extension to fuzzy objects can be done in two ways. One way consists in computing a weighted centroid, where the contribution of each object point is equal to its membership value. The second way consists in applying the definition for binary objects on each $\alpha$-cut and then aggregating the results using a summation [56], or the extension principle [77]. However, this second method may be computationally expensive, depending on the quantization of the object membership values.

### 6.3. Histogram of angles: compatibility method

The method proposed in Refs. [10,78] consists in computing the normalized histogram of angles and in defining a fuzzy set in [0,1] representing the compatibility between this histogram and the fuzzy relation. More precisely, the angle histogram is computed from the angle between any two points in both objects as defined before, and normalized by the maximum frequency. Let us denote by $H^R(A)$ this normalized histogram, where $R$ is the reference object and $A$ the object the position of which with respect to $R$ is evaluated. $H^R(A)$ represents the spatial directional relations of the object $A$ with respect to the reference object $R$. Expressing $H^R(A)$ in terms of the basic relations can be performed using compatibility (see Section 2) of the two fuzzy sets $\mu_\alpha$ (for some direction $\alpha$ of interest) and $H^R(A)$. The center of gravity of the compatibility fuzzy set provides then a global evaluation.

Another solution is to use a fuzzy pattern matching approach [51,52] (between $\mu_\alpha$ and $H^R(A)$) by computing both the degree of inclusion of $\mu_\alpha$ in $H^R(A)$ and their degree of intersection using the definitions of Section 3, as suggested in Ref. [79]. Then, the global evaluation is given in the form of a pair necessity/possibility.

The fuzzy extension of this method is based on a weighted histogram [10] which is equivalent to compute a histogram on each $\alpha$-cut and to combine the obtained results by summation as in Ref. [56].

### 6.4. Aggregation method

An aggregation method has been proposed in Refs. [13, 76], which uses all points of both objects instead of only one characteristic point. For any pair of points $i$ in $R$ and $j$ in $A$, the angle $\theta(i, j)$ is computed, and the corresponding membership value for a direction $\alpha$ (being one of the four considered relations) is computed as previously: $\mu_{ij} = \mu_\alpha(\theta(i,j))$. All these values are then aggregated. The aggregation operator suggested in Ref. [76] is a weighted mean.

### 6.5. Histogram of forces

Instead of considering pairs of points as in angle histogram approaches, pairs of longitudinal sections are considered in Ref. [80], where the concept of F-histogram is introduced. The degree to which an object $A$ is in the direction $\alpha$ with respect to a reference object $R$ is computed using successively points, segments, and longitudinal sections. This leads to a so called ‘histogram of forces’ which allows to compute the weight supporting a proposition like ‘object $A$ is in direction $\alpha$ from object $R$’. Basically, this approach amounts to considering a weighted angle histogram

$$H^R(A)(\theta) = \sum_{a,b,\theta(a,b)=0} \varphi(||\vec{ab}||), \tag{61}$$

where $\varphi$ is a decreasing function. Typically, $\varphi(x) = 1/x^r$. For $r=0$, the weighted histogram is equal to the angle histogram, and for $r \geq 1$, it gives more importance to points of $A$ that are close to some points of $R$. This allows to deal with situations where $A$ and $R$ have very different partial extents, and to account only for the closest parts of them.

### 6.6. Projection based approach

The approach proposed in Ref. [81] is very different from the previous ones since it does not use any histogram. It is based on a projection of the considered object on the axis related to the direction to be assessed (e.g. the $x$-axis for evaluating the relations ‘left to’ and ‘right to’). Let us detail the computation for the relation ‘$A$ is left from $R$’. The same construction applies for any direction. Let us denote by $R^f(x)$ the normalized projection of the set $R$ on the $x$-axis. The degree for a point $x$ to be left to $R$ is defined as:

$$R^{\leftarrow}(x) = \frac{\int_x^{+\infty} R^f(y)\mathrm{d}y}{\int_{-\infty}^{+\infty} R^f(y)\mathrm{d}y}. \tag{62}$$

This definition provides a degree of 1 for points that are completely on the left of the support of $R^f$ and a degree of 0 for points that are completely on the right of the support of $R^f$, and the degree decreases in-between.

Let us now introduce a second set $A$. The degree $(A \leftarrow R)^f(x)$ to which $x$ is in the projection of $A$ and to the left of $R$ is expressed as a conjunction of $A^f(x)$ and $R^{\leftarrow}(x)$. The conjunction is taken as a product in Ref. [81]. The degree to which $A$ is left from $R$ is then deduced as the ratio of the areas below $(A \leftarrow R)^f$ and $A^f$:

$$\mu_\alpha^R(A) = \frac{\int_{-\infty}^{+\infty} A^f(x) \int_x^{+\infty} R^f(y)\mathrm{d}y\, \mathrm{d}x}{\int_{-\infty}^{+\infty} A^f(y)\mathrm{d}y \int_{-\infty}^{+\infty} R^f(y)\mathrm{d}y}. \tag{63}$$

This approach can be generalized to fuzzy sets [81] by taking each point into account in the projection to the amount of its membership function, leading to similar properties as in the crisp case.

### 6.7. Morphological approach

In Refs. [79,82,83], a morphological approach has been proposed in order to evaluate the degree to which an object $A$ is in some direction with respect to a reference object $R$, consisting of two steps (note that this approach applies directly in 3D and on fuzzy objects):

(1) A *fuzzy landscape* is first defined around the reference object $R$ as a fuzzy set such that the membership value of each point corresponds to the degree of satisfaction of the spatial relation under examination. This makes use here of a spatial representation of the relation. Therefore, the fuzzy landscape $\mu_\alpha(R)$ is directly defined in the same space as the considered objects, contrary to the projection method [81], where the fuzzy area is defined on a one-dimensional axis.
(2) Then the object $A$ is compared to the fuzzy landscape $\mu_\alpha(R)$, in order to evaluate how well the object matches with the areas having high membership values (i.e. areas that are in the desired direction). This is done using a fuzzy pattern matching approach, which provides an evaluation as an interval or a pair of numbers instead of one number only.

The evaluation of relative position of $A$ with respect to $R$ is given by a function of $\mu_\alpha(R)(x)$ and $\mu_A(x)$ for all $x \in \mathcal{S}$. The histogram of $\mu_\alpha(R)$ conditionally to $\mu_A$ is such a function. While this histogram gives the most complete information about the relative spatial position of two objects, it is difficult to reason in an efficient way with it. A summary of the contained information could be more useful in practice. An appropriate tool for defining this summary is the fuzzy pattern matching approach [52], which provides an evaluation as two numbers: a necessity degree $N$ (a pessimistic evaluation) defined as a degree of inclusion and a possibility degree $\Pi$ (an optimistic evaluation) defined as a degree of intersection. They can also be interpreted in terms of fuzzy mathematical morphology, since the possibility is equal to the dilation of $\mu_A$ by $\mu_\alpha(R)$ at origin, while the necessity is equal to the erosion, as shown in Ref. [7]. These two interpretations, in terms of set theoretic operations and in terms of morphological ones, explain how the shape of the objects is taken into account.

Several other functions combining $\mu_\alpha(R)$ and $\mu_A(x)$ can be constructed. The extreme values provided by the fuzzy pattern matching are interesting because of their morphological interpretation, and because they provide a pair of extreme values and not only a single value and may better capture the ambiguity of the relation if any. One drawback of these measures is that they are sensitive to noise, since they rely on infimum and supremum computation. An average measure can also be useful from a practical point of view (it is much less sensitive to noise).

The key point in the previous definition relies in the definition of $\mu_\alpha(R)$. The requirements stated above for this fuzzy set are not strong and leave room for a large spectrum of possibilities. This flexibility allows the user to define any membership function according to the application at hand and the context requirements. The definition proposed in Ref. [82] looks precisely at the domains of space that are visible from a reference object point in the desired direction. This applies to any kind of objects, including those having strong concavities.

This amounts to dilate $\mu_R$ by a fuzzy structuring element defined on $\mathcal{S}$ as

$$\forall P \in \mathcal{S}, \quad \nu(P) = \max\left[0, 1 - \frac{2}{\pi}\arccos\frac{\vec{OP}\cdot\vec{u}_\alpha}{||\vec{OP}||}\right], \tag{64}$$

(or another function having the same variations) where $O$ is the center of the structuring element. This morphological definition is shown in Fig. 5.

### 6.8. Surround

From directional relations, more complex relations such as ‘surround’ can be defined. In Ref. [78], the tangent lines to an object $A$ originating from a point of an object $B$ are computed, as well as the angle $\theta$ between these lines. Then the degree of surroundness is defined as:

$$\mu_{\text{surround}}(\theta) = \begin{cases} \cos^2\dfrac{\theta}{2} & \text{if } 0 \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

Fig. 5. Left: a fuzzy reference object. Right: fuzzy landscape representing the relationship ‘to the left of’ obtained by a fuzzy dilation by a directional structuring element.

If $B$ is not reduced to a point, $\theta$ is defined for each point of $B$ and the set of obtained values is handled as for the directional relations. A similar approach has been proposed in Ref. [13], but with a linear function of $\theta$ instead of a trigonometric one. The extension to fuzzy sets is done by integration over $\alpha$-cuts, which can be computationally heavy. This type of definition is not necessarily adapted to any types of shapes and is difficult to extend to 3D objects.

Slightly different approaches for visual surroundness and topological surroundness have been proposed in Ref. [29].

Another approach consists in combining in a disjunctive way several directional relations such as left, right, above, below, etc. [78,82,83], which extends directly to 3D and to fuzzy objects. But the difficulty is to distinguish between situations where $A$ surrounds $B$ and situations where $B$ surrounds $A$.

### 6.9. Discussion

A formal comparison of these approaches is given in Ref. [75], based on their properties, the type of basic elements on which they rely, their behavior in extreme situations, in case of concavities, of distant objects, and on their computational cost. The main conclusions that can be drawn are as follows. While most approaches reduce the representation of objects to points, segments or projections, only the morphological approach considers the objects as a whole and therefore better accounts for their shape. Approaches providing evaluation as intervals or fuzzy numbers are better suited for representing the ambiguity inherent to such relationships. If the distance between objects increases, an object is seen as a point from the reference object, which could be intuitively expected, for all methods but the projection approach. All approaches have a similar complexity, except the centroid method which is computationally simpler but which also reduces too much the information.

The extension to 3D objects requires to represent a direction by two angles, which generally increases the complexity. The extension of the angle histogram method to 3D objects amounts to computing a bi-dimensional histogram, i.e. as a function of these two angles, and then applying the same principle using the relations defined in 3D. The computation of the histogram is heavy in 2D, and becomes even more so in 3D. Another problem when computing bi-histograms is that the domain of possible angle values may be underrepresented, depending on the size and the sampling of the considered objects. This may result in a noisy and hole containing histogram. This effect already appears in 2D. Extension of the force histogram method to 3D objects could be probably done, but with a high complexity. The morphological approach is directly applicable in 3D, without changing the complexity with respect to the number of points. All approaches can been extended to fuzzy objects using their $\alpha$-cuts, at the price of a high computational cost. The morphological approach can be directly applied, without cost increase. The angle

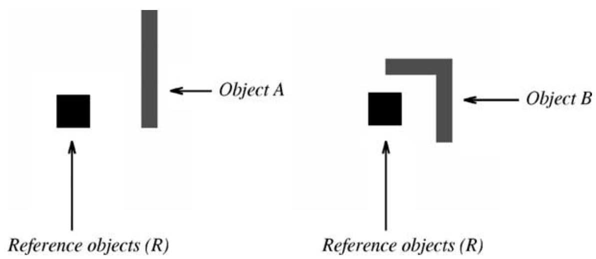



Fig. 6. Two examples where the relative position of objects with respect to the reference object is difficult to define in a 'all-or-nothing' manner.

histogram approach can be extended easily too, by computing weighted histograms.

Another aspect that is very important is the type of questions each definition is able to answer, or dedicated to. This could be the main ingredient for a choice among all possible approaches. These questions can take different forms, e.g.

- what are the spatial relationships between two given objects?
- to which degree a given spatial relation holds between two objects?
- what are the regions of the space where a spatial relationship is satisfied (to some degree) with respect to a reference object?

An important feature of angle histogram and force histogram is that they provide a general description of the directional relationships. From this general information, several ones can be deduced, as the degree of satisfaction of one specific relationship (for a particular direction), or the dominant relationship. This is not easy to obtain with the morphological approach, that needs one computation for each direction of interest. This approach is more dedicated to cases where we are interested in specified relations.

For problems where we have to assess the relative position of several objects with one reference object, the morphological approach may be more appropriate if the computation time is a strong requirement.

Another advantage of the morphological approach is that the first step directly provides a spatial representation of a directional constraint with respect to a reference object (thus answering the third type of question), which can be used in order to guide the search for another object [36,84].

In order to illustrate the reviewed definitions, we choose two simple examples, shown in Fig. 6. Despite their apparent simplicity, they lead to eloquent results, and allow us in particular to show how different parts of the objects can be taken into account.

Table 6 shows the results for object $A$ with respect to object $R$, according to various methods. They all agree to say that $A$ is mainly to the right of $R$. The degree of being to the right increases with the value of $r$, since the part of $A$ which is to the right of $R$ is the closest one to $R$. On the contrary, the degree of being above decreases with $r$. The values are somewhat different for all approaches, but since the ranking and the general behavior is the same, no conclusion concerning a more favorable approach can be derived from this example.

Table 7 shows the results for object $B$ with respect to object $R$. For these objects, two main relations are satisfied: right and above. The centroid method does not account well for the above relation, for which it gives a very low value. This shows one of the limitations of this approach, which is too simple in that it reduces too much the data. Since the part

Table 6
Relative position of object $A$ (rectangle) with respect to object $R$ (square) of Fig. 6, using centroid, aggregation, compatibility methods, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Relation | Centroid | Aggregation | | | Compatibility | | | Morpho. approach, $[N, \Pi]$ average |
|---|---|---|---|---|---|---|---|---|
| | | $r=0$ | $r=2$ | $r=5$ | $r=0$ | $r=2$ | $r=5$ | |
| Left | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | [0.00, 0.00] 0.00 |
| Right | 0.76 | 0.73 | 0.79 | 0.86 | 0.62 | 0.67 | 0.75 | [0.50,1.00] 0.81 |
| Below | 0.00 | 0.00 | 0.01 | 0.01 | 0.05 | 0.06 | 0.06 | [0.00, 0.35] 0.05 |
| Above | 0.24 | 0.27 | 0.20 | 0.13 | 0.38 | 0.33 | 0.25 | [0.00, 0.73] 0.44 |

Angle or force histograms are computed using $r=0$, $r=2$ and $r=5$ (the angle histogram method corresponds to $r=0$).

Table 7
Relative position of object $B$ with respect to object $R$ of Fig. 6, using centroid, aggregation, compatibility methods, and using the morphological approach (the $[N, \Pi]$ intervals are given, as well as the average value)

| Relation | Centroid | Aggregation | | | Compatibility | | | Morpho. approach, $[N, \Pi]$ average |
|---|---|---|---|---|---|---|---|---|
| | | $r=0$ | $r=2$ | $r=5$ | $r=0$ | $r=2$ | $r=5$ | |
| Left | 0.00 | 0.00 | 0.00 | 0.01 | 0.05 | 0.06 | 0.08 | [0.00, 0.44] 0.02 |
| Right | 0.83 | 0.63 | 0.54 | 0.33 | 0.55 | 0.49 | 0.35 | [0.29, 1.00] 0.81 |
| Below | 0.00 | 0.03 | 0.02 | 0.01 | 0.17 | 0.16 | 0.15 | [0.00, 0.60] 0.11 |
| Above | 0.17 | 0.34 | 0.43 | 0.66 | 0.45 | 0.51 | 0.65 | [0.00,1.00] 0.52 |

Angle or force histograms are computed using $r=0$, $r=2$ and $r=5$.

of $B$ which is above $R$ is closer than the one to its right, the values of being right decrease with $r$ while the values of being above increase. The morphological approach highlights the ambiguity of the relations for these objects. Parts of $B$ satisfy completely the above relation for instance, while other parts do not satisfy it at all. The non-zero degrees obtained for the relation below for instance are due to some points of $B$ that are indeed partially below $R$.

## 7. Conclusion

We briefly presented in this paper the main fuzzy approaches to define spatial relationships, as well as some comparative discussions. More details as well as illustrative examples can be found in the referenced papers.

The most representative relationships according to Freeman [2] and Kuipers [3,4] have been addressed. However, other relations may be useful for scene description, interpretation and recognition. For instance, symmetry can play an important role in several applications. Exact symmetry usually does not exist in real objects and one has to deal with approximate symmetries. Many works quantify the degree of symmetry using a symmetry measure often based on a distance. However, except in Ref. [85], most results have been obtained for precisely defined objects. In Ref. [86], a symmetry measure is defined, which characterizes the degree of symmetry of a fuzzy object or between two fuzzy objects with respect to a given plane. This degree is based on a measure of comparison or similarity between the object and its reflection (or the reflection of the second object) with respect to this plane. It would be also interesting to address other relationships such as 'between' (a ternary relation, briefly addressed in 2D in Ref. [13]) or 'along', or any other relation involved in scene description in natural language. Such terms being often context-dependent, additional difficulties are raised.

Digital aspects and algorithms are also worse to be further developed [1]. Crisp relationships are often very sensitive in the digital case. In the fuzzy case, the problem is much less crucial. Indeed, there is no more strict membership, the fuzziness allows us to deal with some gradual transition between objects or between object and background, and relations become then a matter of degree. Therefore, we can expect to gain in robustness when assessing the relationships between two objects. In this respect, the fuzziness, even on digital images, could be interpreted as a partial recovering of the continuity lost during the digitization process. As for algorithms, propagation algorithms have already been developed for directional relative position [82], providing good approximations in a fast way. It could be interesting from a practical point of view to develop similar algorithms for fuzzy distances for instance.

Applications in structural object recognition and scene interpretation have been already developed based on fuzzy relationships and this field certainly deserves to be further investigated. More generally, fuzzy relationships are an important feature in spatial reasoning under imprecision. In the proposed review, mathematical morphology appears as a unifying framework. Moreover, due to its strong algebraic structure, it leads also to qualitative representations, in the context of formal logics [87], which can be useful in qualitative spatial reasoning.

## References


[1] I. Bloch, Fuzzy spatial relationships from mathematical morphology for model-based pattern recognition and spatial reasoning, in: Discrete Geometry for Computer Imagery DGCI 2003, Vol. LNCS 2886, Naples, Italy, 2003, pp. 16–33.

[2] J. Freeman, The modelling of spatial relations, Computer Graphics and Image Processing 4 (2) (1975) 156–171.

[3] B. Kuipers, Modeling spatial knowledge, Cognitive Science 2 (1978) 129–153.

[4] B.J. Kuipers, T.S. Levitt, Navigation and mapping in large-scale space, AI Magazine 9 (2) (1988) 25–43.

[5] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.

[6] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.

[7] I. Bloch, H. Maître, Fuzzy mathematical morphologies: a comparative study, Pattern Recognition 28 (9) (1995) 1341–1387.

[8] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, Information Sciences 8 (1975) 199–249.

[9] L. Zadeh, Fuzzy sets and information granularity, in: M. Gupta, R. Ragade, R. Yager (Eds.), Advances in Fuzzy Set Theory and Applications, North-Holland, Amsterdam, 1979, pp. 3–18.

[10] K. Miyajima, A. Ralescu, Spatial organization in 2D segmented images: representation and recognition of primitive spatial relations, Fuzzy Sets and Systems 65 (1994) 225–236.

[11] G. Gerla, L. Scarpati, Extension principles for fuzzy set theory, Journal of Information Sciences 106 (1998) 49–69.

[12] D. Dubois, H. Prade, Combination of information in the framework of possibility theory, in: M.A. Abidi (Ed.), Data Fusion in Robotics and Machine Intelligence, Academic Press, New York, 1992.

[13] R. Krishnapuram, J.M. Keller, Y. Ma, Quantitative analysis of properties and spatial relations of fuzzy image regions, IEEE Transactions on Fuzzy Systems 1 (3) (1993) 222–233.

[14] D. Sinha, E.R. Dougherty, Fuzzification of set inclusion: theory and applications, Fuzzy Sets and Systems 55 (1993) 15–42.

[15] V.H. Young, Fuzzy subsethood, Fuzzy Sets and Systems 77 (1996) 371–384.

[16] B. Kosko, Fuzziness vs probability, International Journal of General Systems 17 (1990) 211–240.

[17] A.D. Luca, S. Termini, A definition of non-probabilistic entropy in the setting of fuzzy set theory, Information and Control 20 (1972) 301–312.

[18] A. Tversky, Features of similarity, Psychological Review 84 (4) (1977) 327–352.

[19] J.A. Goguen, The Logic of inexact concepts, Synthese 19 (1969) 325–373.

[20] A. Kaufmann, Introduction to the Theory of Fuzzy Subsets, Academic Press, New York, 1975.

[21] M. Ishizuka, Inference methods based on extended Dempster Shafer's theory for problems with uncertainty/fuzziness, New Generation Computing 1 (1983) 159–168.

[22] W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication operators, Fuzzy Sets and Systems 4 (1980) 13–30.

[23] R. Willmott, Two fuzzier implication operators in the theory of fuzzy power sets, Fuzzy Sets and Systems 4 (1980) 31–36.

[24] D. Dubois, H. Prade, Fuzzy sets in approximate reasoning. Part I. Inference with possibility distributions, Fuzzy Sets and Systems 40 (1991) 143–202.

[25] C. Demko, E.H. Zahzah, Image understanding using fuzzy isomorphism of fuzzy structures, in: IEEE International Conference on Fuzzy Systems, Yokohama, Japan, 1995, pp. 1665–1672.

[26] I. Bloch, H. Maître, M. Anvari, Fuzzy adjacency between image objects, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (6) (1997) 615–653.

[27] A. Rosenfeld, The fuzzy geometry of image subsets, Pattern Recognition Letters 2 (1984) 311–317.

[28] J.K. Udupa, S. Samarasekera, Fuzzy connectedness and object definition: theory, algorithms, and applications in image segmentation, Graphical Models and Image Processing 58 (3) (1996) 246–261.

[29] A. Rosenfeld, R. Klette, Degree of adjacency or surroundness, Pattern Recognition 18 (2) (1985) 169–177.

[30] R. Zwick, F. Carlstein, D.V. Budescu, Measures of similarity among fuzzy concepts: a comparative analysis, International Journal of Approximate Reasoning 1 (1987) 221–242.

[31] I. Bloch, On fuzzy spatial distances, in: P. Hawkes (Ed.), Advances in Imaging and Electron Physics vol. 128, Elsevier, Amsterdam, 2003, pp. 51–122.

[32] D. Dubois, H. Prade, On distance between fuzzy points and their use for plausible reasoning, in: International Conference on Systems, Man, and Cybernetics 1983; 300–303.

[33] A. Rosenfeld, Distances between fuzzy sets, Pattern Recognition Letters 3 (1985) 229–233.

[34] E. Benoit, L. Foulloy, Capteurs flous multicomposantes: applications à la reconnaissance des couleurs, in: Les Applications des Ensembles Flous, Nîmes, France, 1993, pp. 167–176.

[35] E. Binaghi, A.D. Ventura, A. Rampini, R. Schettini, Fuzzy reasoning approach to similarity evaluation in image analysis, International Journal of Intelligent Systems 8 (1993) 749–769.

[36] I. Bloch, T. Géraud, H. Maître, Representation and fusion of heterogeneous fuzzy information in the 3D space for model-based structural recognition—application to 3D brain imaging, Artificial Intelligence Journal 148 (2003) 141–175.

[37] S.M. Chen, M.S. Yeh, P.Y. Hsio, A comparison of similarity measures of fuzzy values, Fuzzy Sets and Systems 72 (1995) 79–89.

[38] C.P. Pappis, N.I. Karacapilidis, A comparative assessment of measures of similarity of fuzzy values, Fuzzy Sets and Systems 56 (1993) 171–174.

[39] R. Jain, S.N.J. Murthy, P.L.J. Chen, Similarity measures for image databases, in: IEEE International Conference on Fuzzy Systems, Yokohama, Japan, 1995, pp. 1247–1254.

[40] A. Kaufman, Introduction to the theory of fuzzy subsets: fundamental theoretical elements, Academic Press, New York, 1973.

[41] R. Lowen, W. Peeters, Distances between fuzzy sets representing grey level images, Fuzzy Sets and Systems 99 (2) (1998) 135–150.

[42] R.Y. Yager, Entropy measures under similarity relations, International Journal of General Systems 20 (1992) 341–358.

[43] B. Bouchon-Meunier, R.R. Yager, Entropy of similarity relations in questionnaires and decision trees, in: Second IEEE International Conference on Fuzzy Systems, San Francisco, CA, 1993, pp. 1225–1230.

[44] D. Bhandari, N.R. Pal, D.D. Majumder, Fuzzy divergence, probability measure of fuzzy events and image thresholding, Pattern Recognition Letters 13 (1992) 857–867.

[45] V. Cross, C. Cabello, A mathematical relationship between set—theoretic and metric compatibility measures, in: ISUMA-NAFIPS'95, College Park, MD, 1995, pp. 169–174.

[46] X. Wang, B.D. Baets, E. Kerre, A comparative study of similarity measures, Fuzzy Sets and Systems 73 (1995) 259–268.

[47] B. Bouchon-Meunier, M. Rifqi, S. Bothorel, Towards general measures of comparison of objects, Fuzzy Sets and Systems 84 (2) (1996) 143–153.

[48] E.L. Walker, Fuzzy relations for feature-model correspondence in 3D object recognition, in: NAFIPS, IEEE, Berkeley, CA, 1996, pp. 28–32.

[49] W.-J. Wang, New similarity measures on fuzzy sets and on elements, Fuzzy Sets and Systems 85 (1997) 305–309.

[50] O. Colliot, I. Bloch, A. Tuzikov, Characterization of approximate plane symmetries for 3D fuzzy objects, in: IPMU 2002, Annecy, France, vol. III, 2002, pp. 1749–1756.

[51] M. Cayrol, H. Farreny, H. Prade, Fuzzy pattern matching, Kybernetes 11 (1982) 103–116.

[52] D. Dubois, H. Prade, C. Testemale, Weighted fuzzy pattern matching, Fuzzy Sets and Systems 28 (1988) 313–331.

[53] M.C. Jaulent, A. Yang, Application of fuzzy pattern matching to the flexible interrogation of a digital angiographies database, in: IPMU, Paris, France, 1994, pp. 904–909.

[54] C. Sokic, G. Pavlovic-Lazetic, Homogeneous images in fuzzy databases, in: Seventh IFSA World Congress, Prague, vol. IV, 1997, pp. 297–303.

[55] I. Bloch, H. Maître, M. Minoux, Optimal matching of 3-D convex polyhedra with applications to pattern recognition, Pattern Recognition and Image Analysis 3 (2) (1993) 137–149.

[56] D. Dubois, M.-C. Jaulent, A general approach to parameter evaluation in fuzzy digital pictures, Pattern Recognition Letters 6 (1987) 251–259.

[57] M. Masson, T. Denœux, Multidimensional scaling of fuzzy dissimilarity data, Fuzzy Sets and Systems 128 (2002) 339–352.

[58] D. Huttenlocher, G. Klanderman, W. Rucklidge, Comparing images using the Hausdorff distance, IEEE Transactions on Pattern Analysis and Machine Intelligence 15 (9) (1993) 850–863.

[59] M.L. Pun, D.A. Ralescu, Differentials of fuzzy functions, Journal of Mathematical Analysis and Applications 91 (1983) 552–558.

[60] L.C. de Barros, R.C. Bassanezi, P.A. Tonelli, On the continuity of the Zadeh's extension, in: Seventh IFSA World Congress, Prague, vol. II, 1997, pp. 3–8.

[61] B.B. Chauduri, A. Rosenfeld, On a metric distance between fuzzy sets, Pattern Recognition Letters 17 (1996) 1157–1160.

[62] L. Boxer, On Hausdorff-like metrics for fuzzy sets, Pattern Recognition Letters 18 (1997) 115–118.

[63] J.-L. Fan, Note on Hausdorff-like metrics for fuzzy sets, Pattern Recognition Letters 23 (1998) 793–796.

[64] P. Brass, On the nonexistence of Hausdorff-like metrics for fuzzy sets, Pattern Recognition Letters 23 (2002) 39–43.

[65] I. Bloch, Distances in fuzzy sets for image processing derived from fuzzy mathematical morphology (invited conference), in: Information Processing and Management of Uncertainty in Knowledge-Based Systems, Granada, Spain, 1996, pp. 1307–1312.

[66] I. Bloch, On fuzzy distances and their use in image processing under imprecision, Pattern Recognition 32 (11) (1999) 1873–1895.

[67] K. Sridharan, H.E. Stephanou, Fuzzy distances for proximity characterization under uncertainty, Fuzzy Sets and Systems 103 (1999) 427–434.

[68] G.T. Man, J.C. Poon, A fuzzy-attributed graph approach to handwritten character recognition, in: Second IEEE International Conference on Fuzzy Systems, San Francisco, 1993, pp. 570–575.

[69] A. Kitamoto, M. Takagi, Retrieval of satellite cloud imagery based on subjective similarity, in: Ninth Scandinavian Conference on Image Analysis, Uppsala, Sweden, 1995, pp. 449–456.

[70] A. Perchant, I. Bloch, Fuzzy morphisms between graphs, Fuzzy Sets and Systems 128 (2) (2002) 149–168.

[71] E. Bengoetxea, P. Larranaga, I. Bloch, A. Perchant, C. Boeres, Inexact graph matching by means of estimation of distribution algorithms, Pattern Recognition 35 (2002) 2867–2880.

[72] R. Cesar, E. Bengoetxea, I. Bloch, Inexact graph matching using stochastic optimization techniques for facial feature recognition, in: International Conference on Pattern Recognition ICPR 2002, Québec, vol. 2, 2002, pp. 465–468.

[73] G. Cortelazzo, G. Deretta, G.A. Mian, P. Zamperoni, Normalized weighted Levensthein distance and triangle inequality in the context of similarity discrimination of bilevel images, Pattern Recognition Letters 17 (1996) 431–436.

[74] P.E. Maher, A similarity measure for conceptual graphs, International Journal of Intelligent Systems 8 (1993) 819–837.

[75] I. Bloch, A. Ralescu, Directional relative position between objects in image processing: a comparison between fuzzy approaches, Pattern Recognition 36 (2003) 1563–1582.

[76] J.M. Keller, X. Wang, Comparison of spatial relation definitions in computer vision, in: ISUMA-NAFIPS'95, College Park, MD, 1995, pp. 679–684.

[77] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, Fuzzy Sets and Systems 1 (1978) 3–28.

[78] K. Miyajima, A. Ralescu, Spatial organization in 2D images, in: Third IEEE International Conference on Fuzzy Systems, FUZZ-IEEE'94, Orlando, FL, 1994, pp. 100–105.

[79] I. Bloch, Fuzzy relative position between objects in images: a morphological approach, in: IEEE International Conference on Image Processing ICIP'96, Lausanne, vol. II, 1996, pp. 987–990.

[80] P. Matsakis, L. Wendling, A new way to represent the relative position between areal objects, IEEE Transactions on Pattern Analysis and Machine Intelligence 21 (7) (1999) 634–642.

[81] L.T. Koczy, On the description of relative position of fuzzy patterns, Pattern Recognition Letters 8 (1988) 21–28.

[82] I. Bloch, Fuzzy relative position between objects in image processing: a morphological approach, IEEE Transactions on Pattern Analysis and Machine Intelligence 21 (7) (1999) 657–664.

[83] I. Bloch, Fuzzy relative position between objects in image processing: new definition and properties based on a morphological approach, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 7 (2) (1999) 99–133.

[84] I. Bloch, Spatial representation of spatial relationships knowledge, in: A.G. Cohn, F. Giunchiglia, B. Selman (Eds.), Seventh International Conference on Principles of Knowledge Representation and Reasoning KR 2000, Morgan Kaufmann, San Francisco, CA/Breckenridge, CO, 2000, pp. 247–258.

[85] H. Zabrodsky, S. Peleg, D. Avnir, Symmetry as a continuous feature, IEEE Transactions on Pattern Analysis and Machine Intelligence 17 (1995) 1154–1166.

[86] O. Colliot, A. Tuzikov, R. Cesar, I. Bloch, Approximate reflectional symmetries of fuzzy objects with an application in model-based object recognition, Fuzzy Sets and Systems 147 (2004) 141–163.

[87] I. Bloch, Modal logics based on mathematical morphology for spatial reasoning, Journal of Applied Non Classical Logics 12 (3–4) (2002) 399–424.