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FUZZY ADJACENCY BETWEEN IMAGE OBJECTS

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The notion of adjacency has a strong interest for image processing and pattern recognition, since it denotes an important relationship between objects or regions in an image, widely used as a feature in model-based pattern recognition. A crisp definition of adjacency often leads to low robustness in the presence of noise, imprecision, or segmentation errors. We propose two approaches to cope with spatial imprecision in image processing applications, both based on the framework of fuzzy sets. These approaches lead to two completely different classes of definitions of a degree of adjacency. In the first approach, we introduce imprecision as a property of the adjacency relation, and consider adjacency between two (crisp) objects to be a matter of degree. We represent adjacency by a fuzzy relation whose value depends on the distance between the objects. In the second approach, we introduce imprecision (in particular spatial imprecision) as a property of the objects, and consider objects to be fuzzy subsets of the image space. We then represent adjacency by a relation between fuzzy sets. This approach is, in our opinion, more powerful and general. We propose several ways for extending adjacency to fuzzy sets, either by using α-cuts, or by using a formal translation of binary equations into fuzzy ones. Since set equations are more easily translated into fuzzy terms, we shall privilege set representations of adjacency, particularly in the framework of fuzzy mathematical morphology. Finally, we give some hints on how to compare degrees of adjacency, typically for applications in model-based pattern recognition.

Keywords: Fuzzy image processing, fuzzy adjacency, fuzzy mathematical morphology.

1. Introduction

Adjacency has a large interest in image processing and pattern recognition, since it denotes an important relationship between image objects or regions [25], widely used as a feature in model-based pattern recognition.

A crisp definition of adjacency between crisp objects often leads to a low robustness in case of noise or segmentation errors. Let us consider for instance a problem of model-based pattern recognition, where spatial relationships are an important part of the recognition process. If two model objects are adjacent, we expect the corresponding image objects to be adjacent too, otherwise they will be difficult to recognize. However, if classical crisp adjacency is used, the fact that two objects are adjacent or not may depend on one point only. Figure 1 illustrates the sensitivity of the binary adjacency concept to errors in segmentation or definition of the objects.

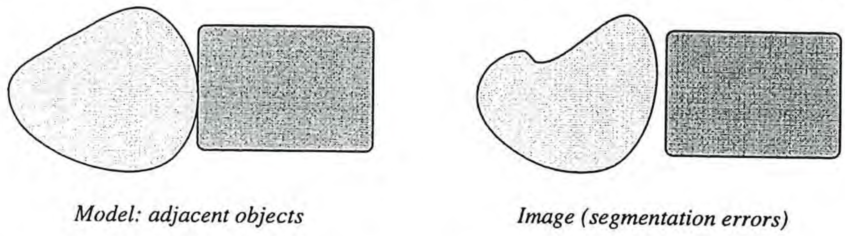

Figure 1: Sensitivity of crisp adjacency: small modifications in the shapes may completely change the adjacency relation, and thus prevent a correct recognition based on this relationship.

In order to include possible errors or imprecision in the processing and in the recognition, we use the framework of fuzzy sets that already proved to be useful for image processing under imprecision (see e.g. [16]). Here we consider two completely different ways for representing imprecision. In the first one, the satisfaction of the adjacency property between two objects is considered to be a matter of degree; this can be more appropriate than a binary index [23], [24]. The second one consists in introducing imprecision in the objects themselves, and to deal with fuzzy objects, i.e. with objects considered as fuzzy sets on the space, with their attached membership function defined for every point. For instance, spatial imprecision due to the limited quality of image information can be represented in an adequate way by considering fuzzy objects. Then obviously adjacency is also a matter of degree.

In both cases, a need arises to find proper definitions for fuzzy adjacency between image regions.

Methods for dealing with uncertainty (in its most general sense) in image processing rely mainly on probabilistic approaches, and more recently on fuzzy sets and possibility theory and Dempster-Shafer evidence theory. They all deal with different types of uncertainty (like randomness, imprecision, or degrees of belief), and have been compared mainly from the point of view of image fusion (see e.g. [5]). The problem of introducing imprecision and uncertainty at the level of adjacency parameters has not been addressed in the other frameworks to our knowledge. We argue that the fuzzy set framework is very appropriate to the management of spatial imprecision. Unfortunately, we could find only a few attempts in the literature to address the problem of fuzzy adjacency. Fuzzy topology was introduced in [23]. In that paper, Rosenfeld defines a fuzzy connectivity between points but without reference to the notion of fuzzy neighborhood, or to fuzzy adjacency. Similar approaches can also be found in [24], [30], [31], where degrees of connectivity in a fuzzy set are also introduced, but neither the connectivity nor the adjacency between two fuzzy sets are defined. Rosenfeld and Klette [26] define a degree of adjacency between two crisp sets, using a geometrical approach based on the notion of "visibility" of a set from another one. This definition is then extended to degree of adjacency between two fuzzy sets. However, this definition is not symmetrical, and probably not easy to transpose to higher dimensions. We propose in this paper a completely different approach. The work the closest to ours is probably the one described in [10], where a degree of adjacency between two fuzzy sets is defined by extending binary definitions of contours, frontiers, and neighborhood. Again, the proposed definition of [10] is not symmetrical, and presents other drawbacks that will be described later.

In this paper we propose several definitions for degree of adjacency coping with spatial imprecision in image processing. This paper is organized as follows. Basic definitions for classical notions of adjacency are given in Section 2, both in the continuous and discrete domains. In Section 3, we propose a definition of degree of adjacency between crisp objects, depending on the distance between them, in order to deal with possible errors or imprecision in the segmentation or representation of these objects. In our opinion, a more powerful approach is to explicitly account for such imprecision in the representation of objects, consequently defined as spatial fuzzy sets, and to define a degree of adjacency between two fuzzy sets. In Section 4, we shortly describe some possible ways for extending adjacency to fuzzy sets. The first one is based on the α-cuts, and is developed in Section 5. The second one is based on a formal translation of binary equations into fuzzy ones and is detailed in Section 6. It leads to the best definitions with respect to properties, intuitive requirements, behavior and interpretation. Since set theoretical expressions are easy to translate into fuzzy terms, we shall privilege set representations of adjacency (instead of geometrical ones as in [26]). Fuzzy mathematical morphology provides a consistent framework for dealing with imprecision in set expressions, and we show how it can be used for defining fuzzy adjacency. Finally, Section 7 provides some hints on how to compare degrees of adjacency, typically for applications in model-based pattern recognition.

2. Crisp adjacency

2.1. *Continuous domain*

We assume here that objects are defined in a finite n-dimensional space, typically $\mathbb{R}^n$ for sake of simplicity. A natural way to define adjacency relies on topology. We consider here the classical topology defined on $\mathbb{R}^n$. We denote by $\mathcal{G}$ the set of open sets, $\mathcal{F}$ the set of closed sets. For any subset X of $\mathbb{R}^n$, we denote by X^C its complementary set ($\mathbb{R}^n - X$), by $\bar{X}$ its topological closure and by $\dot{X}$ its interior (see e.g. [8]). The boundary of X is then defined as:

$$\partial X = \bar{X} - \dot{X} = \bar{X} \cap \bar{X^C}. \tag{1}$$

Using these notations, adjacency between two subsets X and Y can be defined in the following simple way:

Definition 1 *For any two subsets X and Y in $\mathbb{R}^n$, X and Y are adjacent if*

$$\partial X \cap \partial Y \neq \emptyset \text{ and } X \cap \dot{Y} = \emptyset \text{ and } Y \cap \dot{X} = \emptyset. \tag{2}$$

Property 1 *Definition 1 is equivalent to:*

$$\bar{X} \cap \bar{Y} \neq \emptyset \text{ and } X \cap \dot{Y} = \emptyset \text{ and } Y \cap \dot{X} = \emptyset. \quad (3)$$

Property 2 *Definition 1 implies that* $\dot{X} \cap \dot{Y} = \emptyset$.

However, it is not sufficient to verify that $\partial X \cap \partial Y \neq \emptyset$ and $\dot{X} \cap \dot{Y} = \emptyset$ (or, equivalently, $\bar{X} \cap \bar{Y} \neq \emptyset$ and $\dot{X} \cap \dot{Y} = \emptyset$) in order to have a satisfactory definition of adjacency. Indeed, this would lead to consider the two sets in Figure 2 as adjacent, which is intuitively not convenient.

Definition 1 corresponds to the intuitive notion that X and Y are adjacent if they have no overlapping parts (overlap could be expressed as $X \cap \dot{Y} \neq \emptyset$ or $\dot{X} \cap Y \neq \emptyset$) and are not completely separated (separation could be expressed as $\bar{X} \cap \bar{Y} = \emptyset$).

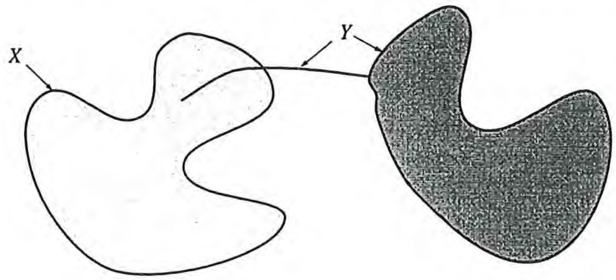

Figure 2: Two sets that are not adjacent but that satisfy $\partial X \cap \partial Y \neq \emptyset$ and $\dot{X} \cap \dot{Y} = \emptyset$.

Since neighborhood is an essential concept in image processing, a definition of adjacency can also be given with respect to it. It can take the following forms, for closed and open sets respectively.

Definition 2 *For any two closed sets F and F' in $\mathcal{F}$, F and F' are adjacent in $\mathcal{F}$ if*

$$F \cap F' \neq \emptyset \text{ and } \forall x \in F \cap F', \forall V(x), \exists (y, y') \in V(x)^2, y \in F, y' \in F', y \notin \dot{F}', y' \notin \dot{F}, \quad (4)$$

where $V(x)$ denotes a neighborhood of x.

This means that any neighborhood of an intersection point contains points both in F and F', as illustrated by Figure 3. The constraint stating that the point y in $V(x)$ that belongs to F should not belong to the interior of F' guarantees that situations like the one in Figure 2 are avoided.

Property 3 *A consequence of definition 2 is that if F and F' are adjacent in $\mathcal{F}$, then $\dot{F} \cap \dot{F}' = \emptyset$ and $F \cap F' = \partial F \cap \partial F'$.*

Property 4 *Definition 2 is equivalent to definition 1 when applied to closed sets.*

Definition 3 *For any two open sets G and G' in $\mathcal{G}$, G and G' are adjacent in $\mathcal{G}$ if*

$$G \cap G' = \emptyset \text{ and } \exists x \in \mathbb{R}^n, \forall V(x), V(x) \cap G \neq \emptyset, V(x) \cap G' \neq \emptyset. \quad (5)$$

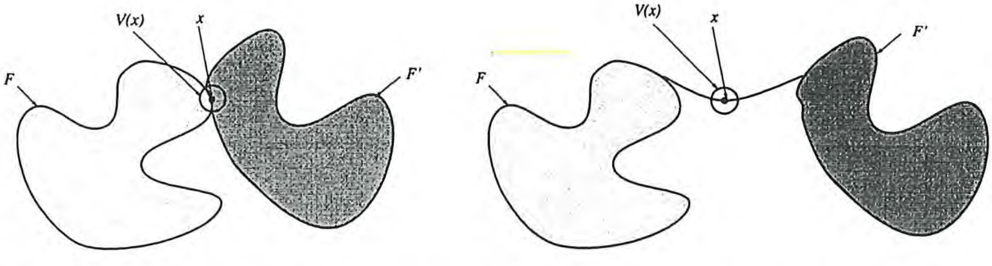

Figure 3: Two examples of adjacency between closed sets. On the right, the closed sets F and F' are adjacent in $\mathcal{F}$ but their interiors $\dot{F}$ and $\dot{F'}$ are not adjacent in $\mathcal{G}$.

This means that there exists at least one point in a neighborhood of both G and G'.

Property 5 *If G and G' are adjacent in $\mathcal{G}$, then any x satisfying the equation of definition 3 belongs to $(G \cup G')^C$ which is a closed set, and moreover $x \in \partial(G \cup G')^C = \partial(G \cup G')$.*

Property 6 *Definition 3 is equivalent to definition 1 when applied to open sets.*

Property 7 *If two open sets G and G' are adjacent in $\mathcal{G}$ according to definition 3, then $\bar{G}$ and $\bar{G'}$ are adjacent in $\mathcal{F}$ according to definition 2.*

However, if two closed sets F and F' are adjacent in $\mathcal{F}$ according to definition 2, their interior $\dot{F}$ and $\dot{F'}$ are not necessarily adjacent in $\mathcal{G}$ according to definition 3. A counter-example is shown on the right example of Figure 3.

In practice, and in particular for morphological applications, mainly closed sets are considered [27]. They are extended to upper semi-continuous membership functions in the fuzzy case.

Moreover, the particular problematic situations illustrated by Figure 2 and Figure 3 (right) are avoided if we restrict ourselves to objects X of $\mathbb{R}^n$ such that:

$$X = \bar{\dot{X}}. \tag{6}$$

Such objects are closed sets and have everywhere a local dimension equal to n [14], [15]. For the 2D example illustrated by Figure 2, X is such an object, whereas Y is not since it contains parts of local dimension 1.

Property 8 *For objects X such that $X = \bar{\dot{X}}$ (purely n-dimensional), the definition of adjacency between X and Y reduces to one of the following forms:*

$$\partial X \cap \partial Y \neq \emptyset \text{ and } \dot{X} \cap \dot{Y} = \emptyset, \tag{7}$$

$$X \cap Y \neq \emptyset \text{ and } \dot{X} \cap \dot{Y} = \emptyset, \tag{8}$$

$$X \cap Y \neq \emptyset \text{ and } \forall x \in X \cap Y, \forall V(x), \exists (y, y') \in V(x)^2, y \in (X - Y), y' \in (Y - X). \tag{9}$$

This property expresses a simpler way to assess the adjacency between objects, and holds for a large class of objects, that may be sufficient in practice.