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Fuzzy morphisms between graphs

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Abstract

A generic definition of fuzzy morphism between graphs (GFM) is introduced that includes classical graph related problem definitions as sub-cases (such as graph and subgraph isomorphism). The GFM uses a pair of fuzzy relations, one on the vertices and one on the edges. Each relation is a mapping between the elements of two graphs. These two fuzzy relations are linked with constraints derived from the graph structure and the notion of association graph. The theory extends the properties of fuzzy relation to the problem of generic graph correspondence. We introduce two complementary interpretations of GFM from which we derive several interesting properties. The first interpretation is the generalization of the notion of association compatibility. The second is the new notion of edge morphism. One immediate application is the introduction of several composition laws. Each property has a theoretical and a practical interpretation in the problem of graph correspondence that is explained throughout the paper. Special attention is paid to the formulation of a non-algorithmical theory in order to propose a first step towards a unified theoretic framework for graph morphisms.

Keywords: Graph morphism; Fuzzy relations; Fuzzy graphs

1. Introduction

Graph representations are widely used for dealing with structural information, in different domains such as networks, psycho-sociology, image interpretation, pattern recognition, etc. One important problem to be solved when using such representations is graph matching. In order to achieve a good correspondence between two graphs, the most used concept is the one of graph isomorphism and a lot of work is dedicated to the search for the best isomorphism between two graphs or subgraphs. However in a number of cases, the bijective condition is too strong, and the problem is expressed rather as an inexact graph matching problem. For instance, inexact graph matching appears as an important area of research in the pattern recognition field [18,29]. Several researches use graphs to represent the knowledge and the information extracted for instance from images, where vertices represent the segments or entities of the image and edges show the relationships between them. Examples of areas in which this type of representation is used are cartography, robotics and autonomous agents, character recognition, and recognition of brain structures

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[28,29]. Graph matching is used when the recognition is based on comparison with a model for instance. One graph represents the model, and another one the image where recognition has to be performed. Because of the schematic aspect of the model (atlas or map for instance) and of the difficulty to segment accurately the image into meaningful entities, no isomorphism can be expected between both graphs. Such problems call for inexact graph matching. Similar examples can be found in other fields.

Most work in inexact graph matching rely on the optimization of some functional. This functional usually measures the adequacy between vertices and between edges, and involves both the similarity between attributes of vertices and of edges, and also the structure of the graph. It is often defined in an ad hoc way, depending on the application at hand. For instance, some constraints can be relaxed or probabilized, correspondence can be defined iteratively by relaxing progressively local constraints, etc. The main focus of papers in this domain is the optimization part of the process. Existing methods include combinatorial optimization techniques [9,33], relaxation techniques [14,16,38], expectation maximization (EM) algorithm [8,15], Bayesian networks and estimation of distribution algorithms [3], genetic algorithms [29], and neural networks. Other methods are more concerned by the structure itself of the graphs and use tree search and propagation techniques [12,32,40], heuristic based graph traversing [6,25,36,37], graph editing [4,5,23], and graph labeling based on probabilistic models of attributes [11,26,34].

Since no bijection can be expected for inexact graph matching problems, the concept of isomorphism has to be replaced by the most generic concept of morphism (or homomorphism). Surprisingly, the increasing literature on inexact graph matching does not focus on the formalization of this concept, and no clear and unified definition of morphism between graphs can be exhibited. Therefore, as opposed to the works mentioned above which concentrate on the optimization, we focus in this paper on the formulation of a non-algorithmical theory in order to propose a first step towards a unified theoretic framework for graph morphisms.

The work which is the closest to our is the contribution of W. Bandler to the area of fuzzy relational products and fuzzy relational morphisms (see [1]). B. Juliano and W. Bandler have been working on fuzzy morphisms between graph structures as applied to cognitive diagnosis [20]. Their formulation for fuzzy morphisms between Hasse diagrams relies on both a vertex mapping and a path (or edge) mapping [21], which is an approach similar to ours. They also apply fuzzy morphisms between fuzzy graph structures called fuzzy cognitive maps. These are used to model chains-of-thought, defined as a "string of cognitive states representing some aspects of an individual's thought processes". Homomorphisms are then used to derive a degree of similarity between chains-of-thought structures. While they focus on inference and decision making (see e.g. [19]), we focus on generic definitions, along with their interpretations and properties.

Also the huge literature on graph or subgraph isomorphisms is of little help for solving inexact matching problems. The aim of this paper is to fill this gap. We propose to define fuzzy morphisms between graphs, and to study their properties. We rely on fuzzy set theory [13,41] to account for imprecision in the matching and non-bijective aspects, and make use of the notion of fuzzy relations [22,30,42]. The proposed approach includes as particular cases the classical notions of graph or subgraph isomorphisms, and of set morphisms. It applies to most kinds of graph (relational, attributed, fuzzy, fuzzy attributed, topological graphs, see e.g. [17,18,24,29,31] for the main definitions), including specific graphs used in image processing and pattern recognition such as adjacency, distance based, hierarchical, and semantic graphs.

After the introduction of a few notations, the definitions of fuzzy relations are recalled in Section 2, and interpreted as fuzzy association graphs. Section 3 is the core of this paper, and introduces the definition of fuzzy morphism between graphs. Two original interpretations are given. In Section 4, we propose composition laws between such morphisms. In Section 5, we detail some properties and interpretations. In Section 6, we conclude and give some hints on possible applications.

2. Preliminaries

We first introduce some notations and recall some basics about graph and subgraph isomorphisms. We denote by $G_i = (N_i, E_i)$ a graph where N_i is the set of vertices and $E_i \subseteq N_i \times N_i$ is the set of edges (or arcs); the integer i indexes the graph. Vertices and edges of the graph G_i are also indexed by i. For instance, u_1 denotes a vertex of graph 1 (G_1). Edges are denoted by (u_i, v_i), where u_i and v_i are the vertices at the endpoints of the edge, or by e_i. We assume in this paper that N_i and E_i are finite sets. When a bijective matching is searched between elements of the graphs G_1 and G_2, the notion of graph (or subgraph) isomorphism is used (see e.g. [6,35]).

Definition 1. A mapping $f : N_1 \to N_2$ is a graph isomorphism iff: $|N_1| = |N_2|$, $|E_1| = |E_2|, \forall u_2 \in N_2$, $\exists! u_1 \in N_1$, $f(u_1) = u_2$, and $\forall (u_2, v_2) \in E_2$, $\exists! (u_1, v_1) \in E_1$, $f(u_1) = u_2$, $f(v_1) = v_2$.

The bijective correspondence between the whole graphs is a very restrictive constraint. Often, only parts of the graphs are expected to be matched bijectively, calling for the notion of subgraph isomorphism, defined as an injective function from a subset of N_1 into N_2.

Definition 2. A mapping $f : N_1 \to N_2$ is a subgraph isomorphism iff: $\exists N_1' \subseteq N_1, \exists N_2' \subseteq N_2, \exists E_1' \subseteq E_1, \exists E_2' \subseteq E_2$ such that the restriction of f to the graphs (N_1', E_1') and (N_2', E_2') is a graph isomorphism.

We now recall some basic notions about fuzzy relations, since the proposed definition of fuzzy morphism will rely on these notions, and on their interpretation as fuzzy association graphs. More details about these concepts can be found in [13,22,30,42].

Let S_1 and S_2 be two sets, and σ_1 and σ_2 the membership functions of two fuzzy subsets of S_1 and S_2, respectively: $\sigma_1 : S_1 \to [0, 1]$, $\sigma_2 : S_2 \to [0, 1]$.

Definition 3. The function $\mu : S_1 \times S_2 \to [0, 1]$ is a fuzzy relation on $\sigma_1 \times \sigma_2$ iff:

$$\forall (x, y) \in S_1 \times S_2, \quad \mu(x, y) \leqslant \sigma_1(x) \wedge \sigma_2(y), \tag{1}$$

where $\wedge$ denotes the minimum. This definition is the most generic one. Particular cases can be found in [30] for $S_1 = S_2$, and in [22] for relations on $S_1 \times S_2$ (instead of $\sigma_1 \times \sigma_2$). This generic definition is mentioned in [42], but a deeper study of its properties can be found only for these two particular cases.

However, several results can be extended to Definition 3 in the general case. In the following, we will only mention the properties that are useful for our aim. Proofs are not given, and can be found in [22,30].

Two definitions of fuzzy graphs can be found in the literature. The first one is a fuzzy subset of $S_1 \times S_2$ [22]:

Definition 4. A fuzzy graph μ is a function $\mu : S_1 \times S_2 \to [0, 1]$.

This corresponds to the definition of a fuzzy relation (Definition 3) in the case where the membership functions σ_i are constant and equal to 1. If S_1 and S_2 are vertices of graphs, then μ can be interpreted as a weighting function on edges joining vertices of S_1 to vertices of S_2.

The second definition is restricted to the case $S_1 = S_2 = S$, and applied on a fuzzy subset σ of S [30]:

Definition 5. A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : S \to [0, 1]$, $\mu : S \times S \to [0, 1]$ which satisfies Eq. (1).

Definition 4 is adapted to cases where two sets of vertices have to be distinguished, these two sets being possibly equal. Definition 5 considers vertices in one unique set, but considers a fuzzy subset of it to define the graph. Both definitions are particular cases of a fuzzy relation (Definition 3) and therefore share its properties.

Definition 6. Let μ_1 and μ_2 be two fuzzy relations on $\sigma_1 \times \sigma_2$ and $\sigma_2 \times \sigma_3$, respectively. For $i \in \{1, 2, 3\}$, σ_i is a function from S_i into $[0, 1]$, and for $i \in \{1, 2\}$, μ_i is a function from $S_i \times S_{i+1}$ into $[0, 1]$. The max–min composition of μ_1 and μ_2, denoted by $\mu_1 \circ \mu_2$, is defined as [42]

$$\forall (u_1, u_3) \in S_1 \times S_3, \ (\mu_1 \circ \mu_2)(u_1, u_3) = \sup_{u_2 \in S_2} \{\mu_1(u_1, u_2) \wedge \mu_2(u_2, u_3)\}, \tag{2}$$

where $\wedge$ is the minimum.

Proposition 7. *The max–min composition $\mu_1 \circ \mu_2$ is a fuzzy relation on $\sigma_1 \times \sigma_3$.*

Proposition 8. *The max–min composition of fuzzy relations is associative*:

$$\forall i \in \{1,2,3,4\},\ \sigma_i : S_i \to [0,1],$$
$$\forall i \in \{1,2,3\}, \mu_i : S_i \times S_{i+1} \to [0,1], \quad (3)$$
$$(\mu_1 \circ \mu_2) \circ \mu_3 = \mu_1 \circ (\mu_2 \circ \mu_3) = \mu_1 \circ \mu_2 \circ \mu_3.$$

The projections of a fuzzy relation on S_1 and S_2 are defined as follows [22].

Definition 9. Let μ be a fuzzy relation on σ. The first projection of μ (or projection of μ onto S_1) is a fuzzy subset of S_1 defined by the following membership function:

$$\forall u_1 \in S_1,\ \mu_{S_1}^{(1)}(u_1) = \sup_{u_2 \in S_2} \mu(u_1, u_2). \quad (4)$$

The second projection of μ (or projection of μ onto S_2) is a fuzzy subset of S_2 defined by the following membership function:

$$\forall u_2 \in S_2,\ \mu_{S_2}^{(2)}(u_2) = \sup_{u_1 \in S_1} \mu(u_1, u_2). \quad (5)$$

The reflexivity, symmetry and transitivity properties are defined in [30] for a fuzzy relation in the case where $S_1 = S_2 = S$ and $\sigma_1 = \sigma_2 = \sigma$, i.e. for Definition 5 of a fuzzy graph.

Definition 10. A fuzzy relation is reflexive iff $\forall u \in S,\ \mu(u,u) = \sigma(u)$.

Definition 11. A fuzzy relation is symmetric iff $\forall (u,v) \in S \times S,\ \mu(u,v) = \mu(v,u)$.

Definition 12. A fuzzy relation is (max–min) transitive iff $\mu \circ \mu \leqslant \mu$.

It is worth noting that symmetry and transitivity do not depend on σ. Several other properties can be found in [30], but are not detailed here.

3. Fuzzy morphisms between graphs

In this section, we introduce a new formalism for morphisms between graphs, and propose a definition of fuzzy morphisms. We also give some interpretations and properties.

3.1. Definition

Definition 13. A fuzzy morphism (ρ_σ, ρ_μ) between graphs G_1 and G_2 is a pair of mappings $\rho_\sigma : N_1 \times N_2 \to [0,1]$ and $\rho_\mu : N_1 \times N_2 \times N_1 \times N_2 \to [0,1]$ which satisfies the following inequality:

$$\forall (u_1, v_1) \in N_1 \times N_1,\ \forall (u_2, v_2) \in N_2 \times N_2,$$
$$\rho_\mu(u_1, u_2, v_1, v_2) \leqslant \rho_\sigma(u_1, u_2) \wedge \rho_\sigma(v_1, v_2). \quad (6)$$

The mapping ρ_σ is called *vertex morphism* and ρ_μ is called *edge morphism*. They are linked by Eq. (6). GFM will stand for graph fuzzy morphism (fuzzy morphism between graphs) in the rest of the paper.

The term morphism refers to algebra, where it denotes a mapping between two spaces endowed with an internal composition law, which is preserved by the mapping. Here the introduced morphism has a meaning which is similar to the one of algebra morphism, but not rigorously the same. The binary relation defining the edges between vertices of the graphs should be kept to some degree by the fuzzy morphism. It is a natural extension of the usual notion of graph isomorphism.

The edge morphism is formalized as a mapping from $N_1 \times N_2 \times N_1 \times N_2$ and not from $(N_1 \times N_2) \times (N_1 \times N_2)$. The main reason is that the first Cartesian product is generic and allows other interpretations (see Section 3.2).

Two simple but fundamental properties can be derived from this definition. If $S_1 = N_1$ and $S_2 = N_2$, we can state from Definition 4:

Proposition 14. *A vertex morphism is a fuzzy relation on a discrete crisp set, and more precisely a fuzzy graph according to Definition* 4.

Proof. The proof is immediate. □

Proposition 15. *A fuzzy morphism* (ρ_σ, ρ_μ) *is a fuzzy graph according to Definition* 5.

Proof. If we set $S = N_1 \times N_2$, ρ_σ is a fuzzy subset of S. The mapping ρ_μ is defined on $S \times S$, and is therefore a fuzzy relation on $\rho_\sigma \times \rho_\sigma$ (since Eq. (6) guarantees that Eq. (1) is satisfied). Thus we recover Definition 5 of a fuzzy graph. □

We define the following partial order relation on fuzzy morphisms (defined on the same graphs), that will be useful in some properties.

Definition 16. Let (ρ_σ, ρ_μ) and (τ_σ, τ_μ) be two fuzzy morphisms between the same graphs G_1 and G_2. The partial ordering on vertex morphisms is defined as follows:

$$\rho_\sigma \leqslant \tau_\sigma \quad \Leftrightarrow \quad \forall (u_1, u_2) \in N_1 \times N_2, \\ \rho_\sigma(u_1, u_2) \leqslant \tau_\sigma(u_1, u_2). \tag{7}$$

In a similar way, the partial ordering on edge morphisms is defined as follows:

$$\rho_\mu \leqslant \tau_\mu \Leftrightarrow \forall (u_1, u_2, v_1, v_2) \in N_1 \times N_2 \times N_1 \times N_2, \\ \rho_\mu(u_1, u_2, v_1, v_2) \leqslant \tau_\mu(u_1, u_2, v_1, v_2). \tag{8}$$

It is straightforward to see that this defines indeed a partial order relation.

Fig. 1 is an example of graphical representation of GFM as a pair of weighted bipartite graphs. The width of the edges of the bipartite graphs is proportional to the membership value. Dashed lines represent a low quality association with respect to what could be intuitively expected. Non represented correspondences have a null value. For instance $\rho_\sigma(e_1, a_2) = 0.6$ and $\rho_\sigma(d_1, b_2) = 0.2$; the second association has a low value with respect to the other one, and Eq. (6) implies: $\rho_\mu(e_1, a_2, d_1, b_2) \leqslant 0.2$. Another example where the association is this time high is: $\rho_\sigma(b_1, b_2) = 0.9$ and $\rho_\sigma(e_1, d_2) = 0.9$; therefore $\rho_\mu(b_1, b_2, e_1, d_2) \leqslant 0.9$.

3.2. Interpretations

One important characteristic of the proposed definition is that it relies on the Cartesian product of sets. Taking the Cartesian product of $S = N_1 \times N_2$ with itself (for the edge morphism) indirectly involves $E_i \subseteq N_i \times N_i$. The edge morphism can be interpreted in two complementary ways:

- $S \times S = (N_1 \times N_2) \times (N_1 \times N_2)$ which corresponds to the classical interpretation of the notion of association compatibility via the edges, and which we call *internal interpretation*. The compatibility between object associations is often used in pattern recognition for quantifying the influence of the matching of two objects on the one of two other objects, by checking the consistency between both associations. This property is illustrated in Fig. 2 (the edges (u_1, v_1) and (u_2, v_2) are not drawn to illustrate the fact that they are not involved directly within this interpretation).

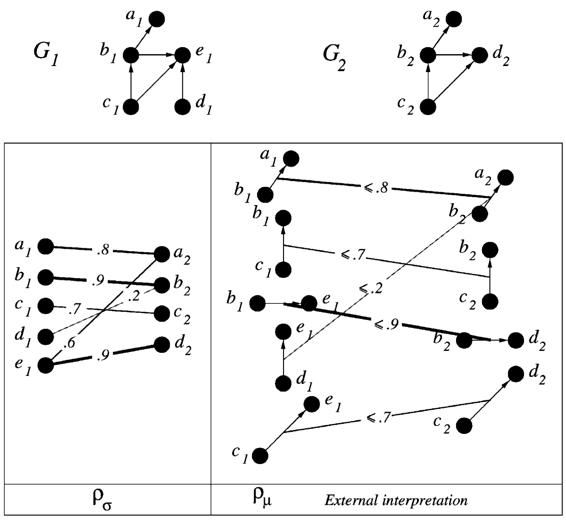

Fig. 1. Example of fuzzy morphism between two directed graphs. Only the edges existing in $N_i \times N_i$ are considered. The values of the vertex morphism ρ_σ are given arbitrarily in this example. For the edge morphism ρ_μ, only an upper bound of its value is given for each edge, according to Eq. (6). The width of the edges of the bipartite graph interpretation is proportional to the membership value. Associations represented by dashed lines are wrong associations with respect to what could be intuitively expected: the values of these associations are very low.

- $S \times S \rightsquigarrow (N_1 \times N_1) \times (N_2 \times N_2) \supseteq E_1 \times E_2$ which is the new notion of edge morphism. The arrow $\rightsquigarrow$ indicates that we make $N_1 \times N_1$ and $N_2 \times N_2$ explicit, but it is not an equality. We call it *external interpretation*. The notion of edge morphism gives importance to the edges themselves by exchanging the order of the sets in the Cartesian product. This interpretation allows us to check the correspondence between edges, and therefore to account for the structure of the graphs. This property is illustrated in Fig. 3 by an horizontal arrow linking both edges.

Fig. 1 represents the external interpretation of the morphism with links between the edges of G_1 and G_2. Fig. 4 represents the internal interpretation of the same

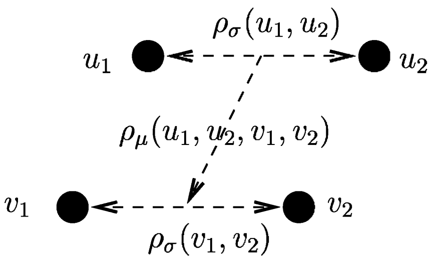

Fig. 2. Internal interpretation of edge morphism as association compatibility. For this example, the question is: is the association of u_1 with u_2 consistent (and to which degree) with the association of v_1 with v_2?

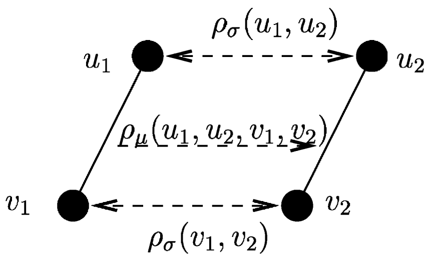

Fig. 3. External interpretation of the edge morphism as the matching between edges themselves. For this example, the question is: what is the degree of association between edge (u_1, v_1) and edge (u_2, v_2)?

example with a representation as a simplified fuzzy graph where each vertex represents an association between an element of N_1 and an element of N_2, and each edge represents a link between two associations formalized by ρ_μ.

A matrix representation of a morphism is also possible: the vertex morphism is a matrix on $N_1 \times N_2$, and the edge morphism is a matrix on $(N_1 \times N_1) \times (N_2 \times N_2)$. But this representation is not useful for our purpose: the other two representations graphically translate the morphism interpretations.

This important distinction allows us to deduce a number of properties from the fuzzy relations and the edge morphisms.

The change of order in the Cartesian product is not made explicitly, but for the benefit of both interpretations, it is enough to consider the two following

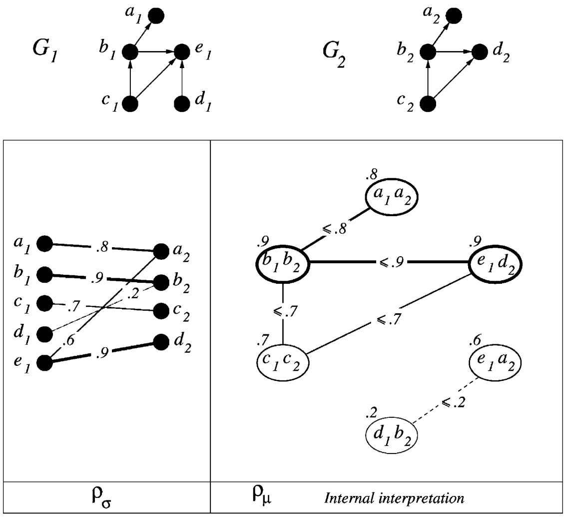

Fig. 4. Internal interpretation of the fuzzy morphism between the graphs of Fig. 1. Each vertex of this graph is an association of one vertex of G_1 with one vertex of G_2 that is weighted by the membership value of ρ_σ. The width of the border of the vertex is proportional to this value. Each edge of this graph represents a non-null value of ρ_μ linking two associations. The corresponding value of ρ_μ is the weight of the edge (only an upper bound is mentioned). The width of the edge is proportional to its value.

decompositions of $(u_1, u_2, v_1, v_2) \in N_1 \times N_2 \times N_1 \times N_2$:

Internal	External
$(\overbrace{\overline{u_1},\ \overline{u_2},}^{N_1 \times N_2}\ \underbrace{\underline{v_1},\ \underline{v_2}}_{N_1 \times N_2})$	$(\overbrace{\overline{u_1}, \underbrace{\underline{u_2},\ \overline{v_1}}_{N_2 \times N_2}, \underline{v_2}}^{N_1 \times N_1})$.

These two interpretations can be considered as two orthogonal views of the same mathematical object, an internal one and an external one. The term internal refers to a composition law that we define in the next Section between morphisms defined on the same sets, and which is therefore an internal composition law.

4. Compositions of fuzzy morphisms

4.1. Internal max–min composition law

Definition 17. Let (ρ_σ, ρ_μ) and (τ_σ, τ_μ) be two graph morphisms: $\rho_\sigma : N_1 \times N_2 \rightarrow [0, 1]$, $\rho_\mu : N_1 \times N_2 \times N_1 \times$

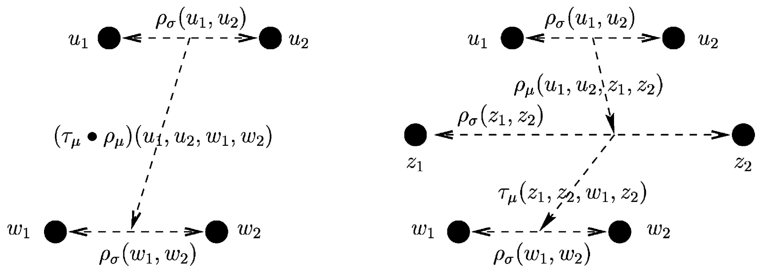

Fig. 5. Interpretation of the internal max–min composition as the search of a path of better compatibility.

$N_2 \to [0,1]$ and $\tau_\sigma : N_1 \times N_2 \to [0,1]$, $\tau_\mu : N_1 \times N_2 \times N_1 \times N_2 \to [0,1]$. The internal max–min composition of these two morphisms is denoted by $(\tau_\sigma, \tau_\mu) \bullet (\rho_\sigma, \rho_\mu) = (\tau_\sigma \bullet \rho_\sigma, \tau_\mu \bullet \rho_\mu)$ and defined as

$$(\tau_\sigma \bullet \rho_\sigma)(u_1, u_2) = \rho_\sigma(u_1, u_2) \vee \tau_\sigma(u_1, u_2), \tag{9}$$

$$(\tau_\mu \bullet \rho_\mu)(u_1, u_2, w_1, w_2) = \sup_{(v_1, v_2) \in N_1 \times N_2} \{\rho_\mu(u_1, u_2, v_1, v_2) \wedge \tau_\mu(v_1, v_2, w_1, w_2)\}, \tag{10}$$

where $(\tau_\sigma \bullet \rho_\sigma)$ is the *internal composition of vertex morphisms*, and $(\tau_\mu \bullet \rho_\mu)$ is the *internal composition of edge morphisms*.

Proposition 18. *The internal max–min composition* $(\tau_\sigma, \tau_\mu) \bullet (\rho_\sigma, \rho_\mu)$ *is a morphism between fuzzy graphs.*

Proof. A vertex morphism is a fuzzy relation. The composition law described by Eq. (9) of Definition 17 is a union of fuzzy relations, which is still a fuzzy relation [22]. This shows the proposition for the composition of vertex morphisms. An edge morphism is also a fuzzy relation. The composition law defined by Eq. (10) of Definition 17 is the one of fuzzy relations (see Definition 6) and the result is a fuzzy relation (see Proposition 7 applied here with $S_1 = S_2 = S_3 = N_1 \times N_2$), which shows the proposition for the composition of edge morphisms. □

The internal composition law can be interpreted as the fusion of two morphisms. Another way to understand this definition is to interpret the composition law as the search for a path of better compatibility in the edge morphism considered as a fuzzy association graph. Let us assume that in the definition of edge morphism composition, the *sup* is obtained for the pair $(z_1, z_2) \in N_1 \times N_2$. Then we have $(\tau_\mu \bullet \rho_\mu)(u_1, u_2, w_1, w_2) = \rho_\mu(u_1, u_2, z_1, z_2) \wedge \tau_\mu(z_1, z_2, w_1, w_2)$. By interpreting $(\tau_\mu \bullet \rho_\mu)(u_1, u_2, w_1, w_2)$ as the compatibility between the association of u_1 and w_1 with the association of u_2 and w_2, it can be said that this compatibility is better than the compatibility of the 4-tuple (u_1, u_2, z_1, z_2) measured by ρ_μ and than the compatibility of the 4-tuple (z_1, z_2, w_1, w_2) measured by τ_μ. Therefore, the composition tries to improve the compatibility between two pairs of associations by going through a third association (Fig. 5).

Proposition 19. *The internal max–min composition law is associative.*

Proof. The composition of vertex morphisms is defined by a max, which is associative. The composition of edge morphisms is defined as the composition of fuzzy relations which is associative (see Proposition 8). □

This shows that the set of fuzzy morphisms endowed with the internal max–min composition law is a monoid.

The composition of vertex morphisms is commutative. But in general, the composition of edge morphisms is not commutative. However, edge morphisms may commute if some symmetry properties are satisfied (this point is addressed in Section 5).

The internal composition law can be iterated several times on the same morphism.

Definition 20. An edge morphism ρ_μ to the power n (in the sense of internal composition) is denoted by ρ_μ^n and defined as the following morphism:

$\forall (u_1,u_2,v_1,v_2)\in N_1\times N_2\times N_1\times N_2,$

$$\rho_\mu^n(u_1,u_2,v_1,v_2) = (\underbrace{\rho_\mu \bullet \cdots \bullet \rho_\mu}_{n-1 \text{ compositions}})(u_1,u_2,v_1,v_2). \tag{11}$$

Note that the power of an edge morphism is defined uniquely since the composition law is associative.

By extension, we define ρ_μ^0 as follows:

$$\begin{aligned} &\rho_\mu^0(u_1,u_2,v_1,v_2) = 0 \quad \text{if } (u_1,u_2)\neq(v_1,v_2),\\ &\rho_\mu^0(u_1,u_2,u_1,u_2) = \rho_\sigma(u_1,u_2). \end{aligned} \tag{12}$$

This definition is consistent with the internal composition law and with the definition of the power of an edge morphism since the following result holds:

Proposition 21. $\rho_\mu^0 \bullet \rho_\mu = \rho_\mu \bullet \rho_\mu^0 = \rho_\mu$.

Proof. $\forall (u_1,u_2,w_1,w_2)\in N_1\times N_2\times N_1\times N_2$, we have

$$\begin{aligned} (\rho_\mu \bullet \rho_\mu^0)(u_1,u_2,w_1,w_2) &= \sup_{(v_1,v_2)\in N_1\times N_2} \{\rho_\mu^0(u_1,u_2,v_1,v_2) \\ &\qquad \wedge \rho_\mu(v_1,v_2,w_1,w_2)\}\\ &= \rho_\mu^0(u_1,u_2,u_1,u_2)\wedge\rho_\mu(u_1,u_2,w_1,w_2)\\ &= \rho_\sigma(u_1,u_2)\wedge\rho_\mu(u_1,u_2,w_1,w_2)\\ &= \rho_\mu(u_1,u_2,w_1,w_2), \end{aligned}$$

since $\rho_\mu(u_1,u_2,w_1,w_2)\leqslant\rho_\sigma(u_1,u_2)$. The second equality can be shown in a similar way. □

Proposition 22. *The internal max–min composition is increasing with respect to the order relations on fuzzy morphisms, i.e.*

$$\rho_\sigma \leqslant \rho'_\sigma \quad \textit{and} \quad \tau_\sigma \leqslant \tau'_\sigma \Rightarrow \rho_\sigma\bullet\tau_\sigma \leqslant \rho'_\sigma\bullet\tau'_\sigma, \tag{13}$$

$$\rho_\mu \leqslant \rho'_\mu \quad \textit{and} \quad \tau_\mu \leqslant \tau'_\mu \Rightarrow \rho_\mu\bullet\tau_\mu \leqslant \rho'_\mu\bullet\tau'_\mu. \tag{14}$$

Proof. Let us just show the relation for edge morphisms. Since $\rho_\mu\leqslant\rho'_\mu$ and $\tau_\mu\leqslant\tau'_\mu$, we have $\forall (u_1,u_2,v_1,v_2,w_1,w_2)\in(N_1\times N_2)^3$:

$$\begin{aligned} &\tau_\mu(u_1,u_2,v_1,v_2)\wedge\rho_\mu(v_1,v_2,w_1,w_2)\\ &\quad\leqslant \tau'_\mu(u_1,u_2,v_1,v_2)\wedge\rho'_\mu(v_1,v_2,w_1,w_2). \end{aligned}$$

Therefore we have, $\forall (u_1,u_2,w_1,w_2)\in N_1\times N_2\times N_1\times N_2$:

$$\begin{aligned} (\rho_\mu \bullet \tau_\mu)(u_1,u_2,w_1,w_2) &\leqslant \sup_{(v_1,v_2)\in N_1\times N_2}\{\tau'_\mu(u_1,u_2,v_1,v_2)\\ &\qquad \wedge\rho'_\mu(v_1,v_2,w_1,w_2)\}\\ &\leqslant (\rho'_\mu\bullet\tau'_\mu)(u_1,u_2,w_1,w_2). \qquad \square \end{aligned}$$

4.2. External max–min composition

Definition 23. Let (ρ_σ,ρ_μ) and (τ_σ,τ_μ) be two graph morphisms: $\rho_\sigma: N_1\times N_2\to[0,1]$, $\rho_\mu: N_1\times N_2\times N_1\times N_2\to[0,1]$ and $\tau_\sigma: N_2\times N_3\to[0,1]$, $\tau_\mu: N_2\times N_3\times N_2\times N_3\to[0,1]$. The external max–min composition of these two morphisms is denoted by $(\tau_\sigma,\tau_\mu)\circ(\rho_\sigma,\rho_\mu)=(\tau_\sigma\circ\rho_\sigma,\tau_\mu\circ\rho_\mu)$ and defined as

$$(\tau_\sigma\circ\rho_\sigma)(u_1,u_3) = \sup_{u_2\in N_2}\{\rho_\sigma(u_1,u_2)\wedge\tau_\sigma(u_2,u_3)\}, \tag{15}$$

$$\begin{aligned} (\tau_\mu\circ\rho_\mu)(u_1,u_3,v_1,v_3) &= \sup_{(u_2,v_2)\in N_2\times N_2}\{\rho_\mu(u_1,u_2,v_1,v_2)\\ &\qquad \wedge\tau_\mu(u_2,u_3,v_2,v_3)\}, \end{aligned} \tag{16}$$

where $(\tau_\sigma\circ\rho_\sigma)$ is the *external composition of vertex morphisms*, and $(\tau_\mu\circ\rho_\mu)$ is the *external composition of edge morphisms*.

Proposition 24. *The external max–min composition* $(\tau_\sigma,\tau_\mu)\circ(\rho_\sigma,\rho_\mu)$ *is a morphism between fuzzy graphs.*

Proof. From Proposition 14, a vertex morphism is a fuzzy relation. This composition law is the one of fuzzy relations, which shows the proposition for the composition of vertex morphisms. The external composition of edge morphisms needs some more derivation: $\forall (u_i,v_i)\in N_i\times N_i$, with $i\in\{1,2,3\}$, we have:

$$\rho_\mu(u_1,u_2,v_1,v_2)\leqslant\rho_\sigma(u_1,u_2)\wedge\rho_\sigma(v_1,v_2),$$

$$\tau_\mu(u_2,u_3,v_2,v_3)\leqslant\tau_\sigma(u_2,u_3)\wedge\tau_\sigma(v_2,v_3).$$

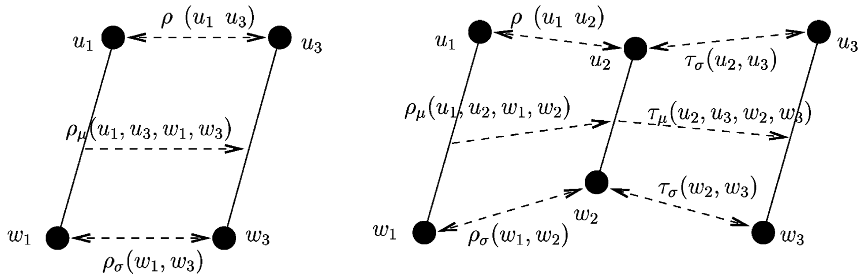

Fig. 6. Interpretation of the external max–min composition law as the search of the best paths over G_2 constituted by an association between an edge of G_1 and an edge of G_2, and an association between this edge of G_2 and an edge of G_3.

Therefore,

$$\begin{aligned}
&\rho_\mu(u_1,u_2,v_1,v_2)\wedge\tau_\mu(u_2,u_3,v_2,v_3)\\
&\quad\leqslant \rho_\sigma(u_1,u_2)\wedge\rho_\sigma(v_1,v_2)\wedge\tau_\sigma(u_2,u_3)\wedge\tau_\sigma(v_2,v_3)\\
&\quad\leqslant (\tau_\sigma\circ\rho_\sigma)(u_1,u_3)\wedge(\tau_\sigma\circ\rho_\sigma)(v_1,v_3),\\
&\qquad\forall(u_2,v_2)\in N_2\times N_2.
\end{aligned}$$

By taking the *sup* over $(u_2,v_2)\in N_2\times N_2$ in the left part, we obtain:

$$\begin{aligned}
&(\tau_\mu\circ\rho_\mu)(u_1,u_3,v_1,v_3)\\
&\quad\leqslant (\tau_\sigma\circ\rho_\sigma)(u_1,u_3)\wedge(\tau_\sigma\circ\rho_\sigma)(v_1,v_3).
\end{aligned}$$

This proves Eq. (6). □

In Definition 23, the quantity $\rho_\mu(u_1,u_2,v_1,v_2)\wedge\tau_\mu(u_2,u_3,v_2,v_3)$ can be interpreted as a "quality" of the path between edge (u_1,v_1) of G_1 and edge (u_3,v_3) of G_3 which goes through edge (u_2,v_2) of G_2. The external edge morphism composition can then be interpreted as the search of the best of such paths over the edges of G_2 (Fig. 6).

Proposition 25. *The external composition of graph morphisms is associative. Let* (ρ_σ,ρ_μ), (τ_σ,τ_μ) *and* (π_σ,π_μ) *be three graph morphisms*: $\rho_\sigma: N_1\times N_2\to[0,1]$, $\rho_\mu: N_1\times N_2\times N_1\times N_2\to[0,1]$, $\tau_\sigma: N_2\times N_3\to[0,1]$, $\tau_\mu: N_2\times N_3\times N_2\times N_3\to[0,1]$ *and* $\pi_\sigma: N_3\times N_4\to[0,1]$, $\pi_\mu: N_3\times N_4\times N_3\times N_4\to[0,1]$:

$$\begin{aligned}
&(\pi_\sigma,\pi_\mu)\circ((\tau_\sigma,\tau_\mu)\circ(\rho_\sigma,\rho_\mu))\\
&\quad=((\pi_\sigma,\pi_\mu)\circ(\tau_\sigma,\tau_\mu))\circ(\rho_\sigma,\rho_\mu)\\
&\quad=(\pi_\sigma,\pi_\mu)\circ(\tau_\sigma,\tau_\mu)\circ(\rho_\sigma,\rho_\mu). \qquad (17)
\end{aligned}$$

Proof. From Proposition 8, the max–min composition of vertex morphisms is associative. So we just have to prove the proposition for the external composition of edge morphisms. $\forall(u_i,v_i)\in N_i\times N_i$, with $i\in\{1,2,3,4\}$ we have

$$\begin{aligned}
&(\pi_\mu\circ(\tau_\mu\circ\rho_\mu))(u_1,u_4,v_1,v_4)\\
&\quad=\sup_{(u_3,v_3)\in N_3\times N_3}\left\{\sup_{(u_2,v_2)\in N_2\times N_2}\{\rho_\mu(u_1,u_2,v_1,v_2)\wedge\tau_\mu(u_2,u_3,v_2,v_3)\}\wedge\pi_\mu(u_3,u_4,v_3,v_4)\right\}\\
&\quad=\sup_{(u_3,v_3)\in N_3\times N_3}\left\{\sup_{(u_2,v_2)\in N_2\times N_2}\{\rho_\mu(u_1,u_2,v_1,v_2)\wedge\tau_\mu(u_2,u_3,v_2,v_3)\wedge\pi_\mu(u_3,u_4,v_3,v_4)\}\right\},
\end{aligned}$$

since $\pi_\mu(u_3,u_4,v_3,v_4)$ is independent of (u_2,v_2). Therefore, we can exchange the sup over (u_2,v_2) and the sup over (u_3,v_3), which shows the associativity. □

Note that other definitions of morphism compositions are possible as max-product, max-average [30], or more generally max-star [22]. Here we restrict to the max–min compositions.

4.3. Preservation of internal composition by external composition

Internal composition of graph morphisms is partially preserved when applying external composition, as expressed by the following proposition, which is a kind of distributivity property:

Proposition 26. *The external composition law is a morphism* (*in the classical sense*) *between sets of vertex morphisms endowed with internal composition laws.*

Let (ρ_σ, ρ_μ), $(\rho'_\sigma, \rho'_\mu)$ *and* (τ_σ, τ_μ) *be three graph morphisms*: $\rho_\sigma : N_1 \times N_2 \rightarrow [0,1]$, $\rho_\mu : N_1 \times N_2 \times N_1 \times N_2 \rightarrow [0,1]$, $\rho'_\sigma : N_1 \times N_2 \rightarrow [0,1]$, $\rho'_\mu : N_1 \times N_2 \times N_1 \times N_2 \rightarrow [0,1]$ *and* $\tau_\sigma : N_2 \times N_3 \rightarrow [0,1]$, $\tau_\mu : N_2 \times N_3 \times N_2 \times N_3 \rightarrow [0,1]$. *The following property holds*:

$$\tau_\sigma \circ (\rho'_\sigma \bullet \rho_\sigma) = (\tau_\sigma \circ \rho'_\sigma) \bullet (\tau_\sigma \circ \rho_\sigma). \tag{18}$$

It should be observed in this equation that the first and second internal compositions are applied on different sets. The first one applies on sets indexed by 1 and 2, while the second one applies on indices 1 and 3.

Proof. The proposition concerns vertex morphisms: $\forall (u_1, u_3) \in N_1 \times N_3$,

$$\begin{aligned}
&\tau_\sigma \circ (\rho'_\sigma \bullet \rho_\sigma)(u_1, u_3)\\
&\quad = \sup_{u_2 \in N_2} \{(\rho'_\sigma \bullet \rho_\sigma)(u_1, u_2) \wedge \tau_\sigma(u_2, u_3)\}\\
&\quad = \sup_{u_2 \in N_2} \{(\rho_\sigma(u_1, u_2) \vee \rho'_\sigma(u_1, u_2)) \wedge \tau_\sigma(u_2, u_3)\}.
\end{aligned}$$

Since max and min are mutually distributive, we have

$$\begin{aligned}
&\tau_\sigma \circ (\rho'_\sigma \bullet \rho_\sigma)(u_1, u_3)\\
&\quad = \sup_{u_2 \in N_2} \{(\rho_\sigma(u_1, u_2) \wedge \tau_\sigma(u_2, u_3)) \vee (\rho'_\sigma(u_1, u_2) \wedge \tau_\sigma(u_2, u_3))\}\\
&\quad = \sup_{u_2 \in N_2} \{\rho_\sigma(u_1, u_2) \wedge \tau_\sigma(u_2, u_3)\} \vee \sup_{u_2 \in N_2} \{\rho'_\sigma(u_1, u_2) \wedge \tau_\sigma(u_2, u_3)\}\\
&\quad = (\tau_\sigma \circ \rho_\sigma)(u_1, u_3) \vee (\tau_\sigma \circ \rho'_\sigma)(u_1, u_3)\\
&\quad = (\tau_\sigma \circ \rho'_\sigma) \bullet (\tau_\sigma \circ \rho_\sigma)(u_1, u_3).
\end{aligned}$$

This shows the proposition. □

Conjecture. *Let us consider the counterpart of Eq.* (18) *for edge morphisms.*

Let (ρ_σ, ρ_μ), $(\rho'_\sigma, \rho'_\mu)$ *and* (τ_σ, τ_μ) *be three graph morphisms*: $\rho_\sigma : N_1 \times N_2 \rightarrow [0,1]$, $\rho_\mu : N_1 \times N_2 \times N_1 \times N_2 \rightarrow [0,1]$, $\rho'_\sigma : N_1 \times N_2 \rightarrow [0,1]$, $\rho'_\mu : N_1 \times N_2 \times N_1 \times N_2 \rightarrow [0,1]$ *and* $\tau_\sigma : N_2 \times N_3 \rightarrow [0,1]$, $\tau_\mu : N_2 \times N_3 \times N_2 \times N_3 \rightarrow [0,1]$. *The following property holds*:

$$\tau_\mu \circ (\rho'_\mu \bullet \rho_\mu) \leqslant (\tau_\mu \circ \rho'_\mu) \bullet (\tau_\mu \circ \rho_\mu). \tag{19}$$

This inequality for edge morphisms (Eq. (19)) has been shown experimentally from a large set of random simulations. So we strongly believe that it holds. We are currently working on a formal proof.

4.4. Projections

The first and second projections of a vertex morphism can be easily defined, similarly as in Definition 9 for fuzzy relations. Let (ρ_σ, ρ_μ) be a graph morphism between G_1 and G_2.

Definition 27. The first (resp. second) projection of the vertex morphism ρ_σ is defined as

$$\forall u_1 \in N_1,\ \rho_{\sigma N_1}^{(1)}(u_1) = \sup_{u_2 \in N_2} \rho_\sigma(u_1, u_2), \tag{20}$$

$$\left(\text{resp. } \forall u_2 \in N_2, \rho_{\sigma N_2}^{(2)}(u_2) = \sup_{u_1 \in N_1} \rho_\sigma(u_1, u_2)\right). \tag{21}$$

Computing the first projection amounts to retain the degree of the best correspondence between a vertex of the first graph and each vertex of the second graph. In a similar way, computing the second projection amounts to retain the degree of the best correspondence between each vertex of the first graph and a vertex of the second graph.

As opposed to vertex morphism, an edge morphism can have internal and external projections.

Definition 28. The first (resp. second) *internal* projection of the edge morphism ρ_μ is defined by

$$\forall(u_1,u_2)\in N_1\times N_2, \rho_{\mu N_1\times N_2}^{(1)}(u_1,u_2) = \sup_{(v_1,v_2)\in N_1\times N_2} \rho_\mu(u_1,u_2,v_1,v_2), \tag{22}$$

$$\left(\text{resp. } \forall(v_1,v_2)\in N_1\times N_2, \rho_{\mu N_1\times N_2}^{(2)}(v_1,v_2) = \sup_{(u_1,u_2)\in N_1\times N_2} \rho_\mu(u_1,u_2,v_1,v_2)\right). \tag{23}$$

This definition consists in eliminating the dependence on one pair of vertices in $N_1\times N_2$. The first projection computes the best degree of compatibility existing between the association (u_1,u_2) and all other vertex associations. The second projection measures the best degree of compatibility existing between all vertex associations and the association (v_1,v_2). These two projections are equal if the morphism is weakly symmetrical (see Section 5.3).

Projections can also be performed with respect to the edges (pairs of vertices in N_i), instead of pairs of vertices in $N_1\times N_2$:

Definition 29. The first (resp. second) *external* projection of the edge morphism ρ_μ is defined by:

$$\forall(u_1,v_1)\in N_1^2, \rho_{\mu N_1\times N_1}^{(1)}(u_1,v_1) = \sup_{(u_2,v_2)\in N_2\times N_2} \rho_\mu(u_1,u_2,v_1,v_2), \tag{24}$$

$$\left(\text{resp. } \forall(u_2,v_2)\in N_2^2, \rho_{\mu N_2\times N_2}^{(2)}(u_2,v_2) = \sup_{(u_1,v_1)\in N_1\times N_1} \rho_\mu(u_1,u_2,v_1,v_2)\right). \tag{25}$$

These projections have similar interpretations as the projections of a vertex morphism, by replacing vertices by edges.

Proposition 30. *The vertex morphism n_σ which is constant and equal to* 1 *is the null element of the internal max–min composition. If the projections of the vertex morphisms are normalized* (*i.e. there exists a vertex such that the projection value for this vertex is equal to* 1), *then the vertex morphism n_σ which is constant and equal to* 1 *is the null element of the external max–min composition. On the subspace of GFM where all the internal projections are normalized for each* (w_1,w_2) (*i.e.* $\forall(w_1,w_2)\in N_1\times N_2,\ \exists(v_1,v_2)\in N_1\times N_2,\ \rho_\mu(v_1,v_2,w_1,w_2)=1$), *the edge morphism n_μ which is constant and equal to* 1 *is the null element of the internal max–min composition. On the subspace of GFM where all the external projections are normalized for each* (u_2,w_2) (*i.e.* $\forall(u_2,w_2)\in N_2\times N_2,\ \exists(v_1,x_1)\in N_1\times N_1,\ \rho_\mu(v_1,u_2,x_1,w_2)=1$), *the edge morphism n_μ which is constant and equal to* 1 *is the null element of the external max–min composition.*

Proof. We just prove the property for the internal composition of edge morphisms.

$$\begin{aligned}(\rho_\mu \bullet n_\mu)(u_1,u_2,w_1,w_2) &= \sup_{(v_1,v_2)\in N_1\times N_2} \{n_\mu(u_1,u_2,v_1,v_2) \\ &\qquad \wedge \rho_\mu(v_1,v_2,w_1,w_2)\} \\ &= \sup_{(v_1,v_2)\in N_1\times N_2} \rho_\mu(v_1,v_2,w_1,w_2) \\ &= \rho_{\mu N_1\times N_2}^{(2)}(w_1,w_2) \\ &= 1\end{aligned}$$

if the second internal projection of ρ_μ is normalized for each (w_1,w_2) (i.e. $\forall(w_1,w_2)\in N_1\times N_2$, $\exists(v_1,v_2)\in N_1\times N_2$, $\rho_\mu(v_1,v_2,w_1,w_2)=1$). In the same way, $n_\mu\bullet\rho_\mu=1$ if the second internal projection of ρ_μ is normalized for each (u_2,w_2). □

Proposition 31. *The vertex morphism $u_\sigma^\bullet$ which is constant and equal to* 0 *is the unit element of the internal max–min composition. The edge morphism $u_\mu^\bullet$ defined by*

$$u_\mu^\bullet(u_1,u_2,v_1,v_2) = \begin{cases} 1 & \text{if } (u_1,u_2)=(v_1,v_2), \\ 0 & \text{otherwise}\end{cases}$$

is the unit element of the internal max–min composition.

Proof. We just prove the property for the internal composition of edge morphisms. $\forall(u_1,u_2,w_1,w_2)\in$

$N_1 \times N_2 \times N_1 \times N_2$:

$$\begin{aligned}(\rho_\mu \bullet u^\bullet_\mu)(u_1,u_2,w_1,w_2) &= \sup_{(v_1,v_2)\in N_1\times N_2} \{u^\bullet_\mu(u_1,u_2,v_1,v_2) \\ &\qquad \wedge \rho_\mu(v_1,v_2,w_1,w_2)\} \\ &= u^\bullet_\mu(u_1,u_2,u_1,u_2)\wedge \rho_\mu(u_1,u_2,w_1,w_2) \\ &= \rho_\mu(u_1,u_2,w_1,w_2). \qquad \square\end{aligned}$$

Proposition 32. *The FGM $(u^\circ_\sigma, u^\circ_\mu)$ which is defined between G_1 and G_1 by*

$$u^\circ_\sigma(u_1,v_1) = \begin{cases} 1 & \text{if } u_1 = v_1, \\ 0 & \text{otherwise},\end{cases}$$

$$u^\circ_\mu(u_1,v_1,u_1',v_1') = \begin{cases} 1 & \text{if } (u_1,u_1') = (v_1,v_1'), \\ 0 & \text{otherwise}\end{cases}$$

is the unit element to the right of the external max–min composition. The unit element to the left is defined similarly, but between G_2 and G_2.

Proof. For the external composition of vertex morphisms. $\forall (u_1,u_2,w_1,w_2)\in N_1\times N_2\times N_1\times N_2$:

$$\begin{aligned}(\rho_\sigma \circ u^\circ_\sigma)(u_1,u_2) &= \sup_{v_1\in N_1}\{u^\circ_\sigma(u_1,v_1)\wedge \rho_\sigma(v_1,u_2)\} \\ &= u^\circ_\sigma(u_1,u_1)\wedge\rho_\sigma(u_1,u_2) \\ &= \rho_\sigma(u_1,u_2).\end{aligned}$$

The proof for the external composition of edge morphism works identically. $\square$

4.5. Inverse graph morphism

Definition 33. Let (ρ_σ,ρ_μ) be a graph morphism between G_1 and G_2. We define the inverse of (ρ_σ,ρ_μ) by $(\rho_\sigma,\rho_\mu)^{-1} = (\rho_\sigma^{-1},\rho_\mu^{-1})$ (from G_2 into G_1):

$$\forall (u_1,u_2)\in N_1\times N_2,\ \rho_\sigma^{-1}(u_2,u_1) = \rho_\sigma(u_1,u_2), \quad (26)$$

$$\begin{aligned}&\forall (u_1,u_2,v_1,v_2)\in N_1\times N_2\times N_1\times N_2, \\ &\rho_\mu^{-1}(u_2,u_1,v_2,v_1) = \rho_\mu(u_1,u_2,v_1,v_2).\end{aligned} \quad (27)$$

This intuitive definition of inverse morphism can be explained by expressing $(\rho_\sigma,\rho_\mu)^{-1}\circ(\rho_\sigma,\rho_\mu)$ as a function of the first projection of (ρ_σ,ρ_μ). $\forall (u_1,v_1)\in N_1\times N_1$:

$$\begin{aligned}(\rho_\sigma^{-1}\circ\rho_\sigma)(u_1,v_1) &= \sup_{u_2\in N_2}\{\rho_\sigma(u_1,u_2)\wedge\rho_\sigma^{-1}(u_2,v_1)\} \\ &= \sup_{u_2\in N_2}\{\rho_\sigma(u_1,u_2)\wedge\rho_\sigma(v_1,u_2)\} \\ &\leqslant \rho^{(1)}_{\sigma N_1}(u_1)\wedge\rho^{(1)}_{\sigma N_1}(v_1).\end{aligned}$$

In the particular case where $u_1 = v_1$ we have

$$(\rho_\sigma^{-1}\circ\rho_\sigma)(u_1,u_1) = \sup_{u_2\in N_2}\rho_\sigma(u_1,u_2) = \rho^{(1)}_{\sigma N_1}(u_1).$$

The function $(\rho_\sigma^{-1}\circ\rho_\sigma)$ is maximal at least at the pairs (u_1,u_1). However nothing more can be said in the general case.

In a similar way for the edge morphism we have $\forall (u_1,u_1',v_1,v_1')\in N_1^4$,

$$\begin{aligned}(\rho_\mu^{-1}\circ\rho_\mu)(u_1,u_1',v_1,v_1') &= \sup_{(u_2,v_2)\in N_2\times N_2}\{\rho_\mu(u_1,u_2,v_1,v_2) \\ &\qquad \wedge \rho_\mu^{-1}(u_2,u_1',v_2,v_1')\} \\ &= \sup_{(u_2,v_2)\in N_2\times N_2}\{\rho_\mu(u_1,u_2,v_1,v_2) \\ &\qquad \wedge \rho_\mu(u_1',u_2,v_1',v_2)\} \\ &\leqslant \rho^{(1)}_{\mu N_1\times N_1}(u_1,v_1)\wedge\rho^{(1)}_{\mu N_1\times N_1}(u_1',v_1').\end{aligned}$$

In the particular case where $(u_1,v_1) = (u_1',v_1')$ we get

$$\begin{aligned}(\rho_\mu^{-1}\circ\rho_\mu)(u_1,u_1,v_1,v_1) &= \sup_{(u_2,v_2)\in N_2\times N_2}\rho_\mu(u_1,u_2,v_1,v_2) \\ &= \rho^{(1)}_{\mu N_1\times N_1}(u_1,v_1).\end{aligned}$$

It is consistent to recover the external projections since this corresponds to the maximal degree of morphism for a given vertex or edge with another element on the other graph. Therefore, the value of the composed morphism $(\rho_\sigma,\rho_\mu)^{-1}\circ(\rho_\sigma,\rho_\mu)$ of a vertex or an edge with itself is necessarily at least equal to the other values.

5. Properties and interpretations

In this section, we develop some more properties, in particular in relation to the internal max–min composition. Several ones are extensions of properties proved in [30].

5.1. Smallest vertex morphism and largest edge morphism

Proposition 34. *Let ρ_σ be a vertex morphism on $N_1 \times N_2$. The largest edge morphism satisfying the edge morphism condition with ρ_σ is*

$$\rho_\mu(u_1,u_2,v_1,v_2) = \rho_\sigma(u_1,u_2) \wedge \rho_\sigma(v_1,v_2). \tag{28}$$

Proof. Straightforward by imposing the equality in Eq. (6). □

Proposition 35. *Let ρ_μ be an edge morphism on $N_1 \times N_2 \times N_1 \times N_2$. The smallest vertex morphism satisfying the morphism condition with ρ_μ is*

$$\rho_\sigma(u_1,u_2) = \sup_{(v_1,v_2)\in N_1\times N_2} \{\rho_\mu(u_1,u_2,v_1,v_2) \vee \rho_\mu(v_1,v_2,u_1,u_2)\}. \tag{29}$$

Proof. First, it is easy to check that this morphism ρ_σ satisfies Eq. (6). Now let us consider another vertex morphism τ_σ that also satisfies this condition. We have

$$\tau_\sigma(u_1,u_2) \geqslant \rho_\mu(u_1,u_2,v_1,v_2),$$
$$\tau_\sigma(u_1,u_2) \geqslant \rho_\mu(v_1,v_2,u_1,u_2)$$

and therefore (since the previous inequalities hold $\forall (v_1,v_2) \in N_1 \times N_2$):

$$\begin{aligned}\tau_\sigma(u_1,u_2) &\geqslant \rho_\mu(u_1,u_2,v_1,v_2) \vee \rho_\mu(v_1,v_2,u_1,u_2)\\ &\geqslant \sup_{(v_1,v_2)\in N_1\times N_2} \{\rho_\mu(u_1,u_2,v_1,v_2) \vee \rho_\mu(v_1,v_2,u_1,u_2)\}\\ &\geqslant \rho_\sigma(u_1,u_2).\end{aligned}$$

This shows that any vertex morphism satisfying Eq. (6) with respect to ρ_μ is greater than ρ_σ. □

These two properties correspond to limit cases of Eq. (6) in the definition of a fuzzy morphism between graphs. These definitions allow one to construct an edge morphism from the knowledge of a vertex morphism and the converse. Therefore, the formalism developed here can be used in classical methods where only the vertex morphism is used.

5.2. Reflexivity

Definition 36. A graph morphism (ρ_σ, ρ_μ) is reflexive if ρ_μ is reflexive in the sense of fuzzy relations, i.e.

$$\forall (u_1,u_2) \in N_1 \times N_2,\quad \rho_\mu(u_1,u_2,u_1,u_2) = \rho_\sigma(u_1,u_2). \tag{30}$$

This definition relies on the internal interpretation of the edge morphism as a fuzzy relation on fuzzy subsets, and therefore as a fuzzy graph as introduced in [30].

Reflexive edge morphisms have interesting properties, that are extensions of fuzzy relations on fuzzy sets [30].

Proposition 37. *If ρ_μ is a reflexive edge morphism, then we have*: $\forall (u_1,u_2,v_1,v_2) \in N_1 \times N_2 \times N_1 \times N_2$,

$$\rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\mu(u_1,u_2,u_1,u_2)$$
$$\textit{and}\quad \rho_\mu(v_1,v_2,u_1,u_2) \leqslant \rho_\mu(u_1,u_2,u_1,u_2). \tag{31}$$

This property expresses that the morphism between any two edges is bounded by the value of the morphism between one of these edges and itself.

Proof. Straightforward from Eq. (6) and reflexivity. □

Proposition 38. *For any reflexive edge morphism ρ_μ and any edge morphism τ_μ, defined on the same graphs and attached to the same vertex morphism ρ_σ, we have*:

$$(\rho_\mu \bullet \tau_\mu) \geqslant \tau_\mu \quad \textit{and} \quad (\tau_\mu \bullet \rho_\mu) \geqslant \tau_\mu. \tag{32}$$

This property expresses an extensivity property of the internal max–min composition, when an edge morphism is composed with a reflexive edge morphism.

Proof. $\forall (u_1,u_2,w_1,w_2)\in N_1\times N_2\times N_1\times N_2$,

$$\begin{aligned}(\rho_\mu\bullet\tau_\mu)&(u_1,u_2,w_1,w_2)\\ &=\sup_{(v_1,v_2)\in N_1\times N_2}\{\tau_\mu(u_1,u_2,v_1,v_2)\\ &\qquad\qquad\wedge\rho_\mu(v_1,v_2,w_1,w_2)\}\\ &\geqslant \tau_\mu(u_1,u_2,w_1,w_2)\wedge\rho_\mu(w_1,w_2,w_1,w_2)\\ &\geqslant \tau_\mu(u_1,u_2,w_1,w_2)\wedge\rho_\sigma(w_1,w_2).\end{aligned}$$

Since $\tau_\mu(u_1,u_2,w_1,w_2)\leqslant\rho_\sigma(u_1,u_2)\wedge\rho_\sigma(w_1,w_2)$, we have

$$\tau_\mu(u_1,u_2,w_1,w_2)\wedge\rho_\sigma(w_1,w_2)=\tau_\mu(u_1,u_2,w_1,w_2)$$

and therefore

$$(\rho_\mu\bullet\tau_\mu)(u_1,u_2,w_1,w_2)\geqslant\tau_\mu(u_1,u_2,w_1,w_2).$$

The second inequality of the proposition can be shown in the same way. □

Proposition 39. *Any reflexive edge morphism ρ_μ satisfies*

$$\rho_\mu^0\leqslant\rho_\mu\leqslant\rho_\mu^2\leqslant\cdots\leqslant\rho_\mu^\infty, \tag{33}$$

where $\rho_\mu^\infty=\lim_{n\to\infty}\rho_\mu{}^n=\sup_{n\in N}\rho_\mu^n$.

Proof. It follows directly from the extensivity property (see Eq. (32)). □

Proposition 40. *Any reflexive edge morphism ρ_μ satisfies* $\forall(u_1,u_2)\in N_1\times N_2$,

$$\begin{aligned}\rho_\mu^0(u_1,u_2,u_1,u_2)&=\rho_\mu(u_1,u_2,u_1,u_2)\\ &=\rho_\mu^2(u_1,u_2,u_1,u_2)\\ &=\cdots=\rho_\mu^\infty(u_1,u_2,u_1,u_2)\\ &=\rho_\sigma(u_1,u_2).\end{aligned}\tag{34}$$

Proof. For $n=0$ and 1 the proposition holds by definition. Let us assume that it holds for n, and let us prove it for $n+1$. We have

$$\rho_\mu^n(u_1,u_2,u_1,u_2)=\rho_\sigma(u_1,u_2).$$

From Eq. (6) we have

$$\rho_\mu(v_1,v_2,u_1,u_2)\leqslant\rho_\sigma(u_1,u_2).$$

Therefore,

$$\begin{aligned}&\sup_{(v_1,v_2)\in N_1\times N_2}\{\rho_\mu^n(u_1,u_2,v_1,v_2)\wedge\rho_\mu(v_1,v_2,u_1,u_2)\}\\ &\quad\leqslant\rho_\mu^n(u_1,u_2,u_1,u_2)\wedge\rho_\mu(u_1,u_2,u_1,u_2)\\ &\quad\leqslant\rho_\sigma(u_1,u_2)\wedge\rho_\sigma(u_1,u_2).\end{aligned}$$

So we have

$$\rho_\mu^{n+1}(u_1,u_2,u_1,u_2)\leqslant\rho_\sigma(u_1,u_2)$$

and since $\rho_\mu^{n+1}\geqslant\rho_\mu$, we also have

$$\rho_\mu^{n+1}(u_1,u_2,u_1,u_2)\geqslant\rho_\sigma(u_1,u_2).$$

This proves the property for $n+1$, and the proposition by recurrence. □

Proposition 41. *The internal composition $\rho_\mu\bullet\tau_\mu$ of two reflexive edge morphisms ρ_μ and τ_μ defined on the same graphs and with the same vertex morphism is reflexive.*

This shows that internal composition preserves reflexivity. It follows also that any power of a reflexive morphism is reflexive.

Proof. $\forall(u_1,u_2)\in N_1\times N_2$, we have

$$\begin{aligned}(\rho_\mu\bullet\tau_\mu)&(u_1,u_2,u_1,u_2)\\ &=\sup_{(v_1,v_2)\in N_1\times N_2}\{\tau_\mu(u_1,u_2,v_1,v_2)\\ &\qquad\qquad\wedge\rho_\mu(v_1,v_2,u_1,u_2)\}\\ &\geqslant\tau_\mu(u_1,u_2,u_1,u_2)\wedge\rho_\mu(u_1,u_2,u_1,u_2)\\ &\geqslant\rho_\sigma(u_1,u_2)\wedge\rho_\sigma(u_1,u_2)\\ &\geqslant\rho_\sigma(u_1,u_2).\end{aligned}$$

On the other hand, since the result of internal composition is a fuzzy morphism and from Eq. (6), we also have $(\rho_\mu\bullet\tau_\mu)(u_1,u_2,u_1,u_2)\leqslant\rho_\sigma(u_1,u_2)$, which shows the property. □

5.3. Symmetries

Symmetry in graph morphisms only concern edge morphisms. Indeed, for vertex morphisms defined on $N_1\times N_2$, symmetry does not make sense (except if

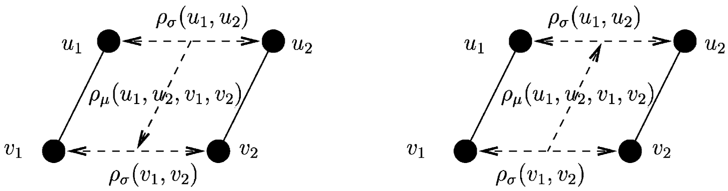

Fig. 7. Symmetrical morphism in the sense of a fuzzy graph for a non-directed graph: $\rho_\mu(u_1, u_2, v_1, v_2) = \rho_\mu(v_1, v_2, u_1, u_2)$.

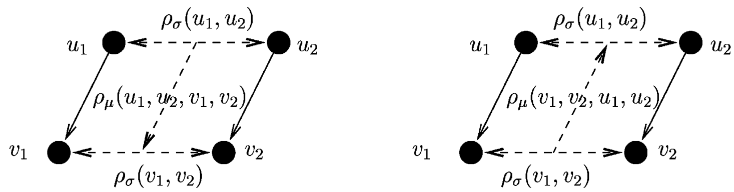

Fig. 8. Non-symmetrical morphism in the sense of a fuzzy graph for a directed graph: $\rho_\mu(u_1, u_2, v_1, v_2) \neq \rho_\mu(v_1, v_2, u_1, u_2)$.

$N_1 = N_2$). We can define two types of symmetry. The first one is expressed as:

Definition 42. A graph morphism (ρ_σ, ρ_μ) is weakly symmetrical if ρ_μ is symmetrical in the sense of symmetry of a fuzzy relation, i.e. $\forall(u_1, u_2, v_1, v_2) \in N_1 \times N_2 \times N_1 \times N_2$:

$$\rho_\mu(u_1, u_2, v_1, v_2) = \rho_\mu(v_1, v_2, u_1, u_2). \tag{35}$$

This definition of symmetry means that the matrix representing ρ_μ on $S \times S$ is symmetrical. This amounts to consider that the compatibilities between edge associations are not ordered. Here we consider that it is equivalent to consider the vertex association (u_1, u_2) and then the association (v_1, v_2), or to consider (v_1, v_2) and then (u_1, u_2) (see Fig. 7). To differentiate between both orders makes sense if the graph is directed. If (u_1, v_1) and (u_2, v_2) are directed, then v_1 is the successor of u_1 and v_2 is the successor of u_2. If u_1 and u_2 are associated, we consider the association of the successors in the graph by considering $\rho_\mu(u_1, u_2, v_1, v_2)$ (see Fig. 8). A similar reasoning applies for the predecessors.

Proposition 43. *If ρ_μ and τ_μ are two weakly symmetrical edge morphisms defined on the same graphs, their internal composition is symmetrical if and only if the composition commutes, i.e. $\rho_\mu \bullet \tau_\mu = \tau_\mu \bullet \rho_\mu$.*

This property is weaker than the one we have for reflexive morphisms. Here internal composition preserves symmetry if and only if it commutes.

Proof. $\forall(u_1, u_2, w_1, w_2) \in N_1 \times N_2 \times N_1 \times N_2$, we have

$$\begin{aligned}
&(\rho_\mu \bullet \tau_\mu)(u_1, u_2, w_1, w_2) \\
&\quad = \sup_{(v_1, v_2) \in N_1 \times N_2} \{\tau_\mu(u_1, u_2, v_1, v_2) \\
&\qquad\qquad \wedge \rho_\mu(v_1, v_2, w_1, w_2)\} \\
&\quad = \sup_{(v_1, v_2) \in N_1 \times N_2} \{\rho_\mu(w_1, w_2, v_1, v_2) \\
&\qquad\qquad \wedge \tau_\mu(v_1, v_2, u_1, u_2)\} \\
&\quad = (\tau_\mu \bullet \rho_\mu)(w_1, w_2, u_1, u_2).
\end{aligned}$$

This shows that $(\rho_\mu \bullet \tau_\mu)(u_1, u_2, w_1, w_2) = (\rho_\mu \bullet \tau_\mu)(w_1, w_2, u_1, u_2)$ iff $(\tau_\mu \bullet \rho_\mu)(w_1, w_2, u_1, u_2) = (\rho_\mu \bullet \tau_\mu)(w_1, w_2, u_1, u_2)$, i.e. iff the internal composition of ρ_μ and τ_μ commutes. $\square$

The second definition of symmetry we propose is as follows:

Definition 44. A graph morphism (ρ_σ, ρ_μ) is strongly symmetrical if $\forall(u_1, u_2, v_1, v_2) \in N_1 \times N_2 \times N_1 \times N_2$ we have

$$\begin{aligned}
\rho_\mu(u_1, u_2, v_1, v_2) &= \rho_\mu(u_1, v_2, v_1, u_2) \\
&= \rho_\mu(v_1, u_2, u_1, v_2).
\end{aligned} \tag{36}$$

Proposition 45. *Strong symmetry implies weak symmetry.*

Proof. The proof is straightforward by exchanging first the two vertices of N_1, and then the two vertices of N_2. $\square$

Strong symmetry is quite constraining, since Eq. (6) becomes $\forall(u_1,v_1)\in N_1\times N_1$, $\forall(u_2,v_2)\in N_2\times N_2$:

$$\rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\sigma(u_1,u_2)\wedge\rho_\sigma(v_1,v_2) \wedge\rho_\sigma(u_1,v_2)\wedge\rho_\sigma(v_1,u_2). \quad (37)$$

This property does not seem interesting globally, but rather locally.

Definition 46. A graph morphism (ρ_σ,ρ_μ) is *locally* strongly symmetrical if $\exists(u_1,u_2,v_1,v_2)\in N_1\times N_2\times N_1\times N_2$ such that

$$\rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\sigma(u_1,u_2)\wedge\rho_\sigma(v_1,v_2) \wedge\rho_\sigma(u_1,v_2)\wedge\rho_\sigma(v_1,u_2). \quad (38)$$

5.4. Transitivity

Definition 47. A graph morphism (ρ_σ,ρ_μ) is transitive if ρ_μ is transitive in the sense of fuzzy relations:

$$\rho_\mu\bullet\rho_\mu \leqslant \rho_\mu. \quad (39)$$

This definition relies on the internal representation and can be assimilated to an anti-extensivity property of the internal composition. Moreover, this property does not depend on the vertex morphism. A morphism is transitive if there does not exist a path with a better compatibility. It means that the morphism already encodes all association compatibility relations. It follows that $\lim_{n\to\infty}\rho_\mu^n$ is transitive. This definition also implies directly that any power of ρ_μ is less than ρ_μ.

Proposition 48. *A graph fuzzy morphism (ρ_σ,ρ_μ) that is both reflexive and transitive has a simple idempotence property*:

$$\rho_\mu\bullet\rho_\mu = \rho_\mu. \quad (40)$$

It follows that any power of ρ_μ is equal to ρ_μ.

Proof. Straightforward since reflexivity implies $\rho_\mu\bullet\rho_\mu\geqslant\rho_\mu$ and transitivity implies $\rho_\mu\bullet\rho_\mu\leqslant\rho_\mu$. $\square$

Let us note that for the vertex morphisms, it always holds that $\rho_\sigma\bullet\rho_\sigma=\rho_\sigma$.

Proposition 49. *If an edge morphism ρ_μ is both weakly symmetrical and transitive, then its values are bounded*:

$$\forall(u_1,u_2,v_1,v_2)\in N_1\times N_2\times N_1\times N_2, \quad \rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\mu(u_1,u_2,u_1,u_2). \quad (41)$$

Note that this property is the same as the one for reflexive morphisms.

Proof.

$$\begin{aligned}
&\rho_\mu(u_1,u_2,v_1,v_2)\\
&\quad\leqslant \sup_{(w_1,w_2)\in N_1\times N_2}\{\rho_\mu(u_1,u_2,w_1,w_2) \wedge\rho_\mu(u_1,u_2,w_1,w_2)\}\\
&\quad\leqslant \sup_{(w_1,w_2)\in N_1\times N_2}\{\rho_\mu(u_1,u_2,w_1,w_2) \wedge\rho_\mu(w_1,w_2,u_1,u_2)\}\\
&\quad\leqslant (\rho_\mu\bullet\rho_\mu)(u_1,u_2,u_1,u_2)\\
&\quad\leqslant \rho_\mu(u_1,u_2,u_1,u_2). \qquad \square
\end{aligned}$$

The values of a symmetrical and transitive morphism are even more constrained, according to the following property.

Proposition 50. *For any triplet of edges, the two smallest values of the edge morphism (supposed to be weakly symmetrical and transitive) computed on pairs of edges in this triplet are necessarily equal.*

Proof. Let us consider any three edges (u_1,u_2), (v_1,v_2), (w_1,w_2). Let us assume for instance (without loss of generality because of symmetry) that

$$\rho_\mu(u_1,u_2,w_1,w_2) \leqslant \rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\mu(v_1,v_2,w_1,w_2).$$

Since ρ_μ is transitive, we have

$$\begin{aligned}
\rho_\mu(u_1,u_2,w_1,w_2) & \\
&\geqslant (\rho_\mu \bullet \rho_\mu)(u_1,u_2,w_1,w_2) \\
&\geqslant \sup_{(v_1',v_2')\in N_1\times N_2} \{\rho_\mu(u_1,u_2,v_1',v_2') \\
&\qquad\qquad \wedge \rho_\mu(v_1',v_2',w_1,w_2)\} \\
&\geqslant \rho_\mu(u_1,u_2,v_1,v_2) \wedge \rho_\mu(v_1,v_2,w_1,w_2) \\
&\geqslant \rho_\mu(u_1,u_2,v_1,v_2),
\end{aligned}$$

since $\rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\mu(v_1,v_2,w_1,w_2)$. And therefore $\rho_\mu(u_1,u_2,w_1,w_2) = \rho_\mu(u_1,u_2,v_1,v_2)$. □

Proposition 51. *If ρ_μ and τ_μ are two transitive edge morphisms on the same graphs and if their internal composition commutes, then $\rho_\mu \bullet \tau_\mu$ is transitive.*

The reverse is in general not true, and the transitivity of the composition is not sufficient to guarantee its commutativity. This contrasts with the case of symmetric morphisms.

Proof. It follows directly from the fact that $\rho_\mu \bullet \tau_\mu$ commutes, from the associativity of the internal composition and from its increasingness. □

Proposition 52. *It ρ_μ is a transitive morphism, for any two edge morphisms τ_μ and τ_μ' on the same graphs satisfying $\tau_\mu \leqslant \rho_\mu$ and $\tau_\mu' \leqslant \rho_\mu$, we have*

$$\tau_\mu \bullet \tau_\mu' \leqslant \rho_\mu. \tag{42}$$

If τ_μ is moreover reflexive, we have

$$\rho_\mu \bullet \tau_\mu = \tau_\mu \bullet \rho_\mu = \rho_\mu. \tag{43}$$

Proof. The first equation follows directly from the increasingness of internal composition and from transitivity of ρ_μ. The second one follows from the first one and from the extensivity of the internal composition if one morphism is reflexive. □

5.5. *α-Cuts of a fuzzy morphism*

Cuts are useful in graph theory to extract a subgraph satisfying some properties. On the other hand, the notion of α-cuts is used in fuzzy set theory for several purposes, including defuzzification and decision making. Therefore, we introduce the notion of α-cuts of a fuzzy morphism. It is directly inspired by the notion of α-cut of a fuzzy set [13,41] and of a fuzzy relation [30].

Definition 53. The α-cut of a fuzzy morphism (ρ_σ, ρ_μ), for $\alpha \in [0,1]$ *is defined as the pair* $(\rho_{\sigma_\alpha}, \rho_{\mu_\alpha})$, where ρ_{σ_α} is a crisp subset of pairs of vertices such that

$$\rho_{\sigma_\alpha} = \{(u_1,u_2) \in N_1 \times N_2,\ \rho_\sigma(u_1,u_2) \geqslant \alpha\} \tag{44}$$

and ρ_{μ_α} is a crisp subset of pairs of edges such that

$$\begin{aligned}\rho_{\mu_\alpha} = \{&(u_1,u_2,v_1,v_2) \in N_1 \times N_2 \times N_1 \times N_2, \\ &\rho_\mu(u_1,u_2,v_1,v_2) \geqslant \alpha\}.\end{aligned} \tag{45}$$

This definition corresponds to a threshold on the morphisms values, keeping only correspondences that have degrees greater than α.

Proposition 54. *We have the following stability and preservation properties*:

- *ρ_{μ_α} is a relation (in the classical sense) on ρ_{σ_α}.*
- *Taking α-cuts of morphisms commutes with the internal composition*:

$$(\rho_\sigma \bullet \tau_\sigma)_\alpha = \rho_{\sigma_\alpha} \cup \tau_{\sigma_\alpha}, \tag{46}$$

$$(\rho_\mu \bullet \tau_\mu)_\alpha = \rho_{\mu_\alpha} \bullet \tau_{\mu_\alpha}. \tag{47}$$

- *If ρ_μ is reflexive, any α-cut of ρ_μ is reflexive.*
- *If ρ_μ is weakly symmetrical, any α-cut of ρ_μ is symmetrical.*
- *If ρ_μ is transitive, any α-cut of ρ_μ is transitive.*

Proof. If $\rho_\mu(u_1,u_2,v_1,v_2) \geqslant \alpha$, we have

$$\alpha \leqslant \rho_\mu(u_1,u_2,v_1,v_2) \leqslant \rho_\sigma(u_1,u_2) \wedge \rho_\sigma(v_1,v_2)$$

and therefore $\rho_\sigma(u_1,u_2) \geqslant \alpha$ and $\rho_\sigma(v_1,v_2) \geqslant \alpha$, which shows the first property.

Let us now consider the α-cuts of the internal composition:

$$\begin{aligned}
&(\rho_\mu \bullet \tau_\mu)(u_1,u_2,w_1,w_2) \geqslant \alpha \\
&\quad\Leftrightarrow \sup_{(v_1,v_2)\in N_1\times N_2} \{\tau_\mu(u_1,u_2,v_1,v_2) \\
&\qquad\qquad \wedge \rho_\mu(v_1,v_2,w_1,w_2)\} \geqslant \alpha \\
&\quad\Leftrightarrow \exists (v_1,v_2) \in N_1 \times N_2,\ \tau_\mu(u_1,u_2,v_1,v_2) \\
&\qquad \wedge \rho_\mu(v_1,v_2,w_1,w_2) \geqslant \alpha
\end{aligned}$$

$$\Leftrightarrow \exists (v_1, v_2) \in N_1 \times N_2,\ \tau_\mu(u_1, u_2, v_1, v_2) \geqslant \alpha$$
$$\text{and } \rho_\mu(v_1, v_2, w_1, w_2) \geqslant \alpha$$
$$\Leftrightarrow (u_1, u_2, w_1, w_2) \in \rho_{\mu_\alpha} \bullet \tau_{\mu_\alpha}.$$

The proof for the vertex morphisms is straightforward.

The proofs of the preservation of edge morphism properties by taking α-cuts are also straightforward. □

5.6. Fuzzy cluster

A fuzzy cluster generalizes to fuzzy graphs the usual notion of clique and the notion of connected components [30]. We adapt here its definition to the formalism of edge morphism:

Definition 55. The set $\mathscr{C} \subseteq N_1 \times N_2$ is a fuzzy cluster of order k on the weakly symmetrical edge morphism ρ_μ: $N_1 \times N_2 \times N_1 \times N_2$ if

$$\inf_{((u_1,u_2),(v_1,v_2)) \in \mathscr{C}^2} \rho_\mu^k(u_1, u_2, v_1, v_2) > \sup_{(w_1,w_2) \notin \mathscr{C}} \left\{ \inf_{(x_1,x_2) \in \mathscr{C}} \rho_\mu^k(x_1, x_2, w_1, w_2) \right\}. \tag{48}$$

The *membership degree of the cluster* is defined as

$$\rho_\mu(\mathscr{C}) = \inf_{((u_1,u_2),(v_1,v_2)) \in \mathscr{C}^2} \rho_\mu^k(u_1, u_2, v_1, v_2). \tag{49}$$

When $k = 1$, the cluster is the generalization of the notion of clique, a maximal complete subgraph. When $k \to \infty$ a cluster becomes a connected component where each pair of vertices is joined by a path.

A fuzzy clique can be interpreted as a group of associations, the minimum compatibility degree in the group being greater than all other compatibilities between an association in the group and an association outside the group. This corresponds to the notion of a strong association group.

Fuzzy cliques can be used in several situations. Let us mention decision making: a classical method for finding the best associations in an association graph is to search for maximal cliques [7]. The defuzzification of a graph morphism can then be obtained from the fuzzy cliques of maximal order. Another idea is to define quality criteria of a morphism, as a function of the number, the size, or the minimal degree of its fuzzy cliques. This idea can be further developed for optimizing a graph morphism, leading to the "best" matching between two graphs.

6. Conclusion

We proposed in this paper a new formalism for defining fuzzy morphisms between graphs (GFM). It extends previous works on fuzzy relations and fuzzy graphs. The main feature of the GFM is the use of a pair of functions: one is a mapping between vertices, the other is a mapping between edges. Moreover, these two functions are linked as the vertices and the edges are linked in a graph. We have introduced two complementary interpretations of GFM with the internal and the external views. The conjunction of these properties constitutes a link between the classical notion of association compatibility and the new notion of edge morphism. The first use is the definition of several composition laws which have several interesting properties. All these properties and others have been interpreted in terms of graph matching and association graphs.

Further work is twofold. Firstly, other properties are worth investigating. Other composition laws can be defined using, for instance, max-star combination, for which different properties can be expected. Particular graphs can be considered (such as topological graphs for instance), and definitions and properties can be specified for such graphs, in order to exploit more deeply their particular structure. The notion of fuzzy cluster should also be further exploited.

Secondly, based on this formalism, the use of GFM for inexact graph matching can now be addressed. This calls for three main steps: defining the quality of a GFM, designing appropriate optimization techniques, and designing a decision rule. The morphism itself is a part of quality evaluation since it provides degrees of correspondence between vertices and between edges. But is not sufficient for this evaluation, and similarities between vertex attributes and between edge attributes should be combined with the degrees of correspondence. Also the composition laws may be useful at this step, as well as the order of maximal fuzzy cliques. Concerning optimization techniques, a first step has been proposed based on this GFM formalism, using genetic algorithms [29] and estimation distribution algorithms [3]. They will be further exploited in our future work, along with relaxation-based techniques. Algorithmic complexity has to be addressed for large size problems. Parallelizing optimization algorithms could be foreseen. The decision rule is necessary in cases where a crisp assignment is needed. The notions of α-cuts of a morphism and of fuzzy cliques may be useful at this level.

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