Computer Vision and Image Understanding 132 (2015) 1–2

Contents lists available at ScienceDirect

Computer Vision and Image Understanding

journal homepage: www.elsevier.com/locate/cviu

Corrigendum

Corrigendum to "Mathematical morphology on hypergraphs, application to similarity and positive kernel" [Comput. Vis. Image Understanding 117 (2013) 342–354] ☆

Isabelle Bloch [a,*], Alain Bretto [b]

[a] Institut Mines-Telecom, Telecom ParisTech, CNRS LTCI, Paris, France
[b] NormandieUniv-Unicaen, Greyc Cnrs-Umr 6072, Caen, France

This note provides a better and detailed proof of Theorem 2 of [1], as well as some corollaries of the main result.

In [1], we developed a theory of mathematical morphology on hypergraphs. Let $H = (V, E)$ be a hypergraph with $E = \{e_1 \dots e_m\}$ ($|E| = m$), and $\delta : (\mathcal{P}(E), \subseteq) \longrightarrow (\mathcal{P}(E), \subseteq)$ be a dilation such that $\delta(e) \neq \emptyset$ for all $e \in E$. We defined a similarity function between hyperedges as:

$$s : E \times E \longrightarrow \mathbb{R}^+$$
$$(e_i, e_j) \mapsto s(e_i, e_j) = \frac{|\delta(e_i) \cap \delta(e_j)|}{|\delta(e_i) \cup \delta(e_j)|}. \quad (1)$$

We stated that the matrix $M = (s(e_i, e_j))_{i,j \in \{1 \dots m\}}$, for s defined as in Eq. (1) from a morphological dilation, is positive definite. In the proof we used the Schur product theorem (i.e. the Hadamard product of two positive semi definite matrices is positive semi definite), which actually holds for symmetric matrices. However the proof involved the sum of anti-symmetric and diagonal matrices, so the argument cannot be applied in this case (although the result was correct).

In this note, we provide a better and more detailed proof of this result, by proving the following theorem. This proof has also the advantage of suggesting extensions to other similarity matrices, has shown by the two subsequent corollaries.

Theorem 1. *Let $H = (V, E)$ be a hypergraph with $E = \{e_1, e_2, e_3, \dots e_m\}$ and let $\delta : (\mathcal{P}(E), \subseteq) \longrightarrow (\mathcal{P}(E), \subseteq)$ be a dilation such that $\delta(e_i) \neq \emptyset$ for all $i \in \{1, 2, \dots, m\}$. Then the following matrices:*

(i) $A = (|\delta(e_i) \cap \delta(e_j)|)_{i,j \in \{1,2,\dots,m\}}$,

(ii) $B = \left(\frac{1}{|\delta(e_i) \cup \delta(e_j)|}\right)_{i,j \in \{1,2,\dots,m\}}$,

(iii) *and* $M = \left(\frac{|\delta(e_i) \cap \delta(e_j)|}{|\delta(e_i) \cup \delta(e_j)|}\right)_{i,j \in \{1,2,\dots,m\}}$

are positive semi definite.

Proof. (*i*) Let $|\delta(e_i)| = a_{i,i}$ for all $i \in \{1, 2, \dots, m\}$ and $|\delta(e_i) \cap \delta(e_j)| = a_{i,j}$ for all $i, j \in \{1, 2, \dots, m\}$, $i \neq j$.

We obtain the matrix:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{pmatrix}$$

To E we can associate a vector space on a field $\mathbb{K}$ in the following way: to each element e_i we associate a vector: $\vec{e_i} = (\underbrace{0, 0, \dots, 0}_{i-1}, 1, \underbrace{0, 0, \dots 0, 0}_{m-i})$. This family of vectors is a basis of a vector space denoted by $V(E)$. So for $\vec{u} \in V(E), \vec{u} = \sum_{i=1}^{m} \lambda_i \vec{e_i}$.

We have the inner product:

$$\begin{array}{llll} b: & V(E) \times V(E) & \rightarrow & \mathbb{R} \\ & (\vec{u_i}; \vec{u_j}) & \mapsto & \langle \vec{u_i}; \vec{u_j} \rangle = \overrightarrow{u_i^{\top}} \vec{u_j} \end{array}$$

For all $\delta(e_i), i \in \{1, 2, \dots, m\}$ we define the vector $\vec{u}_{\delta(e_i)} \in V(E)$ by

$$\vec{u}_{\delta(e_i)} = \sum_{k=1}^{m} \chi_{\delta(e_i)}(e_k) \vec{e_k}$$

where for all $i \in \{1, 2, \dots, m\}$

$$\chi_{\delta(e_i)}(e_k) = \begin{cases} 1 & \text{if } e_k \in \delta(e_i) \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\langle \vec{u}_{\delta(e_i)}; \vec{u}_{\delta(e_j)} \rangle = \sum_{k=1}^{m} \chi_{\delta(e_i)}(e_k).\chi_{\delta(e_j)}(e_k) = \sum_{k=1}^{m} \chi_{\delta(e_i) \cap \delta(e_j)}(e_k)$$
$$= |\delta(e_i) \cap \delta(e_j)|.$$

Therefore,

$$A = (|\delta(e_i) \cap \delta(e_j)|)_{i,j \in \{1,2,\dots,m\}} = (\langle \vec{u}_{\delta(e_i)}; \vec{u}_{\delta(e_j)} \rangle)_{i,j \in \{1,2,\dots,m\}}.$$

DOI of original article: http://dx.doi.org/10.1016/j.cviu.2012.10.013

☆ This work was partially funded by a grant from Institut Mines-Telecom/Telecom ParisTech, and was initiated during the sabbatical stay of A. Bretto at Telecom ParisTech.

* Corresponding author.
E-mail addresses: isabelle.bloch@telecom-paristech.fr (I. Bloch), alain.bretto@info.unicaen.fr (A. Bretto).

http://dx.doi.org/10.1016/j.cviu.2014.12.003
1077-3142/

It is the Gram matrix of $\{\vec{u}_{\delta(e_1)}, \vec{u}_{\delta(e_2)}, \vec{u}_{\delta(e_3)} \ldots, \vec{u}_{\delta(e_m)}\}$ which is positive semi definite. Hence, A is positive semi definite.

(ii) Let

$$b_{i,j} = \frac{1}{|\delta(e_i) \cup \delta(e_j)|}, \text{ for } i,j \in \{1,2,\ldots,m\}$$

and let us note $\delta(e_i) = C_i^c$, consequently:

$$\delta(e_i) \cup \delta(e_j) = C_i^c \cup C_j^c = \left(C_i \cap C_j\right)^c \text{ and } |\delta(e_i) \cup \delta(e_j)| = |E| - |C_i \cap C_j|.$$

We have:

$$b_{i,j} = \frac{1}{|E|(1 - \frac{|C_i \cap C_j|}{|E|})}, \quad \text{for } i,j \in \{1,2,\ldots,m\}.$$

So

$$B = \frac{1}{|E|}\left(\frac{1}{1 - c_{i,j}}\right)_{i,j\in\{1,2,\ldots,m\}} \quad \text{with } c_{i,j} = \frac{|C_i \cap C_j|}{|E|}$$

Since $\delta(e_i) \neq \emptyset$ for all $i \in \{1,2,\ldots,m\}$, we have: $0 \leqslant c_{i,j} < 1$ and $\frac{1}{1-c_{i,j}} = \sum_{k=0}^{+\infty} c_{i,j}^k$, hence,

$$B = \frac{1}{|E|}\sum_{k=0}^{+\infty} C^{\circ k}$$

where $C^{\circ k} = \underbrace{C \circ C \circ \ldots \circ C}_{k}$ is a k times Hadamard product.

Now the matrix

$$C = (c_{i,j}) = \frac{1}{|E|}(|C_i \cap C_j|)_{1\leqslant i,j\leqslant m}$$

is positive semi definite by (i) applied to C_i^c

From the Schur product theorem, if M and N are two positive semi definite matrices, then $M \circ N = N \circ M$ is also positive semi definite, hence, $C^{\circ k}$ is positive semi definite for all $k \in \mathbb{N}$. From this, for all $x_1, x_2, \ldots, x_m \in \mathbb{R}$, we have:

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}^{\circ k} x_i x_j \geqslant 0.$$

So, for all $N \geqslant 0$:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=0}^{N} c_{i,j}^{\circ k} x_i x_j \geqslant 0.$$

Consequently for $N \longrightarrow +\infty$:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=0}^{+\infty} c_{i,j}^{\circ k} x_i x_j \geqslant 0;$$

and we have:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=0}^{+\infty} c_{i,j}^{\circ k} x_i x_j = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{1}{1 - c_{i,j}} x_i x_j \geqslant 0.$$

Both the matrix $\left(\frac{1}{1-c_{i,j}}\right)_{i,j\in\{1,2,\ldots,m\}}$ and $|E|$. B are definite semi positive, so B is.

(iii) By applying again the Schur product theorem, the matrix

$$M = A \circ B$$

is positive semi definite. □

Corollary 1. *Let $E = \{e_1, e_2, e_3, \ldots e_n\}$ a set and let $\{A_i, i \in \{1,2,\ldots,m\}\}$ be a set of non-empty subsets of E, with $\forall i \in \{1,2,\ldots,m\}, A_i \neq \emptyset$. Then the following matrices:*

(i) $A = \left(|A_i \cap A_j|\right)_{i,j\in\{1,2,\ldots,m\}}$,

(ii) $B = \left(\frac{1}{|A_i \cup A_j|}\right)_{i,j\in\{1,2,\ldots,m\}}$,

(iii) and the Jaccard *index* $M = \left(\frac{|A_i \cap A_j|}{|A_i \cup A_j|}\right)_{i,j\in\{1,2,\ldots,m\}}$

are positive semi definite.

Corollary 2. *Let $E = \{e_1, e_2, e_3, \ldots e_n\}$ a set and let $\{A_i, i \in \{1,2,\ldots,m\}\}$ be a set of non-empty subsets of E, with $\forall i \in \{1,2,\ldots,m\}, A_i \neq \emptyset$. Then the following matrices, for $\alpha \in \mathbb{R}^{+*}$:*

(i) $A = \left(|A_i \cap A_j|\right)_{i,j\in\{1,2,\ldots,m\}}$,

(ii) $B = \left(\frac{1}{(|A_i| + |A_j|)^\alpha}\right)_{i,j\in\{1,2,\ldots,m\}}$,

(iii) and $M = \left(\frac{|A_i \cap A_j|}{(|A_i| + |A_j|)^\alpha}\right)_{i,j\in\{1,2,\ldots,m\}}$

are positive semi definite.

Proof. The proof is similar to the previous one. For (ii), we have:

$$\frac{1}{(|A_i| + |A_j|)^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-(|A_i|+|A_j|)t} t^{\alpha-1} dt$$

with

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$$

Let

$$f_{A_i}(t) = e^{-|A_i|t} t^{\frac{\alpha-1}{2}} \in L^2([0;\infty[).$$

We introduce the following scalar product:

$$\langle f_{A_i}; f_{A_j}\rangle = \int_0^\infty f_{A_i}(t) f_{A_j}(t) dt.$$

So

$$B = \frac{1}{\Gamma(\alpha)}\langle f_{A_i}; f_{A_j}\rangle$$

which a Gram matrix. □

Reference

[1] I. Bloch, A. Bretto, Mathematical morphology on hypergraphs, application to similarity and positive kernel, Comput. Vis. Image Understanding 117 (4) (2013) 342–354.