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Belief revision, minimal change and relaxation: A general framework based on satisfaction systems, and applications to description logics

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A B S T R A C T

Belief revision of knowledge bases represented by a set of sentences in a given logic has been extensively studied but for specific logics, mainly propositional, and also recently Horn and description logics. Here, we propose to generalize this operation from a model-theoretic point of view, by defining revision in the abstract model theory of satisfaction systems. In this framework, we generalize to any satisfaction system the characterization of the AGM postulates given by Katsuno and Mendelzon for propositional logic in terms of minimal change among interpretations. In this generalization, the constraint on syntax independence is partially relaxed. Moreover, we study how to define revision, satisfying these weakened AGM postulates, from relaxation notions that have been first introduced in description logics to define dissimilarity measures between concepts, and the consequence of which is to relax the set of models of the old belief until it becomes consistent with the new pieces of knowledge. We show how the proposed general framework can be instantiated in different logics such as propositional, first-order, description and Horn logics. In particular for description logics, we introduce several concrete relaxation operators tailored for the description logic $\mathcal{ALC}$ and its fragments $\mathcal{EL}$ and $\mathcal{ELU}$, discuss their properties and provide some illustrative examples.

1. Introduction

Belief change, the process that makes an agent's beliefs evolve with newly acquired knowledge, is one of the classical but still challenging problems in artificial intelligence. It is gaining more and more interest these days, due to the emergence of new logical-based knowledge representation frameworks enjoying good complexity properties, allowing them to tackle large scale knowledge bases, and to reason on massive datasets. Among these logical frameworks, one can mention Description Logics (DLs) and Horn Clause theories. Description logics, for instance, are now pervasive in many knowledge-based representation systems such as ontological reasoning, semantic web, scene understanding, cognitive robotics, to mention a few. In all these domains, the expert knowledge is not fixed, but rather a flux evolving over time, hence requiring the definition of rational change operators.

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Studying the rationality of belief change operators, when knowledge bases are logical theories, i.e. sets of sentences in a given logic, goes back to the seminal work of Alchourròn, Gardenfors and Makinson [1], that gave birth to what is now known as AGM theory. Three change operations are studied within this framework, *expansion, contraction and revision*. Belief expansion consists in adding new knowledge without checking consistency, while both contraction and revision consist in consistently removing and adding new knowledge, respectively. We focus in this paper on belief revision.

Although defined in the abstract framework of logics given by Tarski [40] (so called Tarskian logics), postulates of the AGM theory make strong assumptions on the considered logics. Indeed, in [1] the considered logics have to be closed under the standard propositional connectives in $\{\wedge, \vee, \neg, \Rightarrow\}$, to be compact (i.e. inference depends on a finite set of axioms), and to satisfy the deduction theorem (i.e. entailment and implication are equivalent). While compactness is a standard property of logics, to be closed under the standard propositional connectives is more questionable. Indeed, many logics (called hereafter non-classical logics) such as description logics, equational logic or Horn clause logic, widely used for various modern applications in computing science, do not satisfy such a constraint. Recently, in many works, belief change has been studied in such non-classical logics [12,17,34,35]. For instance, Ribeiro et al. in [35] studied contraction at the abstract level of Tarskian logics, and recently Zhuang et al. in [42] proposed an extension of AGM contraction to arbitrary logics. The adaptation of the AGM postulates for revision for non-classical logics has been studied but only for specific logics, mainly description logics [16,17,28,29,31,33,41] and Horn logics [11,43]. The reason is that revision can be abstractly defined in terms of expansion and retraction following the Levi identity [23], but this requires the use of negation, which rules out some non-classical logics that do not consider this connective [34].

The AGM postulates were interpreted in terms of minimal change in [22], in the sense that the models of the revision should be as close as possible, according to some metric, to the models of the initial knowledge set. However, to the best of our knowledge, the generalization of the AGM theory with minimality criteria on the set of models of knowledge bases has never been proposed. The reason is that semantics is not explicit in the abstract framework of logics defined by Tarski.

We propose here to generalize AGM revision but in the abstract model theory of satisfaction systems, which formalizes the intuitive notion of logical systems, including syntax, semantics and the satisfaction relation. This notion was introduced in [18] under the name of "rooms", and then of "satisfaction systems" in [38]. See also [26]. Then, we propose to generalize to any satisfaction system the approach developed in [22] for propositional logic and in [30] for description logics. In this abstract framework, we will also show how to define revision operators from the relaxation notion that has been introduced in description logics to define dissimilarity measures between concepts [14,15]. The main idea is to relax the set of models of the old belief until it becomes consistent with the new pieces of knowledge. This notion of relaxation, defined in an abstract way through a set of properties, turns out to generalize several revision operators introduced in different contexts e.g. [9,20,25,29]. This is another key contribution of our work.

To concretize our abstract framework, we provide examples of relaxations in propositional logics, first order logics, and Horn logic. The case of description logics (DLs) is more detailed. This is motivated, as mentioned above, by their broad scope of applications, including reasoning on large web data.

The paper is organized as follows. Section 2 reviews some concepts, notations and terminology about satisfaction systems which are used in this work. In Section 3, we adapt the AGM theory in the framework of satisfaction systems, and then give an abstract model-theoretic rewriting of the AGM postulates. We then show in Section 3.2 that any revision operator satisfying such postulates accomplishes an update with minimal change to the set of models of knowledge bases. In Section 3.3, we introduce a general framework of relaxation-based revision operators and show that our revision operators lead to faithful assignments and then also satisfy the AGM postulates. In Section 4, we illustrate our abstract approach by providing revision operators in different logics, including classical logics (propositional and first order logics) and non-classical ones (Horn and description logics). The case of DL is further developed in Section 4.4, with several examples. Finally, Section 5 is dedicated to related works.

2. Satisfaction systems

Satisfaction systems [26] generalize Tarski's classical "semantic definition of truth" [39] and Barwise's "Translation Axiom" [4]. For the sake of generalization, sentences are simply required to form a set. All other contingencies such as inductive definition of sentences are not considered. Similarly, models are simply seen as elements of a class, i.e. no particular structure is imposed on them.

2.1. Definition and examples

Definition 1 *(Satisfaction system).* A **satisfaction system** $\mathcal{R} = (Sen, Mod, \models)$ consists of

- a set Sen of **sentences**,
- a class Mod of **models**, and
- a satisfaction relation $\models \subseteq Mod \times Sen$.

Let us note that the non-logical vocabulary, so-called *signature*, over which sentences and models are built, is not specified in Definition 1.[1] Actually, it is left implicit. Hence, as we will see in the examples developed in the paper, a satisfaction system always depends on a signature.

Example 1. The following examples of satisfaction systems are of particular importance in computer science and in the remainder of this paper.

Propositional Logic (PL) Given a set of propositional variables Σ, we can define the satisfaction system $\mathcal{R}_\Sigma = (Sen, Mod, \models)$ where Sen is the least set of sentences finitely built over propositional variables in Σ and Boolean connectives in $\{\neg, \vee\}$, Mod contains all the mappings $\nu : \Sigma \to \{0, 1\}$ (0 and 1 are the usual truth values), and the satisfaction relation $\models$ is the usual propositional satisfaction.

Horn Logic (HCL) A *Horn clause* is a sentence of the form $\Gamma \Rightarrow \alpha$ where Γ is a finite (possibly empty) conjunction of propositional variables and α is a propositional variable. The satisfaction system of Horn clause logic is then defined as for **PL** except that sentences are restricted to be conjunctions of Horn clauses.

First Order Logic (FOL) and Many-sorted First Order Logic We detail here only the many-sorted variant of FOL, FOL being a particular case. Signatures are triplets (S, F, P) where S is a set of sorts, and F and P are a set of functions and a set of predicate names, respectively, both with arities in $S^* \times S$ and S^+ respectively (S^+ is the set of all non-empty sequences of elements in S and $S^* = S^+ \cup \{\epsilon\}$ where ϵ denotes the empty sequence). In the following, to indicate that a function name $f \in F$ (respectively a predicate name $p \in P$) has for arity $(s_1 \dots s_n, s)$ (respectively $s_1 \dots s_n$), we will note $f : s_1 \times \dots \times s_n \to s$ (resp. $p : s_1 \times \dots \times s_n$).

Given a signature $\Sigma = (S, F, P)$, we can define the satisfaction system $\mathcal{R}_\Sigma = (Sen, Mod, \models)$ where:

- Sen is the least set of sentences built over atoms of the form $p(t_1, \dots, t_n)$ where $p : s_1 \times \dots \times s_n \in P$ and $t_i \in T_F(X)_{s_i}$ for every i, $1 \le i \le n$ ($T_F(X)_s$ is the term algebra of sort s built over F with sorted variables in a given set X) by finitely applying Boolean connectives in $\{\neg, \vee\}$ and the quantifier $\forall$.
- Mod is the class of models $\mathcal{M}$ defined by a family $(M_s)_{s \in S}$ of sets (one for every $s \in S$), each one equipped with a function $f^\mathcal{M} : M_{s_1} \times \dots \times M_{s_n} \to M_s$ for every $f : s_1 \times \dots \times s_n \to s \in F$ and with an n-ary relation $p^\mathcal{M} \subseteq M_{s_1} \times \dots \times M_{s_n}$ for every $p : s_1 \times \dots \times s_n \in P$.
- Finally, the satisfaction relation $\models$ is the usual first-order satisfaction.

As for **PL**, we can consider the logic **FHCL** of first-order Horn Logic whose models are those of **FOL** and sentences are restricted to be conjunctions of universally quantified Horn sentences (i.e. sentences of the form $\Gamma \Rightarrow \alpha$ where Γ is a finite conjunction of atoms and α is an atom).

Description logic (DL) Signatures are triplets (N_C, N_R, I) where N_C, N_R and I are nonempty pairwise disjoint sets where elements in N_C, N_R and I are called concept names, role names and individuals, respectively.

Given a signature $\Sigma = (N_C, N_R, I)$, we can define the satisfaction system $\mathcal{R}_\Sigma = (Sen, Mod, \models)$ where:

- Sen contains[2] all the sentences of the form $C \sqsubseteq D$, $x : C$ and $(x, y) : r$ where $x, y \in I$, $r \in N_R$ and C is a concept inductively defined from $N_C \cup \{\top\}$ and binary and unary operators in $\{\_ \sqcap \_, \_ \sqcup \_\}$ and in $\{\_^c, \forall r.\_, \exists r.\_\}$, respectively.
- Mod is the class of models $\mathcal{I}$ defined by a set $\Delta^\mathcal{I}$ equipped for every concept name $A \in N_C$ with a set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, for every relation name $r \in N_R$ with a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and for every individual $x \in I$ with a value $x^\mathcal{I} \in \Delta^\mathcal{I}$.
- The satisfaction relation $\models$ is then defined as:
  - · $\mathcal{I} \models C \sqsubseteq D$ iff $C^\mathcal{I} \subseteq D^\mathcal{I}$,
  - · $\mathcal{I} \models x : C$ iff $x^\mathcal{I} \in C^\mathcal{I}$,
  - · $\mathcal{I} \models (x, y) : r$ iff $(x^\mathcal{I}, y^\mathcal{I}) \in r^\mathcal{I}$,

  where $C^\mathcal{I}$ is the evaluation of C in $\mathcal{I}$ inductively defined on the structure of C as follows:
  - · if $C = A$ with $A \in N_C$, then $C^\mathcal{I} = A^\mathcal{I}$;
  - · if $C = \top$ then $C^\mathcal{I} = \Delta^\mathcal{I}$;
  - · if $C = C' \sqcup D'$ (resp. $C = C' \sqcap D'$), then $C^\mathcal{I} = C'^\mathcal{I} \cup D'^\mathcal{I}$ (resp. $C^\mathcal{I} = C'^\mathcal{I} \cap D'^\mathcal{I}$);
  - · if $C = C'^c$, then $C^\mathcal{I} = \Delta^\mathcal{I} \setminus C'^\mathcal{I}$;
  - · if $C = \forall r.C'$, then $C^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \forall y \in \Delta^\mathcal{I}, (x, y) \in r^\mathcal{I} \text{ implies } y \in C'^\mathcal{I}\}$;
  - · if $C = \exists r.C'$, then $C^\mathcal{I} = \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I}, (x, y) \in r^\mathcal{I} \text{ and } y \in C'^\mathcal{I}\}$.

2.2. Knowledge bases and theories

Let us now consider a fixed but arbitrary satisfaction system $\mathcal{R} = (Sen, Mod, \models)$ (since the signature Σ is supposed fixed, the subscript Σ will be omitted from now on).

[1] The set of logical symbols is defined in each particular logic, and does not depend on a theory.

[2] The description logic defined here is better known under the acronym $\mathcal{ALC}$.

Notation 1. Let $T \subseteq Sen$ be a set of sentences.

- $Mod(T)$ is the sub-class of Mod whose elements are models of T, i.e. for every $\mathcal{M} \in Mod(T)$ and every $\varphi \in T$, $\mathcal{M} \models \varphi$. When T is restricted to a formula φ (i.e. $T = \{\varphi\}$), we will denote $Mod(\varphi)$, the class of model of $\{\varphi\}$, rather than $Mod(\{\varphi\})$.
- $Cn(T) = \{\varphi \in Sen \mid \forall \mathcal{M} \in Mod(T),\ \mathcal{M} \models \varphi\}$ is the set of *semantic consequences of* T.
- Let $\mathbb{M} \subseteq Mod$. Let us note $\mathbb{M}^* = \{\varphi \in Sen \mid \forall \mathcal{M} \in \mathbb{M}, \mathcal{M} \models \varphi\}$. Therefore, we have for every $T \subseteq Sen$, $Cn(T) = Mod(T)^*$. When $\mathbb{M}$ is restricted to one model $\mathcal{M}$, $\mathbb{M}^*$ will be equivalently noted $\mathcal{M}^*$.
- Let us note $Triv = \{\mathcal{M} \in Mod \mid \mathcal{M}^* = Sen\}$, i.e. the set of models in which all formulas are satisfied. In **PL** and **FOL**, $Triv$ is empty because the negation is considered. Similarly, the complementation is involved in the **DL** $\mathcal{ALC}$, hence $Triv$ is empty. In **HCL**, $Triv$ only contains the unique model where all propositional variables have a truth value equal to 1. In **FHCL**, $Triv$ contains all models $\mathcal{M}$ where for every predicate name $p : s_1 \times \ldots \times s_n \in P$, $p^{\mathcal{M}} = M_{s_1} \times \ldots \times M_{s_n}$.

Let us note that for every $T \subseteq Sen$, $Triv \subseteq Mod(T)$.

From the above notations, we obviously have:

$$Cn(T) = Cn(T') \Leftrightarrow Mod(T) = Mod(T'). \quad (1)$$

The two functions $Mod(\_)$ from $\mathcal{P}(Sen)$ into $\mathcal{P}(Mod)$ and $\_^*$ from $\mathcal{P}(Mod)$ into $\mathcal{P}(Sen)$ form what is known as a Galois connection in that they satisfy the following properties: for all $T, T' \subseteq Sen$ and $\mathbb{M}, \mathbb{M}' \subseteq Mod$, we have (see [13] and the proof of Proposition 1 below)

(1) $T \subseteq T' \Longrightarrow Mod(T') \subseteq Mod(T)$
(2) $\mathbb{M} \subseteq \mathbb{M}' \Longrightarrow \mathbb{M}'^* \subseteq \mathbb{M}^*$
(3) $T \subseteq Mod(T)^*$
(4) $\mathbb{M} \subseteq Mod(\mathbb{M}^*)$

Definition 2 *(Knowledge base and theory).* A **knowledge base** T is a set of sentences (i.e. $T \subseteq Sen$). A knowledge base T is said to be a **theory** if and only if $T = Cn(T)$.

A theory T is **finitely representable** if there exists a finite set $T' \subseteq Sen$ such that $T = Cn(T')$.

Proposition 1. *For every satisfaction system $\mathcal{R}$, we have:*

Inclusion $\forall T \subseteq Sen, T \subseteq Cn(T)$*;*
Iteration $\forall T \subseteq Sen, Cn(T) = Cn(Cn(T))$*;*
Monotonicity $\forall T, T' \subseteq Sen, T \subseteq T' \Longrightarrow Cn(T) \subseteq Cn(T')$.

Proof. For the sake of completeness, let us first show that Mod is decreasing (Property 1): let us assume $T \subseteq T'$, then $\forall \mathcal{M} \in Mod(T')$ we have $\forall \varphi \in T, \varphi \in T'$, and thus $\mathcal{M} \models \varphi$. Hence $\mathcal{M} \in Mod(T)$.

Let us now show that Cn is increasing (monotonicity property): let us assume $T \subseteq T'$, then $\forall \varphi \in Cn(T)$ we have $\forall \mathcal{M} \in Mod(T'), \mathcal{M} \in Mod(T)$ since Mod is decreasing, and $\mathcal{M} \models \varphi$. Hence $\varphi \in Cn(T')$.

We have $T \subseteq Mod(T)^*$ (Property 3): indeed, $\forall \varphi \in T$ we have $\forall \mathcal{M} \in Mod(T), \mathcal{M} \models \varphi$ by definition of $Mod(T)$. Hence $\varphi \in Mod(T)^*$.

It is then easy to see that Cn is extensive (inclusion property) from the previous property and $Cn(T) = Mod(T)^*$.

Let us finally show that Cn is idempotent (iteration property): extensivity implies $\forall T, Cn(T) \subseteq Cn(Cn(T))$. Since $T \subseteq Mod(T)^*$ and Cn is increasing, we have $Cn(T) \subseteq Cn(Mod(T)^*) = Cn(Cn(T))$. □

Hence, satisfaction systems are *Tarskian* according to the definition of logics given by Tarski: a logic is a pair $(\mathcal{L}, Cn)$ where $\mathcal{L}$ is a set of expressions (formulas) and $Cn : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a mapping that satisfies the inclusion, iteration and monotonicity properties [40]. Indeed, from any satisfaction system $\mathcal{R}$ we can define the following Tarskian logic $(\mathcal{L}, Cn)$ where $\mathcal{L} = Sen$ and Cn is the mapping that associates to every $T \subseteq Sen$, the set $Cn(T)$ of semantic consequences of T.

Classically, the consistency of a theory T is defined as $Mod(T) \neq \emptyset$. The problem of such a definition of consistency is that its significance depends on the considered logic. Hence, this consistency is significant for **FOL**, while in **FHCL** it is a trivial property since each set of sentences is consistent because $Mod(T)$ always contains $Triv$ which is non-empty. Here, for the notion of consistency to be more appropriate for our purpose of defining revision for the largest family of logics, we propose a more general definition of consistency, the meaning of which is that there is at least a sentence which is not a semantic consequence.

Definition 3 *(Consistency).* $T \subseteq Sen$ is **consistent** if $Cn(T) \neq Sen$.

Proposition 2. *For every $T \subseteq Sen$, T is consistent if and only if $Mod(T) \setminus Triv \neq \emptyset$.*

Proof. Let us prove that $Cn(T) = Sen$ iff $Mod(T) \setminus Triv = \emptyset$. Let us first assume that $Mod(T) \setminus Triv = \emptyset$. Therefore, this means that the only models that satisfy T are $\mathcal{M}$ such that $\mathcal{M}^* = Sen$ (if they exist). Hence, we have $Cn(T) = Mod(T)^* = Sen$.

Conversely, let us assume that $Cn(T) = Sen$. This means that every model $\mathcal{M}$ such that $\mathcal{M}^* \neq Sen$ does not belong to $Mod(T)$, and $Mod(T) \setminus Triv = \emptyset$. □

Corollary 1. *For every $T \subseteq Sen$, T is inconsistent is equivalent to $Mod(T) = Triv$.*

3. AGM postulates for revision in satisfaction systems

3.1. AGM postulates and weakened AGM postulates

The AGM postulates for knowledge base revision in satisfaction systems are easily adaptable. We build upon the model-theoretic characterization introduced by Katsuno and Mendelzon (KM) [22] for propositional logic. Note, however, that in propositional logic, a belief base can be represented by a formula, and then the KM postulates exploit this property. This is no longer the case in our context, but we argue that the postulates are still appropriate, except the one on syntax independence, as discussed next. Given two knowledge bases $T, T' \subseteq Sen$, $T \circ T'$ denotes the **revision of** T **by** T', that is, $T \circ T'$ is obtained by adding consistently new knowledge T' to the old knowledge base T. Note that $T \circ T'$ cannot be defined as $T \cup T'$ because nothing ensures that $T \cup T'$ is consistent. The revision operator has then to minimally change T so that $T \circ T'$ is consistent. This is what the AGM postulates ensure.

Here we use the following weakened AGM postulates[3]:

(G1) If T' is consistent, then so is $T \circ T'$.
(G2) $Mod(T \circ T') \subseteq Mod(T')$.
(G3) if $T \cup T'$ is consistent, then $T \circ T' = T \cup T'$.
(G5) $Mod((T \circ T') \cup T'') \subseteq Mod(T \circ (T' \cup T''))$.
(G6) if $(T \circ T') \cup T''$ is consistent, then $Mod(T \circ (T' \cup T'')) \subseteq Mod((T \circ T') \cup T'')$.

In the literature such as in [22,30], an additional postulate concerns the independence of the syntax:

(**G4**) If $Cn(T_1) = Cn(T'_1)$ and $Cn(T_2) = Cn(T'_2)$, then $Mod(T_1 \circ T_2) = Mod(T'_1 \circ T'_2)$.

This postulate states a complete independence of the syntactical forms of both the original knowledge base and the newly acquired knowledge. The problem with Postulate (G4) is that it is almost never satisfied when we want to preserve the structure of knowledge bases and then apply revision operators over the formulas that compose knowledge bases. Indeed, let us consider in the logic **PL** the following knowledge bases $T_1 = \{p, q\}$ and $T_2 = \{q \Rightarrow p, q\}$ over the signature $\{p, q\}$. Obviously, we have that $Mod(T_1) = Mod(T_2) = \{\nu : p \mapsto 1, q \mapsto 1\}$. Let us consider the knowledge base $T' = \{\neg q\}$. We have now that $T_1 \cup T'$ (and then $T_2 \cup T'$) is inconsistent. A way to retrieve the consistency is to replace in T_1 and T_2 the atomic formula q by $\neg q$. Hence, $T_1 \circ T' = \{p, \neg q\}$ and $T_2 \circ T' = \{q \Rightarrow p, \neg q\}$. Then $Mod(T_1 \circ T') = \{\nu : p \mapsto 1, q \mapsto 0\}$, $Mod(T_2 \circ T') = \{\nu : p \mapsto 1, q \mapsto 0; \nu' : p \mapsto 0, q \mapsto 0\}$, and $Mod(T_1 \circ T') \neq Mod(T_2 \circ T')$. This example shows that syntax independence may be too strong a requirement.

In [22], the authors bypass the problem by representing any knowledge base K (which is a theory in [22]) by a propositional formula ψ such that $K = Cn(\psi)$. Hence, they apply their revision operator on ψ and not on K, and so they lose the structure of the knowledge base K.

A weaker form of this postulate could be written as:

(**G'4**) If $Cn(T'_1) = Cn(T'_2)$, then $Mod(T \circ T'_1) = Mod(T \circ T'_2)$,

which ensures a partial independence of the syntax, only on the new knowledge. Remarkably, this weaker form can be derived from the other postulates (as expressed in Proposition 3), and is hence not used in the subsequent proofs (see e.g. Theorem 1 below).

Proposition 3. *Postulates (G1)–(G3), (G5) and (G6) imply Postulate (G'4).*

Proof. See Appendix. □

Based on this result, the only weakened AGM postulates (G1)–(G3), (G5) and (G6) are considered next.

[3] The numbering is kept consistent with the ones in previous works.

3.2. Faithful assignment and weakened AGM postulates

Intuitively, any revision operator $\circ$ satisfying the weakened AGM postulates above induces minimal change, that is the models of $T \circ T'$ are the models of T that are the closest to models of T', according to some distance for measuring how close are models. This is what is now shown in this section by establishing a correspondence between the weakened AGM postulates and binary relations over models with minimality conditions.

Let $\mathbb{M} \subseteq Mod$ and $\preceq$ be a binary relation over $\mathbb{M}$. We define $\prec$ as $\mathcal{M} \prec \mathcal{M}'$ if and only if $\mathcal{M} \preceq \mathcal{M}'$ and $\mathcal{M}' \not\preceq \mathcal{M}$. We also define $Min(\mathbb{M}, \preceq) = \{\mathcal{M} \in \mathbb{M} \mid \forall \mathcal{M}' \in \mathbb{M}, \mathcal{M}' \not\prec \mathcal{M}\}$.

Definition 4 *(Faithful assignment).* An **assignment** is a mapping that assigns to each knowledge base T a binary relation $\preceq_T$ over Mod. We say that this assignment is **faithful (FA)** if the following two conditions are satisfied:

(1) if $\mathcal{M}, \mathcal{M}' \in Mod(T)$, $\mathcal{M} \not\prec_T \mathcal{M}'$.
(2) for every $\mathcal{M} \in Mod(T)$ and every $\mathcal{M}' \in Mod \setminus Mod(T)$, $\mathcal{M} \prec_T \mathcal{M}'$.

A binary relation $\preceq_T$ assigned to a knowledge base T by a faithful assignment will be also said **faithful**.

This definition of FA differs from the one originally given in [22] on two points:

(1) In [22], a third condition is stated:

$$\forall T, T' \subseteq Sen, Mod(T) = Mod(T') \Rightarrow \preceq_T = \preceq_{T'} .$$

As for (G4), this condition expresses a syntactical independence.
(2) It is not required for $\preceq_T$ to be a pre-order. As shown below, the only important feature to have to make a correspondence between a FA and the fact that $\circ$ satisfies the weakened AGM Postulates is that there is a minimal model for $\preceq_T$ in $Mod(T')$ as expressed by Theorem 1.

Theorem 1. *Let $\circ$ be a revision operator. The operator $\circ$ satisfies the weakened AGM Postulates (as defined in Section 3.1) if and only if there exists a FA that maps each knowledge base $T \subseteq Sen$ to a binary relation $\preceq_T$ such that for every knowledge base $T' \subseteq Sen$:*

- $Mod(T \circ T') \setminus Triv = Min(Mod(T') \setminus Triv, \preceq_T)$;
- *if T' is consistent, then* $Min(Mod(T') \setminus Triv, \preceq_T) \neq \emptyset$;
- *for every $T'' \subseteq Sen$, if $(T \circ T') \cup T''$ is consistent, then* $Min(Mod(T') \setminus Triv, \preceq_T) \cap Mod(T'') = Min(Mod(T' \cup T'') \setminus Triv, \preceq_T)$.

Proof. See Appendix. □

Note that if T' is inconsistent, then so is $T \circ T'$, and we can set arbitrarily $T \circ T' = T'$, which corresponds to a cautious revision. The case where T is inconsistent is not considered in this paper (and is usually excluded from the scope of revision procedures), since in that case other operators could be more relevant than revision, in particular debugging methods (see e.g. [36] for debugging of terminologies, or [32] for base revision for ontology debugging, both in description logics).

Given a revision operator $\circ$ satisfying the weakened AGM postulates, any FA satisfying the supplementary conditions of Theorem 1 will be called FA+. To a revision operator $\circ$ satisfying the weakened AGM postulates, we can associate many FA+. An example of such a FA+ was given in the proof of Theorem 1. Another example is the mapping f that associates to every $T \subseteq Sen$ the binary relation $\preceq_T$ defined as follows:

Given $T' \subseteq Sen$, let us start by defining $\preceq_T^{T'} \subseteq Mod(T') \times Mod(T')$ as:

$$\mathcal{M} \preceq_T^{T'} \mathcal{M}' \Longleftrightarrow \mathcal{M} \in Mod(T \circ T') \text{ and } \mathcal{M}' \notin Mod(T \circ T').$$

Let us then set $f(T) = \preceq_T = \bigcup_{T'} \preceq_T^{T'}$ (i.e. $\mathcal{M} \preceq_T \mathcal{M}' \Leftrightarrow \exists T', \mathcal{M} \preceq_T^{T'} \mathcal{M}'$).

Theorem 2. *If $\circ$ satisfies the weakened AGM postulates, then the mapping f defined above is a FA+.*

Proof. See Appendix. □

Actually, the set of FA+ associated with a revision operator satisfying the weakened AGM postulates has a lattice structure, as shown by the following definition and propositions.

Definition 5. Let f_1, f_2 be two FA. Let us denote $f_1 \sqcup f_2$ (resp. $f_1 \sqcap f_2$) the mapping that assigns to each knowledge base $T \subseteq Sen$ the binary relation $\preceq_T = \preceq_T^1 \cup \preceq_T^2$ (resp. $\preceq_T = \preceq_T^1 \cap \preceq_T^2$) where $f_i(T) = \preceq_T^i$ for $i = 1, 2$.

Proposition 4. *If f_1 and f_2 are FA+ for a same revision operator $\circ$, then so are $f_1 \sqcup f_2$ and $f_1 \sqcap f_2$.*

Proof. See Appendix. □

Proposition 5. *The relation $\leq$ defined on FA+ by:*

$$f \leq g \iff \forall T \subseteq Sen,\ f(T) \subseteq g(T)$$

is a partial ordering.

Given a revision operator $\circ$ which satisfies the weakened AGM postulates, the poset (FA+($\circ$), $\leq$) of FA+ associated with $\circ$ is a lattice. For any $f, g \in$ FA+($\circ$), $f \sqcup g$ (respectively $f \sqcap g$) is the least upper bound (respectively the greatest lower bound) of $\{f, g\}$. The lattice (FA+($\circ$), $\leq$) is further complete.

Proof. The fact that the relation $\leq$ actually defines a partial order is straightforward. The fact that $f \sqcup g$ and $f \sqcap g$ are the least upper bound and greatest lower bound of $\{f, g\}$ is also easy to show.

Given a subset $S \subseteq$ FA+($\circ$), its least upper bound is the mapping $\sqcup S : T \mapsto \bigcup_{f \in S} f(T)$, and its greatest lower bound is the mapping $\sqcap S : T \mapsto \bigcap_{f \in S} f(T)$. By extending the proof of Proposition 4, it is easy to show that $\sqcup S$ and $\sqcap S$ are FA+. □

3.3. Relaxation and AGM postulates

Relaxations have been introduced in [14,15] in the framework of description logics with the aim of defining dissimilarity between concepts. Here, we propose to generalize this notion in the framework of satisfaction systems.

Definition 6 *(Relaxation).* A **relaxation** is a mapping $\rho : Sen \to Sen$ satisfying the following properties:

Extensivity $\forall \varphi \in Sen, Mod(\varphi) \subseteq Mod(\rho(\varphi))$.
Exhaustivity $\exists k \in \mathbb{N}, Mod(\rho^k(\varphi)) = Mod$, where ρ^0 is the identity mapping, and for all $k > 0$, $\rho^k(\varphi) = \rho(\rho^{k-1}(\varphi))$.

Let us observe that relaxations exist if and only if the underlying satisfaction system $(Sen, Mod, \models)$ has tautologies (i.e. formulas $\varphi \in Sen$ such that $Mod(\varphi) = Mod$). Indeed, when the satisfaction system has tautologies, we can define the trivial relaxation $\rho : \varphi \mapsto \psi$ where ψ is any tautology.[4] Conversely, all relaxations imply that the underlying satisfaction system has tautologies to satisfy the exhaustivity condition.

The interest of relaxations is that they give rise to revision operators which have demonstrated their usefulness in practice (see Section 4).

Notation 2. Let $T \subseteq Sen$ be a knowledge base. Let $\mathcal{K} = \{k_\varphi \in \mathbb{N} \mid \varphi \in T\}$, and $\mathcal{K}' = \{k'_\varphi \in \mathbb{N} \mid \varphi \in T\}$. Let us note:

- $\rho^{\mathcal{K}}(T) = \{\rho^{k_\varphi}(\varphi) \mid k_\varphi \in \mathcal{K}, \varphi \in T\}$,
- $\sum \mathcal{K} = \sum_{k_\varphi \in \mathcal{K}} k_\varphi$,
- $\mathcal{K} \leq \mathcal{K}'$ when for every $\varphi \in T$, $k_\varphi \leq k'_\varphi$,
- $\mathcal{K} < \mathcal{K}'$ if $\mathcal{K} \leq \mathcal{K}'$ and $\exists \varphi \in T$, $k_\varphi < k'_\varphi$.

In this notation, k_φ is a number associated with each formula φ of the knowledge base (equivalently it can be considered as a function of φ taking values in $\mathbb{N}$), which intuitively represents the degree to which φ is relaxed.

Definition 7 *(Revision based on relaxation).* Let ρ be a relaxation. A **revision operator over** ρ is a mapping $\circ : \mathcal{P}(Sen) \times \mathcal{P}(Sen) \to \mathcal{P}(Sen)$ satisfying for every $T, T' \subseteq Sen$:

$$T \circ T' = \begin{cases} \rho^{\mathcal{K}}(T) \cup T' & \text{if } T' \text{ is consistent} \\ T' & \text{otherwise} \end{cases}$$

for some $\mathcal{K} = \{k_\varphi \in \mathbb{N} \mid \varphi \in T\}$ such that:

(1) if T' is consistent, then $T \circ T'$ is consistent;
(2) for every $\mathcal{K}'$ such that $\rho^{\mathcal{K}'}(T) \cup T'$ is consistent, $\sum \mathcal{K} \leq \sum \mathcal{K}'$ (minimality on the number of applications of the relaxation);
(3) for every T'' such that $Mod(T') \subseteq Mod(T'')$, if $T \circ T'' = \rho^{\mathcal{K}'}(T) \cup T''$, then $\mathcal{K}' \leq \mathcal{K}$.

[4] Note that most systems have tautologies. An example without tautology would be a non-complete logic where the only connective is $\vee$.

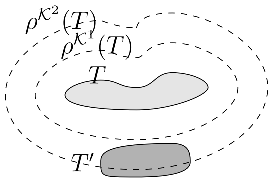

Fig. 1. Successive relaxations of T until it becomes consistent with T'.

Revision based on relaxation is illustrated in Fig. 1 where theories are represented as sets of their models. Intermediate steps to define the revision operators are then the definitions of formula and theory relaxations.

It is important to note that given a relaxation ρ, several revision operators can be defined. Without Condition 3 of Definition 7, we could accept revision operators $\circ$ that do not satisfy Postulates (G5) and (G6). Hence, Condition 3 allows us to exclude such operators. To illustrate this, let us consider in **FOL** the satisfaction system $\mathcal{R} = (Sen, Mod, \models)$ over the signature (S, F, P) where $S = \{s\}$, $F = \emptyset$ and $P = \{=: s \times s\}$. Let us consider $T, T' \subseteq Sen$ such that:

$$T = \left\{ \begin{array}{l} \exists x.\exists y.(\neg x = y) \wedge \forall z(z = x \vee z = y) \\ \exists x.\exists y.\exists z.(\neg x = y \wedge \neg y = z \wedge \neg x = z) \wedge \\ \qquad\qquad \forall w(w = x \vee w = y \vee w = z) \end{array} \right\}$$

$$T' = \left\{ \begin{array}{l} \forall x.x = x \\ \forall x.\forall y.x = y \Rightarrow y = x \\ \forall x.\forall y.\forall z.x = y \wedge y = z \Rightarrow x = z \end{array} \right\}$$

Obviously, T' is consistent. As T does not contain the axioms for equality, it is also consistent. Indeed, the model $\mathcal{M}$ with its associated set $M_s = \{0, 1, 2\}$ and the binary relation $=^{\mathcal{M}} \subseteq M_s \times M_s$, defined by the following set $\{(0, 0), (1, 1), (2, 0)\}$, satisfies T.

But $T \cup T'$ is not consistent. The reason is that when the meaning of $=$ is the equality, the first axiom of T can only be satisfied by models with two values while the second axiom is satisfied by models with three values. A way to retrieve the consistency is to remove one of the two axioms. This can be modeled by the relaxation ρ that maps each formula to a tautology.[5] But in this case, we have then two options depending on whether we remove and change the first or the second axiom by a tautology, which give rise to two revision operators $\circ_1$ and $\circ_2$. The first two conditions of Definition 7 are satisfied by both $\circ_1$ and $\circ_2$.

Now, let us take $T'' = \{\exists x.\exists y.\neg x = y\}$ which is satisfied, when added to the axioms in T', by any model with at least two elements. Hence, $(T \circ_1 T') \cup T''$ and $(T \circ_2 T') \cup T''$ are consistent. Without the third condition, nothing would prevent to define $T \circ_1 (T' \cup T'')$ (respectively $T \circ_2 (T' \cup T'')$) by removing and change in T the second (respectively the first) axiom by a tautology which would be a counter-example to Postulates (G5) and (G6). Actually, as shown by the result below, this third condition of Definition 7 entails Postulates (G5) and (G6), and then, by Proposition 3, entails Postulate (G'4).

However in some situations Condition 3 may be considered as too strong, forcing to relax more than what would be needed to satisfy only Condition 2. This could typically be the case when Condition 2 could be obtained in two different ways, for instance for $\mathcal{K}' = \{0, 1, 0, 0...\}$ or for $\mathcal{K}'' = \{1, 0, 0, 0...\}$. Then taking $Cn(T') = Cn(T'')$, and revising $T \circ T'$ using $\mathcal{K}'$ and $T \circ T''$ using $\mathcal{K}''$ would not meet Condition 3. To satisfy it, relaxation should be done for instance with $\mathcal{K} = \{1, 1, 0, 0...\}$. Therefore in concrete applications, we will have to find a compromise between Condition 3 and (G5)–(G6) at the price of potential larger relaxations on the one hand, and less relaxation but potentially the loss of (G5)–(G6) on the other hand.

Notation 3. In the context of Definition 7, let $T, T' \subseteq Sen$ be two knowledge bases. If $T \circ T' = \rho^{\mathcal{K}}(T) \cup T'$ with $\mathcal{K} = \{k_\varphi \in \mathbb{N} \mid \varphi \in T\}$, then we note $\mathcal{K}_T^{T'} = \mathcal{K}$.

Theorem 3. *Any revision operator $\circ$ based on a relaxation (Definition 7) satisfies the weakened AGM postulates.*

Proof. See Appendix. □

So far we showed that several FA+ can be associated with a given revision operator $\circ$ satisfying the weakened AGM postulates. Here, we define a particular one, which is more specific to revision operators based on relaxation. Let ρ be a relaxation and f_ρ be the mapping that associates to every $T \subseteq Sen$ the binary relation $\preceq_T$ defined as follows:

[5] We will see in Section 4.3 a less trivial but more interesting relaxation in **FOL** that consists in changing universal quantifiers into existential ones.

Given $T' \subseteq Sen$, let us start by defining $\preceq_T^{T'} \subseteq Mod(T') \times Mod(T')$ as:

$$\mathcal{M} \preceq_T^{T'} \mathcal{M}' \Longleftrightarrow \forall \mathcal{K}'' \geq \mathcal{K}_T^{T'}, \mathcal{M}' \in Mod(\rho^{\mathcal{K}''}(T)) \Rightarrow \exists \mathcal{K}' \geq \mathcal{K}_T^{T'}, \begin{cases} \mathcal{K}' < \mathcal{K}'' \text{ and} \\ \mathcal{M} \in Mod(\rho^{\mathcal{K}'}(T)) \end{cases}$$

Let us then set $\preceq_T = \bigcup_{T'} \preceq_T^{T'}$ (i.e. $\mathcal{M} \preceq_T \mathcal{M}' \Leftrightarrow \exists T', \mathcal{M} \preceq_T^{T'} \mathcal{M}'$). We have $\preceq_T \subseteq Mod \times Mod$ because $\preceq_T^{\emptyset} \subseteq \preceq_T$.

Intuitively, it means that T has to be relaxed more to be satisfied by $\mathcal{M}'$ than to be satisfied by $\mathcal{M}$.

Theorem 4. *For any revision operator $\circ$ based on a relaxation ρ as defined in Definition 7, the mapping f_ρ is a FA+.*

Proof. See Appendix. □

4. Applications

In this section, we illustrate our general approach by defining revision operators based on relaxations for the logics **PL**, **HCL**, and **FOL**. We further develop the case of DLs in Section 4.4, by defining several concrete relaxation operators for different fragments of the DL $\mathcal{ALC}$.

4.1. Revision in **PL**

Here, inspired by the work in [7,8] on Morpho-Logics, we define relaxations based on dilations from mathematical morphology [6]. In **PL**, knowing a formula is equivalent to knowing the set of its models, and we can identify any propositional formula φ with the set of its interpretations $Mod(\varphi)$. To define relaxations in **PL**, we will apply set-theoretic morphological operations. First, let us recall a basic definition of dilation in mathematical morphology [6]. Let X and B be two subsets of $\mathbb{R}^n$. The dilation of X by the structuring element B, denoted by $D_B(X)$, is defined as follows:

$$D_B(X) = \{x \in \mathbb{R}^n \mid B_x \cap X \neq \emptyset\}$$

where B_x denotes the translation of B at x. More generally, dilations in any space can be defined in a similar way by considering the structuring element as a binary relationship between elements of this space.[6]

In **PL**, this leads to the following dilation of a formula $\varphi \in Sen$:

$$Mod(D_B(\varphi)) = \{\nu \in Mod \mid B_\nu \cap Mod(\varphi) \neq \emptyset\}$$

where B_ν contains all the models that satisfy some relationship with ν. The relationship standardly used is based on a discrete distance δ between models, and the most commonly used is the Hamming distance d_H where $d_H(\nu, \nu')$ for two propositional models over a same signature is the number of propositional symbols that are instantiated differently in ν and ν'. From any distance δ between models, a distance from models to a formula is derived as follows: $d(\nu, \varphi) = min_{\nu' \models \varphi} \delta(\nu, \nu')$. In this case, we can rewrite the dilation of a formula as follows:

$$Mod(D_B(\varphi)) = \{\nu \in Mod(\Sigma) \mid d(\nu, \varphi) \leq 1\}$$

This consists in using the distance ball of radius 1 as structuring element B. To ensure the exhaustivity condition to our relaxation, we need to add a condition on distances, the *betweenness property* [14].

Definition 8 *(Betweenness property).* Let δ be a discrete distance over a set S. δ has the **betweenness property** if for all x, y in S and all k in $\{0, 1, \ldots, \delta(x, y)\}$, there exists z in S such that $\delta(x, z) = k$ and $\delta(z, y) = \delta(x, y) - k$.

The Hamming distance trivially satisfies the betweenness property. The interest for our purpose of this property is that it allows from any model to reach any other one, and then ensuring the exhaustivity property of relaxation.[7]

Proposition 6. *Let D_B be a dilation applied to formulas $\varphi \in Sen$ for a finite signature, and based on a distance between models that satisfies the betweenness property. Such a dilation D_B is a relaxation.*

[6] Definitions based on the notion of structuring elements are all particular cases of more general algebraic dilations, defined as operators between lattices, which commute with the supremum.

[7] Hence, a dilation of formulas could also be defined by using a distance ball of radius n as structuring element [7].

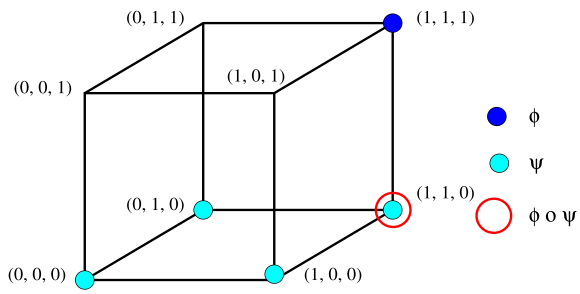

Fig. 2. A simple example of revision based on dilation in **PL** (see text). (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

Proof. It is extensive. Indeed, for every φ and for every model $\nu \in Mod(\varphi)$, we have that $d(\nu, \varphi) = 0$, and then $\varphi \models D_B(\varphi)$. Exhaustivity results from the fact that the considered signature is a finite set and from the betweenness property. □

Using Definition 7, this relaxation allows defining revision operators that include the classical Dalal's revision as a particular case (see [7,8]).

A simple example is illustrated in Fig. 2. Three propositional symbols a, b and c are considered. The set of models is represented by the vertices of a cube, and we assimilate a formula formed by a simple conjunction of symbols with its corresponding model. For instance $a \wedge b \wedge c$ is assimilated to the corresponding world, represented by the point $(1, 1, 1)$ in the cube. The edges link two worlds differing by one instantiation of a propositional symbol, i.e. at a distance 1 for the Hamming distance. For instance vertices representing $a \wedge b \wedge c$ and $\neg a \wedge b \wedge c$ are linked by an edge (we have $d_H(a \wedge b \wedge c, \neg a \wedge b \wedge c) = 1$). Colored dots define φ and ψ: $\varphi = a \wedge b \wedge c$ and $\psi = \neg c$. The red circle represents the result of the revision $\varphi \circ \psi = a \wedge b \wedge \neg c$. Indeed, φ and ψ are inconsistent, hence we relax φ by a dilation of size 1 according to the Hamming distance, leading to $D_B(\varphi) = (a \wedge b \wedge c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c)$, which is now consistent with φ and the conjunction provides the revision. The result here simply amounts to change the old belief which included c, by negating this atom according to the new knowledge expressed by ψ.

4.2. *Revision in* **HCL**

Many works have focused on belief revision involving propositional Horn formulas (cf. [12] to have an overview on these works). Here, we propose to extend relaxations that we have defined in the framework of **PL** to deal with the Horn fragment of propositional theories.

Definition 9 *(Model intersection).* Given a propositional signature Σ and two Σ-models $\nu, \nu' : \Sigma \to \{0, 1\}$, we note $\nu \cap \nu' : \Sigma \to \{0, 1\}$ the Σ-model defined by:

$$p \mapsto \begin{cases} 1 & \text{if } \nu(p) = \nu'(p) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Given a set of Σ-models $\mathcal{S}$, we note

$$cl_\cap(\mathcal{S}) = \mathcal{S} \cup \{\nu \cap \nu' \mid \nu, \nu' \in \mathcal{S}\}$$

$cl_\cap(\mathcal{S})$ is then the closure of $\mathcal{S}$ under intersection of positive atoms.

For any set $\mathcal{S}$ closed under intersection of positive atoms, there exists a Horn sentence φ that defines $\mathcal{S}$ (i.e. $Mod(\varphi) = \mathcal{S}$). Given a distance δ between models, we then define a relaxation ρ as follows: for every Horn formula φ, $\rho(\varphi)$ is any Horn formula φ' such that $Mod(\varphi') = cl_\cap(Mod(D_B(\varphi))$ (by the previous property, we know that such a formula φ' exists).

Proposition 7. *With the same conditions as in Proposition 6, the mapping ρ is a relaxation.*

Then a revision operator can be defined from ρ according to Definition 7.

4.3. *Revision in* **FOL**

A trivial way to define a relaxation in **FOL** is to map any formula to a tautology. A less trivial and more interesting relaxation is to change universal quantifiers to existential ones. Indeed, given a formula φ of the form $\forall x.\psi$, if φ is not consistent with a given theory T, $\exists x.\psi$ may be consistent with T (it is quite intuitive that if it cannot be consistent for all values, it can be for some of them). A similar approach has been adopted for defining merging operators using dilations

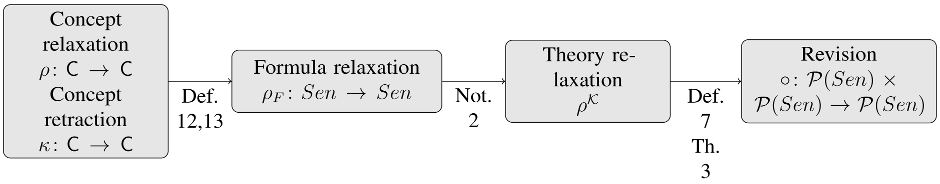

Fig. 3. From concept relaxation and retraction to revision operators in DL.

in **FOL** in [20]. In the following we suppose that given a signature, every formula φ in Sen is a disjunction of formulas in prenex form (i.e. φ is of the form $\bigvee_j Q_1^j x_1^j \ldots Q_{n_j}^j x_{n_j}^j.\psi_j$ where each Q_i^j is in $\{\forall, \exists\}$). Let us define the relaxation ρ as follows, for a tautology τ:

- $\rho(\tau) = \tau$;
- $\rho(\exists_1 x_1 \ldots \exists_n x_n.\varphi) = \tau$;
- Let $\varphi = Q_1 x_1 \ldots Q_n x_n.\psi$ be a formula such that the set $E_\varphi = \{i, 1 \leq i \leq n \mid Q_i = \forall\} \neq \emptyset$. Then, $\rho(Q_1 x_1 \ldots Q_n x_n.\varphi) = \bigvee_{i \in E_\varphi} \varphi_i$ where $\varphi_i = Q'_1 x_1 \ldots Q'_n x_n.\psi$ such that for every $j \neq i$, $1 \leq j \leq n$, $Q'_j = Q_j$ and $Q'_i = \exists$;
- $\rho(\bigvee_j Q_1^j x_1^j \ldots Q_{n_j}^j x_{n_j}^j.\psi) = \bigvee_j \rho(Q_1^j x_1^j \ldots Q_{n_j}^j x_{n_j}^j.\psi)$.

Proposition 8. *ρ is a relaxation.*

Proof. It is obviously extensive, and exhaustivity results from the fact that in a finite number of steps, we always reach the tautology τ. □

Again a revision operator can then be defined from ρ using Definition 7.

4.4. Revision in DL

4.4.1. General construction scheme

The instantiation of our abstract framework to DLs follows the scheme depicted in Fig. 3.

The necessary ingredient is the specialization of formulas relaxations as abstractly defined in Definition 6. To this end, we propose to define a formula relaxation in two ways (other definitions may also exist). For sentences of the form $C \sqsubseteq D$, the first proposed approach consists in relaxing the set of interpretations of D, while the second one amounts to "retracting" the set of interpretations of C. We give hereafter formal definitions of these notions of concept relaxation and retraction.

Definition 10 *(Concept relaxation).* Given a signature (N_C, N_R, I), we note C the set of concepts over this signature. A **concept relaxation** is an operator $\rho : \mathsf{C} \to \mathsf{C}$ that satisfies, in every model, the following properties for all C in C:

(1) ρ is *extensive*, i.e. $C \sqsubseteq \rho(C)$
(2) ρ is *exhaustive*, i.e. $\exists k \in \mathbb{N}, \top \sqsubseteq \rho^k(C)$

A similar notion of concept relaxation has first been introduced in [14,15] but with an additional constraint of non-decreasingness property that we do not need in this work.

A trivial concept relaxation is the operation $\rho_\top$ that maps every concept C to $\top$. Other non-trivial concrete concept relaxations will be discussed in the sequel.

Definition 11 *(Concept retraction).* A ***(concept) retraction*** is an operator $\kappa : \mathsf{C} \to \mathsf{C}$ that satisfies, in every model, the following properties for all C in C:

(1) κ is *anti-extensive*, i.e. $\kappa(C) \sqsubseteq C$, and
(2) κ is *exhaustive*, i.e. $\forall D \in \mathsf{C}, \exists k \in \mathbb{N}$ such that $\kappa^k(C) \sqsubseteq D$.

Note that in this definition, D could be replaced equivalently by $\bot$.

With these definitions at hand, formulas relaxation can be defined as follows, using either concept relaxation (Definition 10) or concept retraction (Definition 11). We suppose that any signature (N_C, N_R, I) always contains in N_R a relation name $r_\top$ the meaning of which is, in any model $\mathcal{O}$, $r_\top^{\mathcal{O}} = \Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$.

Definition 12 *(Formula relaxation based on concept relaxation).* Let ρ a concept relaxation as in Definition 10. A ***formula relaxation based on*** ρ, denoted ${\rho_F}^\rho$ is defined as follows, for any two complex concepts C and D, any individuals a, b, and any role r:

$$\rho_F^\rho(C \sqsubseteq D) \equiv C \sqsubseteq \rho(D),$$
$$\rho_F^\rho(a : C) \equiv a : \rho(C),$$
$$\rho_F^\rho(\langle a, b\rangle : r)) \equiv \langle a, b\rangle : r_\top.$$

Note that the relaxation of the role assertion axiom amounts to delete it from the knowledge base, since a tautology is satisfied by any model.

Proposition 9. *ρ_F^ρ is a formula relaxation in the sense of Definition 6.*

Proof. It directly follows from the extensivity and exhaustivity of ρ. □

Definition 13 *(Formula relaxation based on concept retraction).* A ***formula relaxation based on a concept retraction*** κ, denoted ρ_F^κ, is defined as follows, for any two complex concepts C and D, any individuals a, b, and any role r:

$$\rho_F^\kappa(C \sqsubseteq D) \equiv \kappa(C) \sqsubseteq D,$$
$$\rho_F^\kappa(a : C) \equiv a : \top,$$
$$\rho_F^\kappa(\langle a, b\rangle : r)) \equiv \langle a, b\rangle : r_\top.$$

Similarly, the relaxation of the concept assertion amounts to delete it from the knowledge base.

A similar construction can be found in [29] for sentences of the form $(a : C)$.

Proposition 10. *ρ_F^κ is a formula relaxation in the sense of Definition 6.*

Proof. Extensivity and exhaustivity follow directly from the properties of κ. □

To complete the picture, it remains to define concrete concept relaxation and retraction operators for particular Description Logics families. We consider the logic $\mathcal{ALC}$, as defined in Section 2.1, and its fragments $\mathcal{EL}$ and $\mathcal{ELU}$. $\mathcal{EL}$-concept description constructors are existential restriction ($\exists$), conjunction ($\sqcap$), $\top$ and $\bot$, while $\mathcal{ELU}$-concept constructors are those of $\mathcal{EL}$ enriched with disjunction ($\sqcup$).

4.4.2. Relaxation and retraction in $\mathcal{EL}$

$\mathcal{EL}$-concept retractions. A trivial concept retraction is the operator $\kappa_\bot$ that maps every concept to $\bot$. Note that this operator is also particularly interesting for debugging ontologies expressed in $\mathcal{EL}$ [37]. Let us illustrate this operator for revision through the following example adapted from [29] to restrict the language to $\mathcal{EL}$.

Example 2. Let $T = \{\text{Tweety} \sqsubseteq \text{bird}, \text{bird} \sqsubseteq \text{flies}\}$ and $T' = \{\text{Tweety} \sqcap \text{flies} \sqsubseteq \bot\}$. Clearly $T \cup T'$ is inconsistent. The formula relaxation based on the retraction $\kappa_\bot$ amounts to apply $\kappa_\bot$ to the concept Tweety resulting in the following new knowledge base $\{\bot \sqsubseteq \text{bird}, \text{bird} \sqsubseteq \text{flies}\}$ which is now consistent with T'. An alternative solution is to retract the concept bird in $\text{bird} \sqsubseteq \text{flies}$ which results in the following knowledge base $\{\text{Tweety} \sqsubseteq \text{bird}, \bot \sqsubseteq \text{flies}\}$ which is also consistent with T'. The sets of minimal sum $\mathcal{K}_1$ and $\mathcal{K}_2$ in Condition 2 of Definition 7 are $\mathcal{K}_1 = \{1, 0\}$, (i.e. $k_{\varphi_1} = 1, k_{\varphi_2} = 0$, where $\varphi_1 = \text{Tweety} \sqsubseteq \text{bird}, \varphi_2 = \text{bird} \sqsubseteq \text{flies}$) and $\mathcal{K}_2 = \{0, 1\}$. However, Condition 3 of the same definition is not satisfied: let us take $T'' = T'$. Then a fortiori we have $Mod(T') \subseteq Mod(T'')$. We can then write $T \circ T' = \rho^{\mathcal{K}_1}(T) \cup T'$ and $T \circ T'' = \rho^{\mathcal{K}_2}(T) \cup T'' = \rho^{\mathcal{K}_2}(T) \cup T'$. But we do not have any ordering relation between $\mathcal{K}_1$ and $\mathcal{K}_2$. To ensure Condition 3, we must relax one more time the axioms in T leading to the following knowledge base $\{\bot \sqsubseteq \text{bird}, \bot \sqsubseteq \text{flies}\}$ (for $\mathcal{K} = \{1, 1\}$). The final revision then writes $T \circ T' = \{\bot \sqsubseteq \text{bird}, \bot \sqsubseteq \text{flies}, \text{Tweety} \sqcap \text{flies} \sqsubseteq \bot\}$. This revision satisfies the weakened AGM postulates but may appear too strong, and one may prefer one of the following solutions: $T \circ_1 T' = \{\bot \sqsubseteq \text{bird}, \text{bird} \sqsubseteq \text{flies}, \text{Tweety} \sqcap \text{flies} \sqsubseteq \bot\}$ or $T \circ_2 T' = \{\text{Tweety} \sqsubseteq \text{bird}, \bot \sqsubseteq \text{flies}, \text{Tweety} \sqcap \text{flies} \sqsubseteq \bot\}$ at the price of loosing (G5)–(G6).

Although the results are rather intuitive, one should note that it is pretty hard to figure out what each DL researcher would like to have as a result in such an example, and this enforces the interest of relying on an established theory such as AGM or its extension. In our work we propose operators enjoying a set of properties stemming from our adaptation of the AGM theory. Some of them can meet the requirement of a knowledge engineer, and some other may not completely, depending on the context, the ontology, etc.

$\mathcal{EL}$-concept relaxations. Dually, a trivial relaxation is the operator $\rho_\top$ that maps every concept to $\top$. Other non-trivial $\mathcal{EL}$-concept description relaxations have been introduced in [14]. We summarize here some of these operators.

$\mathcal{EL}$ concept descriptions can appropriately be represented as labeled trees, often called *$\mathcal{EL}$ description trees* [3]. An $\mathcal{EL}$ description tree is a tree whose nodes are labeled with sets of concept names and whose edges are labeled with role names. An $\mathcal{EL}$ concept description

$$C \equiv P_1 \sqcap \cdots \sqcap P_n \sqcap \exists r_1.C_1 \sqcap \cdots \sqcap \exists r_m.C_m, \tag{2}$$

with $P_i \in N_C \cup \{\top\}$, can be translated into a description tree by labeling the root node v_0 with $\{P_1, \ldots, P_n\}$, creating an r_j successor, and then proceeding inductively by expanding C_j for the r_j-successor node for all $j \in \{1, \ldots, m\}$.

An $\mathcal{EL}$-concept description relaxation then amounts to apply simple tree operations. Two relaxations can hence be defined [14]: (i) ρ_{depth} that reduces the role depth of each concept by 1, simply by pruning the description tree, and (ii) ρ_{leaves} that removes all leaves from a description tree.

4.4.3. Relaxations in $\mathcal{ELU}$

The relaxation defined above exploits the strong property that an $\mathcal{EL}$ concept description is isomorphic to a description tree. This is arguably not true for more expressive DLs. Let us try to go one step further in expressivity and consider the logic $\mathcal{ELU}$. Here we only propose some definitions of relaxations. Retractions could be designed similarly. A relaxation operator, as introduced in [14], requires a concept description to be in a special normal form, called normal form with grouping of existentials, defined recursively as follows.

Definition 14 *(Normal form with grouping of existential restrictions).* We say that an $\mathcal{EL}$-concept D is written in **normal form with grouping of existential restrictions** if it is of the form

$$D = \bigsqcap_{A \in N_D} A \sqcap \bigsqcap_{r \in N_R} D_r, \tag{3}$$

where $N_D \subseteq N_C$ is a set of concept names and the concepts D_r are of the form

$$D_r = \bigsqcap_{E \in \mathcal{C}_{D_r}} \exists r.E, \tag{4}$$

where no subsumption relation holds between two distinct conjuncts and $\mathcal{C}_{D_r}$ is a set of complex $\mathcal{EL}$-concepts that are themselves in normal form with grouping of existential restrictions.

The purpose of D_r terms is simply to group existential restrictions that share the same role name. For an $\mathcal{ELU}$-concept C we say that C is in *normal form* if it is of the form $(C \equiv C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k)$ and each of the C_i is an $\mathcal{EL}$-concept in normal form with grouping of existential restrictions.

Definition 15 *(Relaxation from normal form [14]).* Given an $\mathcal{ELU}$-concept description C we define an operator ρ_e recursively as follows.

- For $C = \top$ we define $\rho_e(C) = \top$.
- For $C = D_r$, where D_r is a group of existential restrictions as in Equation (4), we need to distinguish two cases:
  · if $D_r \equiv \exists r.\top$ we define $\rho_e(D_r) = \top$, and
  · if $D_r \not\equiv \exists r.\top$ then we define $\rho_e(D_r) = \bigsqcup_{\mathcal{S} \subseteq \mathcal{C}_{D_r}} \left(\bigsqcap_{E \notin \mathcal{S}} \exists r.E \sqcap \exists r.\rho_e \left(\bigsqcap_{F \in \mathcal{S}} F \right) \right)$.
  Note that in the latter case $\top \notin \mathcal{C}_{D_r}$ since D_r is in normal form.
- For $C = D$ as in Equation (3) we define $\rho_e(D) = \bigsqcup_{G \in \mathcal{C}_D} \left(\rho_e(G) \sqcap \bigsqcap_{H \in \mathcal{C}_D \setminus \{G\}} H \right)$, where $\mathcal{C}_D = N_D \cup \{D_r \mid r \in N_R\}$.
- Finally for $C = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ we set $\rho_e(C) = \rho_e(C_1) \sqcup \rho_e(C_2) \sqcup \cdots \sqcup \rho_e(C_k)$.

Proposition 11. *[14] ρ_e is a relaxation.*

Let us illustrate this operator with an example.

Example 3. Suppose an agent believes that a person Bob is married to a female judge: $T = \{\text{Bob} \sqsubseteq \text{male} \sqcap \exists.\text{MarriedTo}.(\text{female} \sqcap \text{judge})\}$. Suppose now that due to some obscurantist law, it happens that females are not allowed to be judges. This new belief is captured as $T' = \{\text{judge} \sqcap \text{female} \sqsubseteq \bot\}$. By applying ρ_e one can resolve the conflict between the two belief sets. To ease the reading, let us rewrite the concepts as follows: $A \equiv \text{male}$, $B \equiv \text{female}$, $C \equiv \text{judge}$, $m \equiv \text{MarriedTo}$, $D \equiv \exists\text{MarriedTo}.(\text{female} \sqcap \text{judge})$. Hence, from Definition 15 we have $\rho_e(A \sqcap D) \equiv (\rho_e(A) \sqcap D) \sqcup (A \sqcap \rho_e(D))$, with $\rho_e(A) \equiv \top$ and

$$\begin{aligned}\rho_e(D) &\equiv \exists m.\rho_e(B \sqcap C) \sqcup (\exists m.B \sqcap \exists m.\rho_e(C)) \sqcup (\exists m.\rho_e(B) \sqcap \exists m.C)\\ &\equiv \exists m.(B \sqcup C) \sqcup (\exists m.B \sqcap \exists m.\top) \sqcup (\exists m.\top \sqcap \exists m.C)\\ &\equiv \exists m.B \sqcup \exists m.C \sqcup \exists m.(B \sqcup C) \equiv \exists m.B \sqcup \exists m.C\end{aligned}$$

Then

$$\begin{aligned}\rho_e(A \sqcap D) &\equiv (\rho_e(A) \sqcap D) \sqcup (A \sqcap \rho_e(D))\\ &\equiv (\top \sqcap D) \sqcup (A \sqcap (\exists m.B \sqcup \exists m.C))\\ &\equiv D \sqcup (A \sqcap (\exists m.B \sqcup \exists m.C))\end{aligned}$$

The new agent's belief, up to a rewriting, becomes
$\{\text{Bob} \sqsubseteq \exists.\text{MarriedTo}.\,(\text{female} \sqcap \text{judge}) \sqcup (\text{male} \sqcap (\exists\text{Married}.\text{female} \sqcup \exists\text{Married}.\text{judge}))\,,\text{judge} \sqcap \text{female} \sqsubseteq \bot\}$.

One can notice from this example that the relaxation ρ_e leads to a refined revision operator. Indeed, the resulting relaxed axiom in T emphasizes all the minimal possible changes (through the disjunction operator) on Bob's condition. This is due to the fact that the relaxation operator ρ_e corresponds to dilating the set of models of a ball defined from an edit distance on the concept description tree of size one. For more details on the correspondence between this relaxation operator, the set of models and tree edit distances, one can refer to [14].

Another possibility for defining a relaxation in $\mathcal{ELU}$ is obtained by exploiting the disjunction constructor by augmenting a concept description with a set of exceptions.

Definition 16 *(Relaxation from exceptions in $\mathcal{ELU}$).* Given a set of exceptions $\mathcal{E} = \{E_1, \cdots, E_n\}$, we define a relaxation of degree k of an $\mathcal{ELU}$-concept description C as follows: for a finite set $\mathcal{E}^k \subseteq \mathcal{E}$ with $|\mathcal{E}^k| = k$, C is relaxed by adding the sets $E_{i_j} \in \mathcal{E}^k$ such that $E_{i_j} \sqcap C \sqsubseteq \bot$

$$\rho^k_{\mathcal{E}}(C) = C \sqcup E_{i_1} \sqcup \cdots \sqcup E_{i_k}.$$

Proposition 12. *$\rho^k_{\mathcal{E}}$ is extensive.*

Proof. Extensivity of this operator follows directly from the definition. □

However, exhaustivity is not necessarily satisfied unless the exception set includes the $\top$ concept, or the disjunction of some or all of its elements entails the $\top$ concept.

If we consider again Example 2, a relaxation of the formula bird $\sqsubseteq$ flies *using the operator $\rho^k_{\mathcal{E}}$ over the concept* flies *with the exception set $\mathcal{E} = \{\text{Tweety}\}$ results in the formula* bird $\sqsubseteq$ flies $\sqcup$ Tweety*. The new revised knowledge base, if Condition 3 in Definition 7 is not considered, is then* $\{\text{Tweety} \sqsubseteq \text{bird}, \text{bird} \sqsubseteq \text{flies} \sqcup \text{Tweety}, \text{Tweety} \sqcap \text{flies} \sqsubseteq \bot\}$ *which is consistent. This is obviously a more refined revision than the one obtained from the operator $\rho_\bot$, but requires the logic to be equipped with the disjunction connective and the definition of a set of exceptions.*

Another example involving this relaxation will be discussed in the $\mathcal{ALC}$ case (cf. Example 4).

4.4.4. Relaxation and retraction in $\mathcal{ALC}$

We consider here operators suited to $\mathcal{ALC}$ language. Of course, all the operators defined for $\mathcal{EL}$ and $\mathcal{ELU}$ remain valid.

$\mathcal{ALC}$-concept retractions. A first possibility for defining retraction is to remove iteratively from an $\mathcal{ALC}$-concept description one or a set of its subconcepts. A similar construction has been introduced in [29]. Interestingly enough, almost all the operators defined in [20,29] are relaxations.

Definition 17 *(Retraction from exceptions in $\mathcal{ALC}$).* Given a set of exceptions $\mathcal{E} = \{E_1, \cdots, E_n\}$, we retract any $\mathcal{ALC}$-concept description C by constraining it to the elements E^c_i such that $E_i \sqsubseteq C$:

$$\kappa^n_{\mathcal{E}}(C) = C \sqcap E^c_1 \sqcap \cdots \sqcap E^c_n.$$

Proposition 13. *$\kappa^n_{\mathcal{E}}$ is anti-extensive.*

Proof. The proof follows directly from the definition. □

As for its counterpart relaxation ($\rho^k_{\mathcal{E}}$), exhaustivity of $\kappa^n_{\mathcal{E}}$ is not necessarily satisfied unless the exception set includes the $\bot$ concept, or the conjunction of some or all of its elements entails the $\bot$ concept.

Consider again Example 2. We have $\kappa^1_{\mathcal{E}}(\text{BIRD}) = \text{BIRD} \sqcap \text{TWEETY}^c$. *The resulting revised knowledge base, if Condition 3 in Definition 7 is not considered, is then* $\{\text{TWEETY} \sqsubseteq \text{BIRD}, \text{BIRD} \sqcap \text{TWEETY}^c \sqsubseteq \text{FLIES}, \text{TWEETY} \sqcap \text{FLIES} \sqsubseteq \bot\}$ *which is consistent.*

Another possibility, suggested in [20] and related to operators defined in propositional logic as introduced in [7], consists in applying the retraction at the atomic level. This captures somehow Dalal's idea of revision operators in propositional logic [10].

Definition 18. Let C be an $\mathcal{ALC}$-concept description of the form $Q_1 r_1 \cdots Q_m r_m.D$, where Q_i is a quantifier and D is quantifier-free and in CNF form,[8] i.e. $D = E_1 \sqcap E_2 \sqcap \cdots E_n$ with E_i being disjunctions of possibly negated atomic concepts, i.e. $E_i = \sqcup_{k \in \Xi(i)} A_k$, where $\Xi(i)$ is the index set of the atomic (possibly negated) concepts A_k forming E_i. We define, as in the propositional case [7], $\kappa(E_i) = \sqcap_{k \in \Xi(i)} \sqcup_{j \in \Xi(i) \setminus \{k\}} A_j$ and $\kappa^n_p(D) = \sqcap_{i \in \{1 \dots n\}} \kappa(E_i)$. Then we set $\kappa_{\text{Dalal}}(C) = Q_1 r_1 \cdots Q_m r_m.\kappa_p(D)$.

Proposition 14. κ^n_{Dalal} *is a retraction.*

Proof. Exhaustivity and anti-extensivity follow from those of κ_p. Indeed the operator κ_p is exhaustive and anti-extensive, and if applied n times it reaches the $\bot$ concept (see [7] for properties of this operator). □

This idea can be generalized to consider any retraction defined in $\mathcal{ELU}$.

Definition 19. Let C be an $\mathcal{ALC}$-concept description of the form $Q_1 r_1 \cdots Q_m r_m.D$, where Q_i is a quantifier and D is quantifier-free.

Then we define $\kappa_{\cap}(C) = Q_1 r_1 \cdots Q_m r_m.\kappa^n_{\mathcal{E}}(D)$.

Proposition 15. $\kappa^n_{\cap}$ *is anti-extensive.*

Proof. The properties of this operator follows from the ones of $\kappa^n_{\mathcal{E}}(D)$. Hence, anti-extensivity is verified but not necessarily exhaustivity. □

Another possible $\mathcal{ALC}$-concept description retraction is obtained by substituting the existential restriction by an universal one. This idea has been sketched in [20] for defining dilation operators by transforming ∀ into ∃, i.e. special relaxation operators enjoying additional properties [14], and also used for defining revision in **FOL** (see Section 4.3). We adapt it here, by transforming ∃ into ∀, to define retraction in DL syntax.

Definition 20. Let C be an $\mathcal{ALC}$-concept description of the form $Q_1 r_1 \cdots Q_n r_n.D$, where Q_i is a quantifier, D is quantifier-free, then we define

$$\kappa_q(C) = \bigsqcap \{Q'_1 r_1 \cdots Q'_n r_n.D \mid \exists j \leq n \text{ s.t. } Q_j = \exists \text{ and } Q'_j = \forall, \text{ and for all } i \leq n \text{ s.t. } i \neq j, Q'_i = Q_i\}$$

Proposition 16. κ_q *is anti-extensive.*

Proof. See Appendix. □

Note that for κ_q exhaustivity can be obtained by further removing recursively the remaining universal quantifiers and apply at the final step any retraction defined above on the concept D.

$\mathcal{ALC}$-concept relaxations. Let us now introduce some relaxation operators suited to $\mathcal{ALC}$ language.

Definition 21. Let C be an $\mathcal{ALC}$-concept description of the form $Q_1 r_1 \cdots Q_m r_m.D$, where Q_i is a quantifier and D is quantifier-free and in DNF form, i.e. $D = E_1 \sqcup E_2 \sqcup \cdots E_n$ with E_i being a conjunction of possibly negated atomic concepts, i.e. $E_i = \sqcap_{k \in \Xi(i)} A_k$, where $\Xi(i)$ is the index set of the atomic (possibly negated) concepts A_k forming E_i. We define $\rho(E_i) = \sqcup_{k \in \Xi(i)} \sqcap_{j \in \Xi(i) \setminus \{k\}} A_j$ and $\rho^n_p(D) = \sqcup_{i \in \{1 \dots n\}} \rho(E_i)$, as in the propositional case [7], and then $\rho^n_{\text{Dalal}}(C) = Q_1 r_1 \cdots Q_m r_m.\rho^n_p(D)$.

As for retraction, this idea can be generalized to consider any relaxation defined in $\mathcal{ELU}$.

Definition 22. Let C be an $\mathcal{ALC}$-concept description of the form $Q_1 r_1 \cdots Q_n r_n.D$, where Q_i is a quantifier and D is quantifier-free, then we define $\rho^n_{\cup}(C) = Q_1 r_1 \cdots Q_n r_n.\rho^n_{\mathcal{E}}(D)$.

8 Any concept can indeed be written in this prenex form.

Let us consider another example adapted from the literature to illustrate these operators [29].

Example 4. Let us consider the following knowledge bases: $T = \{\textsc{Bob} \sqsubseteq \forall \textsc{hasChild}.\textsc{rich}, \textsc{Bob} \sqsubseteq \exists \textsc{hasChild}.\textsc{Mary}, \textsc{Mary} \sqsubseteq \textsc{rich}\}$ and $T' = \{\textsc{Bob} \sqsubseteq \textsc{hasChild}.\textsc{John}, \textsc{John} \sqsubseteq \textsc{rich}^c\}$ (we consider here individuals as concepts). Relaxing the formula $\textsc{Bob} \sqsubseteq \forall \textsc{hasChild}.\textsc{rich}$ by applying $\rho^n_\cup$ to the concept on the right hand side results in the following formula $\textsc{Bob} \sqsubseteq \forall \textsc{hasChild}.(\textsc{rich} \sqcup \textsc{John})$ which resolves the conflict between the two knowledge bases.

A last possibility, dual to the retraction operator given in Definition 20, consists in transforming universal quantifiers into existential ones (as done for relaxation in **FOL** in Section 4.3).

Definition 23. Let C be an $\mathcal{ALC}$-concept description of the form $Q_1 r_1 \cdots Q_n r_n.D$, where Q_i is a quantifier and D is quantifier-free, then we define a relaxation as:

$$\rho_q(C) = \bigsqcup \{Q'_1 r_1 \cdots Q'_n r_n.D \mid \exists j \leq n \text{ s.t. } Q_j = \forall \text{ and } Q'_j = \exists, \text{ and for all } i \leq n \text{ s.t. } i \neq j, Q'_i = Q_i\}$$

If we consider again Example 4, relaxing the formula $\textsc{Bob} \sqsubseteq \forall \textsc{hasChild}.\textsc{rich}$ by applying ρ_q to the concept on the right hand side results in the following formula $\textsc{Bob} \sqsubseteq \exists \textsc{hasChild}.\textsc{rich}$, which resolves the conflict between the two knowledge bases.

Proposition 17. *The operators ρ_{Dalal} and ρ_q are extensive and exhaustive. The operator $\rho_\cup$ is extensive but not exhaustive.*

Proof. The properties of ρ_{Dalal} and $\rho_\cup$ are directly derived from the definitions and from properties of ρ_p detailed in [7] and $\rho_{\mathcal{E}}$. The proof of ρ_q being extensive and exhaustive can be found in [20]. □

5. Related work

Recently a first generalization of AGM revision has been proposed in the framework of Tarskian logics considering minimality criteria on removed formulas [34] following previous works of the same authors for contraction [35]. Representation results that make a correspondence between a large family of logics containing non-classical logics such as **DL** and **HCL** and AGM postulates for revision with such minimality criteria have then been obtained. Here, the proposed generalization also gives similar representation theorems (cf. Theorem 1) but for a different minimality criterion. Indeed, we showed in Section 3.2 that revision operators satisfying the weakened AGM postulates are precisely the ones that accomplish an update with minimal change to the set of models of knowledge bases, generalizing the approach developed in [22] for the logic **PL** and [30] for **DL**. However, our revision operator based on relaxation also has a minimality criterion on transformed formulas. Indeed, a simple consequence of Definition 7 is the property

(Relevance) Let $T, T' \subseteq Sen$ be two knowledge bases such that $T \circ T' = \rho^{\mathcal{K}}(T) \cup T'$. Then, for every $\varphi \in T$ such that $k_\varphi \neq 0$, $\rho^{\mathcal{K}'}(T) \cup T'$ is inconsistent for $\mathcal{K}' = \mathcal{K} \setminus \{k_\varphi\} \cup \{k'_\varphi = 0\}$.

This property states that only formulas that contribute to inconsistencies with T' are allowed to be transformed. Our property **(Relevance)** is similar to the property with the same name in [34,35], but for contraction operators, and that states that only the formulas that somehow "contribute" to derive the formulas to abandon can be removed.

Since the primary aim of this paper is to show that a more general framework, encompassing different logics, can be useful, it is out of the scope of this paper to provide an overview of all existing relaxation methods. However, some works deserve to be mentioned, since they are based on ideas that show some similarity with the relaxation notion proposed in our framework.

The relaxation idea originates from the work on Morpho-Logics, initially introduced in [7,8]. In this seminal work, revision operators (and explanatory relations) were defined through dilation and erosion operators. These operators share some similarities with relaxation and retraction as defined in this paper. Dilation is a sup-preserving operator and erosion is inf-preserving, hence both are increasing. Some particular dilations and erosions are exhaustive and extensive while relaxation and retraction operators are defined to be exhaustive and extensive but not necessarily sup- and inf-preserving. Dilation has been further exploited for merging first-order theories in [20].

In [1], the notion of partial meet contraction is defined as the intersection of a non-empty family of maximal subsets of the theory that do not imply the proposition to be eliminated. Revision is then defined from the Levi identity. The maximal subsets can also be selected according to some choice function. The authors also define a notion of partial meet revision, which can be seen as a special case of the relaxation operator introduced in this paper. In [21], the author also discusses choice functions and compares the postulates for partial meet revision to the AGM postulates. He also highlights the distinction between belief sets (which can be very large) and belief bases (which are not necessarily closed by Cn). More precisely, A is a belief base of a belief set K iff $K = Cn(A)$. A permissive belief revision is defined in [9], based on the notion of weakening. The beliefs which are suppressed by classical revision methods are replaced by weaker forms, which keep the resulting belief set consistent. This notion of weakening is closed to the one of relaxation developed in this paper. In the last decade, several works have studied revision operators in description logics. While most of them concentrated on the adaptation of the AGM theory, few works have addressed the definition of concrete operators [25,27–29]. For instance, in [25], based on the seminal work in [5], revision in DL is studied by defining strategies to manage inconsistencies and using the notion of knowledge integration (see also the work by Hansson). The authors propose a conjunctive maxi-adjustment, for stratified knowledge bases and lexicographic entailment. In [28], weakening operators, that are in fact relaxation operators, are defined. Our work brings a principled formal flavor to these operators. In [27], revision of ontologies in DL is based on the notion of forgetting, which is also a way to manage inconsistencies. The authors propose a model based approach, inspired by Dalal's revision in **PL**, and based on a distance between terminologies and on the difference set between two interpretations. The models of the revision $T \circ T'$ are then the interpretations $\mathcal{I}$ for which there exists an interpretation $\mathcal{I}'$ such that the cardinality of the difference set between $\mathcal{I}$ and $\mathcal{I}'$ is equal to the distance between T and T'. In [24], updating Aboxes in DL is discussed, and some operators are introduced. The rationality of these operators is not discussed, hence the interest of a formal theory such as the AGM postulates. In [2] an original use of DL revision is introduced for the orchestration of processes. A closely related field is inconsistency handling in ontologies (e.g. [36,37]), with the main difference that the rationality of inconsistency repairing operators is not investigated, as suggested by the AGM theory.

As previously highlighted, some of our DL-based relaxation operators are closely related to the ones introduced in [29] for knowledge bases revision. Our relaxation-based revision framework, being abstract enough (i.e. defined through easily satisfied properties), encompasses these operators. Moreover, the revision operator defined in [29] considers only inconsistencies due to Abox assertions. Our operators are general in the sense that Abox assertions are handled as any formula of the language.

6. Conclusion

The contribution of this paper is threefold. First, we provided a generalization of AGM postulates, in a slightly weaker form from a model-theoretic point of view, in the abstract model theory of satisfaction systems, so as they become applicable to a wide class of non-classical logics. In this framework, we then generalized to any satisfaction systems the characterization of the AGM postulates given by Katsuno and Mendelzon for propositional logic in terms of minimal change with respect to an ordering among interpretations. This work generalizes the previous ones in the area. It also suggests the theory behind satisfaction systems to be a principled framework for dealing with knowledge dynamics with the growing interest on non-classical logics such as DL. We do hope that bridges can thus be built, by working at the cross-road of different areas of theoretical computer science.

Secondly, we proposed a general framework for defining revision operators based on the notion of relaxation. We demonstrated that such a relaxation-based framework for belief revision satisfies the weakened AGM postulates. As a byproduct, we give a principled formal flavor to several operators defined in the literature (e.g. weakening operators defined in DL).

Thirdly, we introduced a number of concrete relaxations within the scope of description logics, discussed their properties and illustrated them through simple examples. It was out of the scope of this paper to discuss languages such as OWL. However, the proposed approach could be applied to SROIQ and implemented in OWL, by augmenting a relaxation with operations on complex constructors.

Future works will concern the study of the complexity of the introduced operators, the comparison of their induced ordering, and their generalization to more expressive DL as well as other non-classical logics such as first-order Horn logics or equational logics.

Finally, there is an extension of satisfaction systems that takes into account explicitly the notion of signatures, the theory of institutions [19], a categorical model theory which has emerged in computing science studies of software specifications and semantics. In this paper, as we have considered logical theories over a same signature, signature morphisms and their interpretation for model classes and sentence sets were not relevant. However, these results carry over to institutions, which are indexed satisfaction systems.

Appendix. Proofs of the main results

Proof of Proposition 3. Let us suppose that $Cn(T'_1) = Cn(T'_2)$. Here, three cases have to be considered:

(1) One of T'_1 and T'_2 is inconsistent (say T'_1 without loss of generality). Since $Cn(T'_1) = Cn(T'_2)$ by hypothesis, T'_2 is also inconsistent. By Postulate (G2), we then have that, for $i = 1, 2$, $Mod(T \circ T'_i) \subseteq Mod(T'_i)$, and $Mod(T'_i) = Triv$ (Corollary 1). Hence $Mod(T \circ T'_i) \subseteq Triv$, and $Mod(T \circ T'_1) = Mod(T \circ T'_2) = Triv$.

(2) Both $T \cup T'_1$ and $T \cup T'_1$ are consistent. Since $Cn(T'_1) = Cn(T'_2)$, we know that $Mod(T'_1) = Mod(T'_2)$ (Equation (1)), and then $Mod(T \cup T'_1) = Mod(T \cup T'_2)$. Therefore, by Postulate (G3), we have that $Mod(T \circ T'_1) = Mod(T \circ T'_2)$.

(3) T'_1 and T'_2 are consistent but $T \cup T'_1$ or $T \cup T'_2$ is not (say $T \cup T'_1$). From $Cn(T'_1) = Cn(T'_2)$, we derive that $T \cup T'_2$ is also inconsistent. By Postulate (G1), both $T \circ T'_1$ and $T \circ T'_2$ are consistent. Let $\mathcal{M} \in Mod(T \circ T'_1)$. If $\mathcal{M} \in Triv$, then obviously $\mathcal{M} \in Mod(T \circ T'_2)$. Therefore, let us suppose that $\mathcal{M} \notin Triv$. By Postulate (G2), $\mathcal{M} \in Mod(T'_1)$, and then $\mathcal{M} \in Mod(T'_2)$. Let $\mathcal{M}' \in Mod(T \circ T'_2) \setminus Triv$. Such a model exists as $T \circ T'_2$ is consistent. By Postulate (G2) and the

hypothesis that $Cn(T'_1) = Cn(T'_2)$, $\{\mathcal{M}, \mathcal{M}'\}^*$ contains both T'_1 and T'_2. Obviously, we have that $(T \circ T'_1) \cup \{\mathcal{M}, \mathcal{M}'\}^*$ and $(T \circ T'_2) \cup \{\mathcal{M}, \mathcal{M}'\}^*$ are consistent. Therefore, by Postulates (G5) and (G6), we have that $Mod((T \circ T'_1) \cup \{\mathcal{M}, \mathcal{M}'\}^*) = Mod((T \circ (T'_1 \cup \{\mathcal{M}, \mathcal{M}'\}^*) = Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$ and $Mod((T \circ T'_2) \cup \{\mathcal{M}, \mathcal{M}'\}^*) = Mod((T \circ (T'_2 \cup \{\mathcal{M}, \mathcal{M}'\}^*) = Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$. We can then derive that $Mod((T \circ T'_1) \cup \{\mathcal{M}, \mathcal{M}'\}^*) = Mod((T \circ T'_2) \cup \{\mathcal{M}, \mathcal{M}'\}^*)$, and conclude that $\mathcal{M} \in Mod(T \circ T'_2)$. Similarly, by reversing the roles of T'_1 and T'_2, if $\mathcal{M} \in Mod(T \circ T'_2)$, we can conclude that $\mathcal{M} \in Mod(T \circ T'_1)$.

Proof of Theorem 1.

(1) Let us suppose that $\circ$ satisfies AGM Postulates. For every knowledge base T, let us define the binary relation $\preceq_T \subseteq Mod \times Mod$ by: for all $\mathcal{M}, \mathcal{M}' \in Mod$,

$$\mathcal{M} \preceq_T \mathcal{M}' \text{ iff } \begin{cases} \text{either } \mathcal{M} \in Mod(T) \\ \text{or } \mathcal{M} \in Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*) \text{ and } \mathcal{M}' \notin Triv \end{cases}$$

Let us first show that $\preceq_T$ satisfies the two conditions of FA.

- The first condition easily follows from the definition of $\preceq_T$.
- To prove the second one, let us assume that $\mathcal{M} \in Mod(T)$ and $\mathcal{M}' \notin Mod(T)$. Since $\mathcal{M} \in Mod(T)$, we have $\mathcal{M} \preceq_T \mathcal{M}'$. Here two cases have to be considered:
  (a) $\mathcal{M} \in Triv$. In this case, we directly have by definition that $\mathcal{M}' \not\preceq_T \mathcal{M}$.
  (b) $\mathcal{M} \notin Triv$. Then $T \cup \{\mathcal{M}, \mathcal{M}'\}^*$ is consistent since $\mathcal{M} \in Mod(T) \setminus Triv$ and $\mathcal{M} \in Mod(\mathcal{M}^*) \subseteq Mod(\{\mathcal{M}, \mathcal{M}'\}^*)$. Then by Postulate (G3), we have that $T \circ \{\mathcal{M}, \mathcal{M}'\}^* = T \cup \{\mathcal{M}, \mathcal{M}'\}^*$. Therefore, we have that $\mathcal{M}' \notin Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$, and $\mathcal{M}' \not\preceq_T \mathcal{M}$.
  Hence $\mathcal{M} \prec_T \mathcal{M}'$ in both cases.

Let us now prove the three supplementary conditions.

- First, let us show that $Mod(T \circ T') = Min(Mod(T') \setminus Triv, \preceq_T)$. If T' is inconsistent, then by Proposition 2 $Mod(T') \setminus Triv = \emptyset$, and by (G2) $Mod(T \circ T') \subseteq Mod(T') \subseteq Triv$, hence $Mod(T \circ T') \setminus Triv = \emptyset = Min(Mod(T') \setminus Triv, \preceq_T)$.
  Let us assume now that T' is consistent.
  · Let us first show that $Mod(T \circ T') \setminus Triv \subseteq Min(Mod(T') \setminus Triv, \preceq_T)$. Let $\mathcal{M} \in Mod(T \circ T') \setminus Triv$. Let us assume that $\mathcal{M} \notin Min(Mod(T') \setminus Triv, \preceq_T)$. By (G2), $\mathcal{M} \in Mod(T') \setminus Triv$. By hypothesis, there exists $\mathcal{M}' \in Mod(T') \setminus Triv$ such that $\mathcal{M}' \prec_T \mathcal{M}$. Here, two cases have to be considered:
    (a) $\mathcal{M}' \in Mod(T)$. As $\mathcal{M}' \in Mod(T') \setminus Triv$, then $T \cup T'$ is consistent, and then by (G3), $T \circ T' = T \cup T'$. Thus, $\mathcal{M} \in Mod(T)$, and then $\mathcal{M} \preceq_T \mathcal{M}'$, which is a contradiction.
    (b) $\mathcal{M}' \notin Mod(T)$. By definition of $\preceq_T$, this means that $\mathcal{M}' \in Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$. As $\mathcal{M}, \mathcal{M}' \in Mod(T')$, by Postulate (G2), $(T \circ T') \cup \{\mathcal{M}, \mathcal{M}'\}^*$ is consistent, and then by Postulates (G5) and (G6), we have that $Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*) = Mod((T \circ T') \cup \{\mathcal{M}, \mathcal{M}'\}^*)$. By the hypothesis that $\mathcal{M}' \prec_T \mathcal{M}$, we can deduce that $\mathcal{M} \notin Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$, whence by Postulate (G6) we have that $\mathcal{M} \notin Mod(T \circ T') \setminus Triv$, which is a contradiction.
    Finally we can conclude that $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$, and then $Mod(T \circ T') \setminus Triv \subseteq Min(Mod(T') \setminus Triv, \preceq_T)$.
  · Let us now show that $Min(Mod(T') \setminus Triv, \preceq_T) \subseteq Mod(T \circ T') \setminus Triv$. Let $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$. Let us assume that $\mathcal{M} \notin Mod(T \circ T') \setminus Triv$. As T' is consistent, by Postulates (G1) and (G2), there exists $\mathcal{M}' \in Mod(T \circ T')$ such that $\mathcal{M}'^* \neq Sen$, and $\mathcal{M}' \in Mod(T')$. Since $T' \subseteq \{\mathcal{M}, \mathcal{M}'\}^*$, we also have that $Mod(T' \cup \{\mathcal{M}, \mathcal{M}'\}^*) = Mod(\{\mathcal{M}, \mathcal{M}'\}^*)$. By Postulates (G5) and (G6), we can write $Mod(T \circ T') \cap Mod(\{\mathcal{M}, \mathcal{M}'\}^*) = Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$, since $(T \circ T') \cup \{\mathcal{M}, \mathcal{M}'\}^*$ is consistent. Hence, $\mathcal{M} \notin Mod(T \circ \{\mathcal{M}, \mathcal{M}'\}^*)$, and then $\mathcal{M}' \prec_T \mathcal{M}$, which is a contradiction. We can conclude that $\mathcal{M} \in Mod(T \circ T') \setminus Triv$, and then $Min(Mod(T') \setminus Triv, \preceq_T) \subseteq Mod(T \circ T') \setminus Triv$.
- Secondly, let us show that $Min(Mod(T') \setminus Triv, \preceq_T) \neq \emptyset$ if T' is consistent. By Postulate (G1), we have that $T \circ T'$ is consistent, and then $Mod(T \circ T') \setminus Triv \neq \emptyset$. We can directly conclude by the previous point that $Min(Mod(T') \setminus Triv, \preceq_T) \neq \emptyset$.
- Finally, let us show that for every $T', T'' \subseteq Sen$, $Min(Mod(T') \setminus Triv, \preceq_T) \cap Mod(T'') = Min(Mod(T' \cup T'') \setminus Triv, \preceq_T)$ if $(T \circ T') \cup T''$ is consistent. By (G5) and (G6), we have that $Mod(T \circ (T' \cup T'')) = Mod((T \circ T') \cup T'')$. Therefore, by the first point, we can directly conclude that $Min(Mod(T') \setminus Triv, \preceq_T) \cap Mod(T'') = Min(Mod(T' \cup T'') \setminus Triv, \preceq_T)$.

(2) Let us now suppose that for a revision operation $\circ$ there exists a FA which maps any knowledge base $T \subseteq Sen$ to a binary relation $\preceq_T \subseteq Mod \times Mod$ satisfying the three conditions of Theorem 1. Let us prove that $\circ$ verifies the AGM Postulates.

(G1) This postulate directly results from the fact that $Min(Mod(T') \setminus Triv, \preceq_T) \neq \emptyset$ when T' is consistent, hence $Mod(T \circ T') \setminus Triv \neq \emptyset$.

(G2) Let $\mathcal{M} \in Mod(T \circ T')$. If $\mathcal{M} \in Triv$, then obviously $\mathcal{M} \in Mod(T')$. Now, if $\mathcal{M} \notin Triv$, then by definition, $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$. This means that $\mathcal{M} \in Mod(T')$.

(G3) Suppose that $T \cup T'$ is consistent (hence $Mod(T \cup T') \setminus Triv \neq \emptyset$).
- Let us first prove that $Mod(T \circ T') \subseteq Mod(T \cup T')$. Let $\mathcal{M} \in Mod(T \circ T')$. Here two cases have to be considered:
  (a) $\mathcal{M} \in Triv$. In this case, we obviously have that $\mathcal{M} \in Mod(T \cup T')$.

(b) $\mathcal{M} \notin Triv$. By definition, $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$. Hence, we have that $\mathcal{M} \in Mod(T')$. Let us suppose now that $\mathcal{M} \notin Mod(T)$. As T is consistent, $Mod(T) \setminus Triv \neq \emptyset$ by Proposition 2. Therefore, there exists $\mathcal{M}' \in Mod(T) \setminus Triv$ such that $\mathcal{M}' \prec_T \mathcal{M}$ (from $\mathcal{M} \notin Mod(T)$ and the second property of FA), which is a contradiction. Hence $\mathcal{M} \in Mod(T)$ and $\mathcal{M} \in Mod(T \cup T')$.

- Let us now prove that $Mod(T \cup T') \subseteq Mod(T \circ T')$. Let $\mathcal{M} \in Mod(T \cup T')$ such that $\mathcal{M} \notin Mod(T \circ T')$. Therefore, $\mathcal{M} \in Mod(T)$. By hypothesis, there exists $\mathcal{M}' \in Mod(T') \setminus Triv$ such that $\mathcal{M}' \prec_T \mathcal{M}$ (since $\mathcal{M} \notin Min(Mod(T') \setminus Triv, \preceq_T)$), and then $\mathcal{M}' \notin Mod(T)$ by the first condition of FA. However, by the second condition of FA, we have that $\mathcal{M} \prec_T \mathcal{M}'$, which is a contradiction.

Finally, we can conclude that $Mod(T \circ T') = Mod(T \cup T')$.

(G5) Let $\mathcal{M} \in Mod(T \circ T') \cap Mod(T'')$. Let us assume that $\mathcal{M} \notin Min(Mod(T' \cup T'') \setminus Triv, \preceq_T)$. This means that $\mathcal{M} \in Triv$ or there exists $\mathcal{M}' \in Mod(T' \cup T'')$ such that $\mathcal{M}'^* \neq Sen$ and $\mathcal{M}' \prec_T \mathcal{M}$. In the first case, we obviously have that $\mathcal{M} \in Mod(T \circ (T' \cup T''))$. In the second case, we then have that $\mathcal{M}' \in Mod(T')$, and then $\mathcal{M}' \not\prec_T \mathcal{M}$ since $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$, which is a contradiction.

(G6) Let us suppose that $(T \circ T') \cup T''$ is consistent. Let $\mathcal{M} \in Mod(T \circ (T' \cup T''))$. By hypothesis, either $\mathcal{M} \in Triv$ and in this case, obviously we have that $\mathcal{M} \in Mod((T \circ T') \cup T'')$, or $\mathcal{M} \in Min(Mod(T' \cup T'') \setminus Triv, \preceq_T)$ as $Mod(T \circ (T' \cup T''))\setminus Triv = Min(Mod(T' \cup T'') \setminus Triv, \preceq_T)$. As $(T \circ T') \cup T''$ is consistent, we have that $Min(Mod(T' \cup T'') \setminus Triv, \preceq_T) = Min(Mod(T') \setminus Triv, \preceq_T) \cap Mod(T'')$ and then $\mathcal{M} \in Mod((T \circ T') \cup T'')$.

Proof of Theorem 2. First, let us show that f is a FA.

- Let $\mathcal{M}, \mathcal{M}' \in Mod(T)$. Let us suppose that $\mathcal{M} \prec_T \mathcal{M}'$. This means that there exists $T' \subseteq Sen$ such that $\mathcal{M}, \mathcal{M}' \in Mod(T')$, $\mathcal{M} \in Mod(T \circ T')$ and $\mathcal{M}' \notin Mod(T \circ T')$. Hence we have that $T \cup T'$ is consistent, and then by Postulate (G3), $T \circ T' = T \cup T'$. We then have that $\mathcal{M}' \in Mod(T \circ T')$ which is a contradiction.
- Let $\mathcal{M} \in Mod(T)$ and let $\mathcal{M}' \in Mod \setminus Mod(T)$. We have that $\mathcal{M} \preceq_T^{\emptyset} \mathcal{M}'$, and then $\mathcal{M} \preceq_T \mathcal{M}'$ by definition of $\preceq_T$. Now, let us suppose that $\mathcal{M}' \preceq_T \mathcal{M}$. This means that there exists $T' \subseteq Sen$ such that $\mathcal{M}, \mathcal{M}' \in Mod(T')$, $\mathcal{M}' \in Mod(T \circ T')$ and $\mathcal{M} \notin Mod(T \circ T')$. But, as $\mathcal{M} \in Mod(T)$, we have that $T \cup T'$ is consistent, and then by Postulate (G3), $T \circ T' = T \cup T'$. Hence, we have that $\mathcal{M} \in Mod(T \circ T')$ which is a contradiction.

Let us show now the supplementary conditions of Theorem 1.

- First, let us show that $Mod(T \circ T') \setminus Triv = Min(Mod(T') \setminus Triv, \preceq_T)$. The case where T' is inconsistent follows the same proof as in Theorem 1.
  Let us suppose that T' is consistent. Let $\mathcal{M} \in Mod(T \circ T') \setminus Triv$. Let us suppose that $\mathcal{M} \notin Min(Mod(T') \setminus Triv, \preceq_T)$. This means that there exists $\mathcal{M}' \in Mod(T') \setminus Triv$ such that $\mathcal{M}' \prec_T \mathcal{M}$. Therefore, there exists $T'' \subseteq Sen$ such that $\mathcal{M}, \mathcal{M}' \in Mod(T'')$, $\mathcal{M}' \in Mod(T \circ T'')$ and $\mathcal{M} \notin Mod(T \circ T'')$. Hence, both $(T \circ T') \cup T''$ and $(T \circ T'') \cup T'$ are consistent, and then by Postulates (G5) and (G6), $Mod((T \circ T') \cup T'') = Mod((T \circ T'') \cup T') = Mod(T \circ (T' \cup T''))$. We can then derive that $\mathcal{M} \in Mod(T \circ T'')$ which is a contradiction.
  Let $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$. Let us suppose that $\mathcal{M} \notin Mod(T \circ T') \setminus Triv$. As T' is consistent, by Postulates (G1) and (G2), there exists $\mathcal{M}' \in Mod(T \circ T') \setminus Triv$. By definition of $\preceq_T^{T'}$, we have that $\mathcal{M}' \preceq_T^{T'} \mathcal{M}$, and then $\mathcal{M}' \preceq_T \mathcal{M}$ which is a contradiction.
- The proof of the two other conditions corresponds to the one given in Theorem 1.

Proof of Proposition 4. It is sufficient to show that $\preceq_T^1 \cup \preceq_T^2$ and $\preceq_T^1 \cap \preceq_T^2$ satisfy Conditions (1) and (2) of Definition 4 plus all the conditions of Theorem 1.

Let us first show that they are FA. Let $T \subseteq Sen$. Let $\mathcal{M}, \mathcal{M}' \in Mod(T)$. By definition of FA, then we have either $\mathcal{M} \not\preceq_T^i \mathcal{M}'$ and $\mathcal{M}' \not\preceq_T^i \mathcal{M}$ or $\mathcal{M} \preceq_T^i \mathcal{M}'$ and $\mathcal{M}' \preceq_T^i \mathcal{M}$ for $i = 1, 2$. We then have four cases to consider, but for $f_1 \sqcap f_2(T) = \preceq_T$ (resp. $f_1 \sqcup f_2(T) = \preceq_T$), we always end up at either $\mathcal{M} \not\preceq_T \mathcal{M}'$ and $\mathcal{M}' \not\preceq_T \mathcal{M}$ or $\mathcal{M} \preceq_T \mathcal{M}'$ and $\mathcal{M}' \preceq_T \mathcal{M}$. Likewise, for every $\mathcal{M} \in Mod(T)$ and every $\mathcal{M}' \in Mod \setminus Mod(T)$, we have that $\mathcal{M} \prec_T^i \mathcal{M}'$ for $i = 1, 2$. Therefore, it is obvious to conclude that $\mathcal{M} \prec_T \mathcal{M}'$.

Now, by the first supplementary condition for $\preceq_T^1$ and $\preceq_T^2$ in Theorem 1, we have for every $T' \subseteq Sen$ that $Min(Mod(T') \setminus Triv, \preceq_T^1) = Min(Mod(T') \setminus Triv, \preceq_T^2) = Mod(T \circ T') \setminus Triv$. Hence, we can write that $Min(Mod(T') \setminus Triv, \preceq_T^1 \cup \preceq_T^2) = Min(Mod(T') \setminus Triv, \preceq_T^1 \cap \preceq_T^2) = Min(Mod(T') \setminus Triv, \preceq_T^i)$ for $i = 1, 2$. The three supplementary conditions are then straightforward, and this allows us to directly conclude that $f_1 \sqcup f_2$ and $f_1 \sqcap f_2$ are FA+.

Proof of Theorem 3. $\circ$ obviously satisfies Postulates (G1), (G2) and (G3). To prove (G5)–(G6), let us suppose $T, T', T'' \subseteq Sen$ such that $(T \circ T') \cup T''$ is consistent (the case where $(T \circ T') \cup T''$ is inconsistent is obvious). This means that $\rho^{\mathcal{K}_T^{T'}}(T) \cup T' \cup T''$ is consistent. Now, obviously we have that $Mod(T' \cup T'') \subseteq Mod(T')$. Hence, by the second and the third conditions of Definition 7, we necessarily have that $T \circ (T' \cup T'') = \rho^{\mathcal{K}_T^{T'}}(T) \cup T' \cup T''$, and then $Mod((T \circ T') \cup T'') = Mod(T \circ (T' \cup T''))$.

Proof of Theorem 4. Let $T \subseteq Sen$. Let us first show that $f_\rho(T) = \preceq_T$ is faithful.

- Obviously, we have for every $\mathcal{M}, \mathcal{M}' \in Mod(T)$ and every $T' \subseteq Sen$ that both $\mathcal{M} \not\preceq_T^{T'} \mathcal{M}'$ and $\mathcal{M}' \not\preceq_T^{T'} \mathcal{M}$. Hence the same relations hold for $\preceq_T$.
- Let $\mathcal{M} \in Mod(T)$ and let $\mathcal{M}' \in Mod \setminus Mod(T)$. Obviously, we have that $\mathcal{M} \preceq_T^{\emptyset} \mathcal{M}'$. Let $T' \subseteq Sen$ such that $\mathcal{M}, \mathcal{M}' \in Mod(T')$ (the case where for all $T' \subseteq Sen$ $\mathcal{M}$ or $\mathcal{M}'$ is not in $Mod(T')$ implies that $\mathcal{M}$ and $\mathcal{M}'$ are incomparable by $\preceq_T^{T'}$, and then we directly have that $\mathcal{M}' \not\preceq_T \mathcal{M}$). Here two cases have to be considered:
  (1) $\mathcal{M} \in Triv$. As $\mathcal{M}' \notin Mod(T)$, then $\mathcal{M}' \notin Triv$. Hence, there does not exist $\mathcal{K}' < \mathcal{K}$ such that $\mathcal{M}' \in Mod(\rho^{\mathcal{K}'}(T))$. Otherwise, $\rho^{\mathcal{K}'}(T) \cup T'$ would be consistent, which would contradict the hypothesis that $T \circ T' = \rho^{\mathcal{K}}(T) \cup T'$.
  (2) $\mathcal{M} \notin Triv$. We have that $\mathcal{M} \in Mod(T \cup T')$ but $\mathcal{M}' \notin Mod(T \cup T')$, and then $\mathcal{M}' \not\preceq_T^{T'} \mathcal{M}$ By definition of $\circ$.
  Hence, in both cases we can conclude that $\mathcal{M}' \not\preceq_T \mathcal{M}$.

Let us prove that $Mod(T \circ T') \setminus Triv = Min(Mod(T') \setminus Triv, \preceq_T)$. This will directly prove that $Min(Mod(T') \setminus Triv, \preceq_T) \neq \emptyset$ when T' is consistent. Indeed, by definition, we have that $T \circ T'$ is consistent when T' is consistent, and then $Min(Mod(T') \setminus Triv, \preceq_T) \neq \emptyset$ if $Mod(T \circ T') \setminus Triv = Min(Mod(T') \setminus Triv, \preceq_T)$.

If T' is inconsistent, then so is $T \circ T'$ by definition. Hence, $Mod(T \circ T') \setminus Triv = Min(Mod(T') \setminus Triv, \preceq_T) = \emptyset$.

Let us now suppose that T' is consistent.

- Let us show that $Mod(T \circ T') \setminus Triv \subseteq Min(Mod(T') \setminus Triv, \preceq_T)$. Let $\mathcal{M} \in Mod(T \circ T') \setminus Triv$. Let $\mathcal{M}' \in Mod(T') \setminus Triv$. Two cases have to be considered:
  (1) $\mathcal{M}' \in Mod(T \circ T')$. Obviously, we have both $\mathcal{M} \not\preceq_T^{T'} \mathcal{M}'$ and $\mathcal{M}' \not\preceq_T^{T'} \mathcal{M}$. Let us show that this is also true for every $T'' \subseteq Sen$ such that $\mathcal{M}, \mathcal{M}' \in Mod(T'')$. Let us suppose that there exists $T'' \subseteq Sen$ such that $\mathcal{M}' \preceq_T^{T''} \mathcal{M}$. By hypothesis, we then have that $(T \circ T') \cup T''$ is consistent. Therefore, by Conditions 2 and 3 of Definition 7, we have that $(T \circ T') \cup T'' = T \circ (T' \cup T'')$. Hence, we also have that $T \circ (T' \cup T'') = \rho^{\mathcal{K}_T^{T'}}(T) \cup T' \cup T''$. Consequently, as $Mod(T' \cup T'') \subseteq Mod(T'')$, we have by Condition 3 of Definition 7 that $\mathcal{K}_T^{T''} \leq \mathcal{K}_T^{T'}$. Therefore, as $\mathcal{M}' \preceq_T^{T''} \mathcal{M}$, we can deduce that there exists $\mathcal{K}'' < \mathcal{K}_T^{T'}$ such that $\mathcal{M}' \in Mod(\rho^{\mathcal{K}''}(T))$. We then have that $\rho^{\mathcal{K}''}(T) \cup T'$ is consistent, and then by Condition 2 of Definition 7, $\sum \mathcal{K}_T^{T'} \leq \sum \mathcal{K}''$, which is a contradiction.
  (2) $\mathcal{M}' \notin Mod(T \circ T')$. By definition of $\preceq_T^{T'}$, we have that $\mathcal{M} \preceq_T^{T'} \mathcal{M}'$, and therefore $\mathcal{M} \preceq_T \mathcal{M}'$.
  Finally, we can conclude that $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$.
- Let us now show that $Min(Mod(T') \setminus Triv, \preceq_T) \subseteq Mod(T \circ T') \setminus Triv$. Let $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$. Let us suppose that $\mathcal{M} \notin Mod(T \circ T') \setminus Triv$. As T' is consistent, then so is $T \circ T'$. Hence, there exists $\mathcal{M}' \in Mod(T \circ T') \setminus Triv$. As $\mathcal{M} \in Mod(T') \setminus Mod(T \circ T')$, we have that $\mathcal{M}' \preceq_T^{T'} \mathcal{M}$, and then as $\mathcal{M} \in Min(Mod(T') \setminus Triv, \preceq_T)$ we also have that $\mathcal{M} \preceq_T \mathcal{M}'$. This means that there exists $T'' \subseteq Sen$ such that $\mathcal{M}, \mathcal{M}' \in Mod(T'')$ and $\mathcal{M} \preceq_T^{T''} \mathcal{M}'$. By hypothesis, we then have that $(T \circ T') \cup T''$ is consistent. Therefore, by Conditions 2 and 3 of Definition 7, we have that $(T \circ T') \cup T'' = T \circ (T' \cup T'')$. Hence, we also have that $T \circ (T' \cup T'') = \rho^{\mathcal{K}_T^{T'}}(T) \cup T' \cup T''$. Consequently, we have by Condition 3 of Definition 7 that $\mathcal{K}_T^{T''} \leq \mathcal{K}_T^{T'}$. Hence, there exists $\mathcal{K}'' \geq \mathcal{K}_T^{T''}$ such that $\mathcal{K}'' < \mathcal{K}_T^{T'}$ and $\mathcal{M} \in Mod(\rho^{\mathcal{K}''}(T))$. We can then deduce that $\rho^{\mathcal{K}''}(T) \cup T'$ is consistent, and then by Condition 2 of Definition 7 we have that $\sum \mathcal{K}_T^{T'} \leq \sum \mathcal{K}''$, which is a contradiction.

Finally, to prove the last point, we follow the same steps as in the proof of Theorem 1.

Proof of Proposition 15. The proof relies on the following general result:

$$\forall C, \forall r, \forall r.C \sqsubseteq \exists r.C$$

Indeed, for each interpretation $\mathcal{I}$, if $r_i^{\mathcal{I}} \neq \emptyset$, we have

$$x \in (\forall r.C)^{\mathcal{I}} \Rightarrow (\forall y, (x, y) \in r^{\mathcal{I}} \Rightarrow y \in C^{\mathcal{I}}) \Rightarrow (\exists y, (x, y) \in r^{\mathcal{I}} \text{and } y \in C^{\mathcal{I}}) \Rightarrow x \in (\exists r.C)^{\mathcal{I}}.$$

Hence $(\forall r.C)^{\mathcal{I}} \subseteq (\exists r.C)^{\mathcal{I}}$ for each $\mathcal{I}$ (if $r_i^{\mathcal{I}} = \emptyset$ it is obvious), and $\forall r.C \sqsubseteq \exists r.C$.

In a similar way, we can show, that for any C_1, C_2, r, and $Q \in \{\exists, \forall\}$:

$$C_1 \sqsubseteq C_2 \Rightarrow Q r.C_1 \sqsubseteq Q r.C_2.$$

Now, let us consider any j such that $Q_j = \exists$, and set $C' = Q_{j+1} r_{j+1} \ldots Q_n r_n.D$. We have from the first result $Q'_j r_j.C' \sqsubseteq Q_j r_j.C'$. Applying the second result recursively on each Q_i for $i < j$, we then have

$$Q_1 r_1 \ldots Q_{j-1} r_{j-1} Q'_j r_j.C' \sqsubseteq Q_1 r_1 \ldots Q_{j-1} r_{j-1} Q_j r_j.C'.$$

The same relation holds for the conjunction over any j such that $Q_j = \exists$, from which we conclude that $\forall C, \kappa_q(C) \sqsubseteq C$, i.e. κ_q is anti-extensive.

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