Spatial Reasoning and model-based image understanding

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Knowledge representation and reasoning on spatial entities and spatial relationships

- largely developed in the artificial intelligence community
  - mainly topological relations
  - formal logics (ex: mereotopology)
  - inference
- less developed in image interpretation
  - need for imprecise knowledge representation
  - (semi-)quantitative framework (⇒ numerical evaluation)
  - examples: structural recognition in images under imprecision
- main ingredients:
  - knowledge representation (including spatial relations)
  - imprecision representation and management
  - fusion of heterogeneous information
  - reasoning and decision making
Philosophy

- From Pythagoras to Zeno: concept of space linked to the first developments in arithmetics and Pythagorean geometry - Problem of infinitely subdivision possibility.
- Descartes: spatial extension is specific to material entities, governed by the only laws of mechanics.
- Newton: notion of absolute space.
- Hume: space reduced to a pure psychological function.
- Leibniz: space cannot be an absolute reality, motion and position are real and detectable only in relation to other objects, not in relation to space itself.
- Kant: objectivity of space.
- Poincaré: empiricist point of view where spatial knowledge is mainly derived from motor experience. Relativity of space.
- Bergson: a position in the space can be considered as an instantaneous cut of the movement, but the movement is more that a sum of positions in the space.
- Einstein: geometry is linked to the sensible and perceptible space. The geometrical configuration of the world itself becomes relative.
- Purely philosophical views of space developed by the phenomenologists and the existentialists.
- Reichenbach: geometry as a theory of relations.
Linguistics

- Rich variety of lexical terms for describing spatial location of entities.
- Concern all lexical categories (nouns, verbs, adjectives, adverbs, prepositions).
  - French, and other Romance languages, shows a typological preference for the lexicalization of the path in the main verb.
  - In Germanic and Slavic languages, the path is rather encoded in satellites associated to the verb (particle or prefix).
- Source of inspiration of many works on qualitative spatial information representation and qualitative spatial reasoning.
- Asymmetry, importance of reference, of context, of functional properties of the considered physical entities
- Imprecision (too precise statements can even become inefficient if they make the message too complex).
Human perception: example of distance

• Purely spatial measures, in a geometric sense, give rise to "metric distances", and are related to intrinsic properties of the objects.

• Temporal measures lead to distances expressed as travel time, and can be considered of extrinsic type, as opposed to the previous class.

• Economic measures, in terms of costs to be invested, are also of extrinsic type.

• Perceptual measures lead to distances of deictic type; they are related to an external point of view, which can be concrete or just a mental representation, which can be influenced by environmental features, by subjective considerations, leading to distances that are not necessarily symmetrical.

• Influence of other objects.
Cognition

Cognitive understanding of a spatial environment is issued from two types of processes:

- route knowledge acquisition (first acquired during child development), which consists in learning from sensori-motor experience (i.e. actual navigation) and implies an order information between visited landmarks,

- survey knowledge acquisition, from symbolic sources such as maps, leading to a global view ("from above") including global features and relationships, which is independent of the order of landmarks.

Neuro-imaging:

- a right hippocampal activation can be observed for both mental navigation and mental map tasks,

- a parahippocampal gyrus activation is additionally observed only for mental navigation, when route information and object landmarks have to be incorporated.

Internal representation of space in the brain:

- egocentric representations,

- allocentric representations ("map in the head").

Intensively used in several works in the modeling and conception of geographic information systems, and in mobile robotics.
Spatial reasoning formalisms

- Quantitative
- Qualitative (QSR)
- Fuzzy representations and reasoning: semi-quantitative / semi-qualitative approaches
Quantitative spatial reasoning

- Precisely defined objects
- Computation of well defined relations
- Many limitations
  - on the objects
  - on the relations
  - on the type of representations
  - for reasoning

But does not always match the usual way of reasoning (e.g. to the north, closer...).
Qualitative spatial reasoning

- Reasoning on space
- Formal logics:
  - abstraction
  - only the necessary distinctions are represented
  - different granularity levels
  - consistency (∃ a model?)
- Key point: compromise between
  - expressiveness
  - completeness with respect to a class of situations
  - complexity
- Representation of space:
  - reference frame
  - types of objects (properties, extended spatial entities)
  - types of relations
- Examples of formalisms:
  - cardinal directions: 9 positions
  - Allen’s intervals (temporal reasoning): 13 relations
  - rectangle calculus (Allen on each axis): 169 relations
  - cube calculus
  - RCC8, mereotopology (based on connection and parthood predicates)
Cardinal directions (Frank, Egenhofer, Ligozat)

Qualitative directions: N, NE, E, SE, S, SW, W, NW

Cone–based

Projection–based

Only few compositions can be exactly determined.
Allen’s intervals

13 basic relations

p
m
o
s
e
d
f
Allen’s intervals

Geometrical / quantitative representation
**Allen’s intervals**

Qualitative representation: lattice

- Smallest element = p
- Largest element = pi
Rectangle, cube algebra

- Allen’s interval in each direction
- 2D (rectangles): $13^2 = 169$ relations
- 3D (cubes): $13^3 = 2197$ relations
- ⇒ high complexity, and fixed shaped objects
RCC-8: Region Connection Calculus (Randell, Cui, Cohn)

- spatial entities, defined in a qualitative way
- no reference to points
- connection predicate $C$
- parthood predicate $P$

$$P(x, y) : \forall z, C(z, x) \rightarrow C(z, y)$$

$DC(a,b)$  $EC(a,b)$  $PO(a,b)$  $TPP(a,b)$  $NTPP(a,b)$
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$DC(x, y)$</td>
<td>$x$ is disconnected from $y$</td>
<td>$\neg C(x, y)$</td>
</tr>
<tr>
<td>$P(x, y)$</td>
<td>$x$ is a part of $y$</td>
<td>$\forall z, C(z, x) \rightarrow C(z, y)$</td>
</tr>
<tr>
<td>$PP(x, y)$</td>
<td>$x$ is a proper part of $y$</td>
<td>$P(x, y) \land \neg P(y, x)$</td>
</tr>
<tr>
<td>$EQ(x, y)$</td>
<td>$x$ is identical with $y$</td>
<td>$P(x, y) \land P(y, x)$</td>
</tr>
<tr>
<td>$O(x, y)$</td>
<td>$x$ overlaps $y$</td>
<td>$\exists z, P(z, x) \land P(z, y)$</td>
</tr>
<tr>
<td>$DR(x, y)$</td>
<td>$x$ is discrete from $y$</td>
<td>$\neg O(x, y)$</td>
</tr>
<tr>
<td>$PO(x, y)$</td>
<td>$x$ partially overlaps $y$</td>
<td>$O(x, y) \land \neg P(x, y) \land \neg P(y, x)$</td>
</tr>
<tr>
<td>$EC(x, y)$</td>
<td>$x$ is externally connected to $y$</td>
<td>$C(x, y) \land \neg O(x, y)$</td>
</tr>
<tr>
<td>$TPP(x, y)$</td>
<td>$x$ is a tangential proper part of $y$</td>
<td>$PP(x, y) \land \exists z [EC(z, x) \land EC(z, y)]$</td>
</tr>
<tr>
<td>$NTPP(x, y)$</td>
<td>$x$ is a non tangential proper part of $y$</td>
<td>$PP(x, y) \land \neg \exists z [EC(z, x) \land EC(z, y)]$</td>
</tr>
</tbody>
</table>
RCC-8: Region Connection Calculus (Randell, Cui, Cohn)
Qualitative trajectory calculus (Cohn et al.)

- Extension of RCC to take time into account (dynamic scenes).
- RCC + Allen
- Example:
  - $X, Y$ objects
  - $I_i$ time intervals

\[(P(X, Y), I_1) \land (PO(X, Y), I_2) \land (DR(X, Y), I_3) \land meet(I_1, I_2) \land meet(I_2, I_3) \land before(I_1, I_3)\]
Modal logics of space

Topology:
• □A: A is locally true (A is true at point x iff A is true in a neighborhood of x).
• ◊A = ¬□¬A: A is true at x iff A is true at least one point of the neighborhood of x.
• Reasoning axioms and inference rules of S4:
  • A → (B → A)
  • (A → (B → C)) → ((A → B) → (A → C))
  • (¬A → ¬B) → (B → A)
  • □(A → B) → (□A → □B)
  • □A → A
  • □A → □□A
Modal logics of space

Other examples:

- Translation of RCC into modal logics.
- Logics of places ($\square = \text{everywhere, } \Diamond = \text{somewhere}$).
- Modal logics of proximity ($\square A = \text{everywhere close to } A$).
- Modal logics of distance ($\square \leq a = \text{everywhere in a neighborhood of radius } a$).
- Logics of inclusion and contact (inference in GIS).
- Modal logics of geometry (affine, projective, parallelism...).
A few important issues

- Context
- Representation issues
- Reasoning (inference, satisfiability, decidability, CSP...)
- Complexity
- Applications

State of the art:
- Focus on representation issues
- Very few applications
- Focus on topology
- Almost nothing on metric relations
- Almost nothing on uncertainty
Example: composition tables

Allen intervals:

<table>
<thead>
<tr>
<th>.</th>
<th>p</th>
<th>m</th>
<th>o</th>
<th>F</th>
<th>D</th>
<th>s</th>
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<td>(P)</td>
</tr>
</tbody>
</table>

full=(pmoFDseSdfOMP) and concur=(oFDseSdfO)

From [http://www.ics.uci.edu/~alspaugh/cls/shr/allen.html](http://www.ics.uci.edu/~alspaugh/cls/shr/allen.html)
### Example: composition tables

**RCC-8:**

<table>
<thead>
<tr>
<th>o</th>
<th>DC</th>
<th>EC</th>
<th>PO</th>
<th>TPP</th>
<th>NTPP</th>
<th>TPPI</th>
<th>NTPPi</th>
<th>EQ</th>
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</thead>
<tbody>
<tr>
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<td>*</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC</td>
<td>DC</td>
</tr>
<tr>
<td>EC</td>
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<td>DC,EC,PO,TPP,TPPLEQ</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>EC,PO,TPP,NTPP</td>
<td>PO,TPP,NTPP</td>
<td>PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC</td>
</tr>
<tr>
<td>PO</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>*</td>
<td>PO,TPP,NTPP</td>
<td>PO,TPP,NTPP</td>
<td>PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
<td>DC,EC,PO,TPP,NTPP</td>
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<tr>
<td>TPP</td>
<td>DC</td>
<td>DC,EC</td>
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<td>TPP,NTPP</td>
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<td>EQ</td>
</tr>
</tbody>
</table>

From wikipedia
Semi-quantitative spatial reasoning: fuzzy approaches

- Limitations of purely qualitative reasoning
- Interest of adding semi-quantitative extension to qualitative value for deriving useful and practical conclusions
- Limitations of purely quantitative representations in the case of imprecise statements, knowledge expressed in linguistic terms, etc.
- Integration of both quantitative and qualitative knowledge using semi-quantitative (or semi-qualitative) interpretation of fuzzy sets
- Freeman (1975): fuzzy sets provide computational representation and interpretation of imprecise spatial constraints, expressed in a linguistic way, possibly including quantitative knowledge
- Granularity, involved in:
  - objects or spatial entities and their descriptions
  - types and expressions of spatial relations and queries
  - type of expected or potential result
Motivation: model-based recognition and spatial reasoning

- representation of imprecision
- spatial relations as structural information
  - topological relationships (set relations, adjacency)
  - distances
  - relative directional relationships
  - more complex relations (between, along...)
- two classes of relations
  - well defined in the crisp case (adjacency, distances...)
  - vague even in the crisp case (directional relationships...)
- fusion of several and heterogeneous pieces of knowledge and information
Motivation: model-based recognition and spatial reasoning

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⇒ Fuzzy set theory, mathematical morphology
Imprecision and fuzziness

- objects (no clear boundaries, coarse segmentation...)
- relations (ex: left of, quite close)
- type of knowledge available (ex: the caudate nucleus is close to the lateral ventricle)
- question to be answered (ex: go towards this object while remaining at some security distance)
Types of representations: example of distances

- number in $\mathbb{R}^+$ (or in $[0, 1]$)
- interval
- fuzzy number, fuzzy interval
- Rosenfeld:
  - distance density: degree to which the distance is equal to $n$
  - distance distribution: degree to which the distance is less than $n$
- linguistic value
- logical formula

$\Rightarrow$ unifying framework of fuzzy set theory
Definitions: fuzzy sets

- Space $S$ (image space, space of characteristics, etc.)
- Fuzzy set: $\mu : S \rightarrow [0, 1] - \mu(x) =$ membership degree of $x$ to $\mu$
- Support: $\text{Supp}(\mu) = \{x \in S, \mu(x) > 0\}$ - Core / kernel: $\{x \in S, \mu(x) = 1\}$
- $\alpha$-cut: $\mu_\alpha = \{x \in S, \mu(x) \geq \alpha\}$
- Cardinality: $|\mu| = \sum_{x \in S} \mu(x)$ (for $S$ finite)
- Convexity: $\forall (x, y) \in S^2, \forall \lambda \in [0, 1], \mu[\lambda x + (1 - \lambda)y] \geq \min[\mu(x), \mu(y)]$
- Fuzzy number: convex fuzzy set on $\mathbb{R}$, u.s.c., unimodal, with compact support. Example: LR-fuzzy sets.
Basic operations (Zadeh, 1965)

- **Equality:** $\mu = \nu \iff \forall x \in S, \mu(x) = \nu(x)$
- **Inclusion:** $\mu \subseteq \nu \iff \forall x \in S, \mu(x) \leq \nu(x)$
- **Intersection:** $\forall x \in S, (\mu \cap \nu)(x) = \min[\mu(x), \nu(x)]$
- **Union:** $\forall x \in S, (\mu \cup \nu)(x) = \max[\mu(x), \nu(x)]$
- **Complementation:** $\forall x \in S, \mu^C(x) = 1 - \mu(x)$
- **Properties:**
  - consistency with binary set operations
  - $\mu = \nu \iff \mu \subseteq \nu$ and $\nu \subseteq \mu$
  - fuzzy complementation is involutive: $(\mu^C)^C = \mu$
  - intersection and union are commutative and associative
  - intersection and union are idempotent and mutually distributive
  - intersection and union are dual with respect to the complementation: $(\mu \cap \nu)^C = \mu^C \cup \nu^C$
  - $(\mu \cup \nu)_{\alpha} = \mu_{\alpha} \cup \nu_{\alpha}$, etc.

**BUT:** $\mu \cap \mu^C \neq \emptyset$, $\mu \cup \mu^C \neq S$
**Definitions: possibility theory**

**Possibility measure:** function $\Pi$ from $2^S$ into $[0, 1]$ such that:

1. $\Pi(\emptyset) = 0$
2. $\Pi(S) = 1$
3. $\forall I \subseteq N, \forall A_i \subseteq S (i \in I), \Pi(\cup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i)$

**Necessity measure:** $\forall A \subseteq S$, $N(A) = 1 - \Pi(A^C)$

1. $N(\emptyset) = 0$
2. $N(S) = 1$
3. $\forall I \subseteq N, \forall A_i \subseteq S (i \in I), N(\cap_{i \in I} A_i) = \inf_{i \in I} N(A_i)$

**Useful properties:**

- $\max(\Pi(A), \Pi(A^C)) = 1$, $\min(N(A), N(A^C)) = 0$
- $\Pi(A) \geq N(A)$
- $N(A) > 0 \Rightarrow \Pi(A) = 1$, $\Pi(A) < 1 \Rightarrow N(A) = 0$
- $N(A) + N(A^C) \leq 1$, $\Pi(A) + \Pi(A^C) \geq 1$

**Possibility distribution:** function $\pi$ from $S$ into $[0, 1]$ with the normalization condition

$\sup_{x \in S} \pi(x) = 1$

In the finite case: $\Pi(A) = \sup\{\pi(x), x \in A\}$

Conversely: $\forall x \in S$, $\pi(x) = \Pi(\{x\})$

$N(A) = 1 - \sup\{\pi(x), x \notin A\} = \inf\{1 - \pi(x), x \in A^C\}$
Semantics

• degree of similarity (notion of distance)
• degree of plausibility (that an object from which only an imprecise description is known is actually the one one wants to identify)
• degree of preference (fuzzy class = set of "good" choices), close to the notion of utility function
Representing different types of imperfection
Fuzzy complementation

function $c$ from $[0, 1]$ into $[0, 1]$ such that:

1. $c(0) = 1$
2. $c(1) = 0$
3. $c$ is involutive, i.e. $\forall x \in [0, 1], c(c(x)) = x$
4. $c$ is strictly decreasing

General form of continuous complementations: $c(x) = \varphi^{-1}[1 - \varphi(x)]$ with $\varphi : [0, 1] \rightarrow [0, 1], \varphi(0) = 0, \varphi(1) = 1$, $\varphi$ strictly increasing.

Example: $\varphi(x) = x^n \Rightarrow c(x) = (1 - x^n)^{1/n}$
**Set theoretical operations**

**Triangular norms** (fuzzy intersection)

A t-norm $t : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a function such that:

1. **Commutativity**, i.e. $\forall (x, y) \in [0, 1]^2$, $t(x, y) = t(y, x)$;
2. **Associativity**, i.e. $\forall (x, y, z) \in [0, 1]^3$, $t[t(x, y), z] = t[x, t(y, z)]$;
3. **1 is unit element**, i.e. $\forall x \in [0, 1]$, $t(x, 1) = t(1, x) = x$;
4. **Increasingness** with respect to the two variables:

   $$\forall (x, x', y, y') \in [0, 1]^4, (x \leq x' \text{ and } y \leq y') \Rightarrow t(x, y) \leq t(x', y').$$

Moreover: $t(0, 1) = t(0, 0) = t(1, 0) = 0$, $t(1, 1) = 1$, and $0$ is **null element** ($\forall x \in [0, 1]$, $t(x, 0) = 0$).

**Examples of t-norms:** $\min(x, y)$, $xy$, $\max(0, x + y - 1)$. 
Set theoretical operations

Triangular conorms (fuzzy union) t-conorm $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ such that:

1. commutativity, i.e. $\forall (x, y) \in [0, 1]^2$, $T(x, y) = T(y, x)$;
2. associativity, i.e. $\forall (x, y, z) \in [0, 1]^3$, $T[T(x, y), z] = T[x, T(y, z)]$;
3. 0 is unit element, i.e. $\forall x \in [0, 1]$, $T(x, 0) = T(0, x) = x$;
4. increasingness with respect to the two variables

Moreover: $T(0, 1) = T(1, 1) = T(1, 0) = 1$, $T(0, 0) = 0$, and 1 is null element ($\forall x \in [0, 1], T(x, 1) = 1$).

Examples of t-conorms: $\max(x, y), x + y - xy, \min(1, x + y)$.

Duality: $\forall (x, y) \in [0, 1]^2$, $T[c(x), c(y)] = c[t(x, y)]$

Other combination operators (mean, symmetrical sums, etc.) $\Rightarrow$ information fusion
Linguistic variable

- size
- \{very small, small, medium, large, very large\}

- syntactic rules
- terms
- semantic rules
- membership functions

Spatial Reasoning – p.28/80
Imprecise reasoning

- Difference between data and knowledge
- Classical logic:
  - language
  - semantics (interpretations, truth values)
  - syntax (axioms and inference rules)
- Human reasoning: flexible, allows for imprecise statements
- Gradual predicates:
  - continuous referential
  - typicality
Uncertainty

= unable to say whether a proposition is true or not

• because information is incomplete, vague, imprecise
  ⇒ possibility

• because information is contradicting or fluctuating
  ⇒ probability

  certainty degree ≠ truth degree

  "It is probable that he is far from his goal"  "He is very far from his goal"

• Fuzzy logic: propositions with truth degrees

• Possibilistic logic: propositions with (un)certainty degrees
Fuzzy logic

- Basic fuzzy propositions: \( X \ is \ P \)
  \( X = \) variable taking values in \( U \)
  \( P = \) fuzzy subset of \( U \)
  Truth degrees in \([0, 1]\) defined from \( \mu_P \)

- Conjunction

  \[ X \ is \ A \ and \ Y \ is \ B \]

  \[ \mu_{A \land B}(x, y) = t[\mu_A(x), \mu_B(y)] \]

- Disjunction

  \[ X \ is \ A \ or \ Y \ is \ B \]

  \[ \mu_{A \lor B}(x, y) = T[\mu_A(x), \mu_B(y)] \]

- Negation

  \[ \mu_{\neg A}(x) = c[\mu_A(x)] \]

- Variables taking values in a product space: \( X \) with values in \( U \), \( Y \) with values \( V \) \( \Rightarrow \)
  conjunction = cartesian product

  \[ X \ is \ A \ and \ Y \ is \ B \]

  \[ \mu_{A \times B}(x, y) = t[\mu_A(x), \mu_B(y)] \]
**Fuzzy implications**

- **Classical logic:**
  \[(A \Rightarrow B) \iff (B \text{ or not } A)\]

- **Fuzzy logic:**
  - \textit{A and B crisp:}
    \[\text{Imp}(A, B) = T[c(A), B]\]
  - \textit{A and B fuzzy:}
    \[\text{Imp}(A, B) = \inf_x T[c(\mu_A(x)), \mu_B(x)]\]

- **Examples \((c(a) = 1 - a)\):**
  \[
  \begin{array}{|c|c|c|}
  \hline
  T(a, b) = \max(a, b) & I(a, b) = \max(1 - a, b) & \text{Kleene-Diene} \\
  T(a, b) = \min(1, a + b) & I(a, b) = \min(1, 1 - a + b) & \text{Lukasiewicz} \\
  T(a, b) = a + b - ab & I(a, b) = 1 - a + ab & \text{Reichenbach} \\
  \hline
  \end{array}
  \]

- **Residual implications from a t-norm:**
  \[I(A, B) = \sup\{X \mid t(X, A) \leq B\}\]

  \text{Adjunction:} \ t(X, A) \leq B \iff X \leq I(A, B)
Fuzzy reasoning

- Classical logic
  - Modus ponens:
    \[(A \land (A \Rightarrow B)) \Rightarrow B\]
  - Modus tollens:
    \[((A \Rightarrow B) \land \neg B) \Rightarrow \neg A\]
  - Syllogism:
    \[((A \Rightarrow B) \land (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)\]
  - Cuntraposition:
    \[(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)\]
- Fuzzy modus ponens
  - Rule:
    \[\text{if } X \text{ is } A \text{ then } Y \text{ is } B\]
  - Knowledge or observation:
    \[X \text{ is } A'\]
  - Conclusion:
    \[Y \text{ is } B'\]
    \[\mu_{B'}(y) = \sup_{x} t[\mu_{A \Rightarrow B}(x, y), \mu_{A'}(x)]\]
Fuzzy rules

IF \((x \text{ is } A \ \text{AND} \ y \text{ is } B)\) THEN \(z \text{ is } C\)

IF \((x \text{ is } A \ \text{OR} \ y \text{ is } B)\) THEN \(z \text{ is } C\)

... 

\(\alpha\): truth degree of \(x \text{ is } A\)
\(\beta\): truth degree of \(y \text{ is } B\)
\(\gamma\): truth degree of \(z \text{ is } C\)

Satisfaction degree of the rule:

\[Imp(t(\alpha, \beta), \gamma) = T[c(t(\alpha, \beta)), \gamma]\]

\[Imp(T(\alpha, \beta), \gamma) = T[c(T(\alpha, \beta)), \gamma]\]

...
Example in image filtering

IF a pixel is darker than its neighbors
THEN increase its grey level
ELSE IF a pixel is brighter than its neighbors
THEN decrease its grey level
ELSE unchanged

F. Russo et al.
**Possibilistic logic**

- **Possibility measure** on a Boolean algebra of logical formulas: \( \Pi : B \rightarrow [0, 1] \) such that:
  - \( \Pi(\bot) = 0 \)
  - \( \Pi(\top) = 1 \)
  - \( \forall \varphi, \phi, \Pi(\varphi \lor \psi) = \max(\Pi(\varphi), \Pi(\psi)) \)
  - \( \forall \varphi, \Pi(\exists x \varphi) = \sup\{\Pi(\varphi[a|x]), a \in D(x)\} \) (with \( D(x) \) = domain of variable \( x \), and \( \varphi[a|x] \) obtained by replacing occurrences of \( x \) in \( \varphi \) by \( a \))

- **Normalized possibility distribution**: \( \pi : \Omega \rightarrow [0, 1] \) such that \( \exists \omega \in \Omega, \pi(\omega) = 1 \) (\( \Omega \) = set of interpretations)

\[
\Pi(\varphi) = \sup\{\pi(\omega), \omega \models \varphi\}
\]

- **Necessity measure**:

\[
N(\varphi) = 1 - \Pi(\neg \varphi)
\]

\( \forall \varphi, \phi, N(\varphi \land \psi) = \min(N(\varphi), N(\psi)) \)

- **Example**: default rule "if \( A \) then \( B \)"

\[
\Pi(A \land B) \geq \Pi(A \land \neg B)
\]
Possibilistic modus ponens

• Rule:

\[ N(A \Rightarrow B) = \alpha \]

• Knowledge or observation:

\[ N(A) = \beta \]

• Conclusion:

\[ \min(\alpha, \beta) \leq N(B) \leq \alpha \]
Stratified knowledge bases

\[ KB = \{(\varphi_i, \alpha_i), i = 1...n\} \]

\( \alpha_i \): certainty degree or priority of formula \( \varphi_i \)

- Representation by a possibility distribution:
  - for one formula \((\varphi, \alpha)\):
    \[
    \pi_{(\varphi, \alpha)}(\omega) = \begin{cases} 
    1 & \text{if } \omega \models \varphi \\
    1 - \alpha & \text{otherwise}
    \end{cases}
    \]
  - more generally:
    \[
    \pi_{KB}(\omega) = \min_{i=1...n} \{1 - \alpha_i, \omega \models \neg \varphi_i\} = \min_{i=1...n} \max(1 - \alpha_i, \varphi_i(\omega))
    \]

- Inconsistency degree of \( KB \): \( 1 - \max_\omega \pi_{KB}(\omega) \)
- Complete base: either \( KB \vdash \varphi \), or \( KB \vdash \neg \varphi \)
- Ignorance on \( \varphi \): \( KB \nvDash \varphi \) and \( KB \nvDash \neg \varphi \)
  \( \Rightarrow \) simplest possibilistic model:
  \[
  \Pi(\varphi) = \Pi(\neg \varphi) = 1
  \]
Spatial fuzzy objects

$S: \mathbb{R}^3$ or $\mathbb{Z}^3$ in the digital case

$\mu : S \rightarrow [0, 1]$

$\mu(x) =$ degree to which $x$ belongs to the fuzzy object
Definition of membership functions

- often based on heuristics and ad hoc procedures
- from intensity function $I$ or gradient

$$\mu(x) = F_1[I(x)]$$

$$\mu(x) = F_2[\nabla I(x)]$$

- from the output values of some detector
- by introducing imprecision at the boundary of a crisp detection

$$\mu(x) = \begin{cases} 
1 & \text{if } x \in E^n(O) \\
0 & \text{if } x \in S - D^m(O) \\
F_3[d(x, E^n(O))] & \text{otherwise}
\end{cases}$$

- from classification algorithms
How can an operation be extended to the fuzzy case?

Extension principle: $f$ from $\mathcal{U}$ into $\mathcal{V}$

$$\forall y \in \mathcal{V}, \mu'(y) = \begin{cases} 0 & \text{if } f^{-1}(y) = \emptyset, \\ \sup_{x \in S} |y = f(x) \cap \mu(x) & \text{otherwise} \end{cases}$$
How can an operation be extended to the fuzzy case?

Using $\alpha$-cuts:

$$R(\mu) = \int_0^1 R_B(\mu_\alpha) d\alpha$$

$$R(\mu) = \sup_{\alpha \in [0,1]} \min(\alpha, R_B(\mu_\alpha))$$

$$R(\mu) = \sup_{\alpha \in [0,1]} (\alpha R_B(\mu_\alpha))$$

... 

Extension principle based on $\alpha$-cuts:

$$\forall n, R(\mu, \nu)(n) = \sup_{R_B(\mu_\alpha, \nu_\alpha) = n} \alpha$$
How can an operation be extended to the fuzzy case?

Formal translation:

<table>
<thead>
<tr>
<th>set $X$</th>
<th>fuzzy set $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>complementation $X^C$</td>
<td>fuzzy complementation $c(\mu)$</td>
</tr>
<tr>
<td>intersection $\cap$</td>
<td>t-norm $t$</td>
</tr>
<tr>
<td>union $\cup$</td>
<td>t-conorm $T$</td>
</tr>
<tr>
<td>$\exists$</td>
<td>sup</td>
</tr>
<tr>
<td>$\forall$</td>
<td>inf</td>
</tr>
</tbody>
</table>

$\Rightarrow$ easy translation of algebraic and logical expressions
Set relationships

Fuzzy sets \( \Rightarrow \) relations become a matter of degree

- Degree of intersection: 
  \[
  \mu_{int}(\mu, \nu) = \sup_{x \in S} t[\mu(x), \nu(x)]
  \]

- Degree of inclusion:
  \[
  \inf_{x \in S} T[c(\nu(x)), \mu(x)]
  \]

or:
\[
\mu_{int}(\mu, \nu) = \frac{V_n[t(\mu,\nu)]}{\min[V_n(\mu),V_n(\nu)]}
\]
Mathematical morphology

Dilation: operation in complete lattices that commutes with the supremum.
Erosion: operation in complete lattices that commutes with the infimum.
Example with a structuring element:

\[
\delta_B(X) = \{ x \in \mathbb{R}^n \mid B_x \cap X \neq \emptyset \}
\]

\[
\epsilon_B(X) = \{ x \in \mathbb{R}^n \mid B_x \subseteq X \}
\]
Fuzzy mathematical morphology

• Dilation as degree of intersection:

\[ D_\nu(\mu)(x) = \sup \{ t[\nu(y - x), \mu(y)], \ y \in S \} \]

• Erosion as degree of inclusion:

\[ E_\nu(\mu)(x) = \inf \{ I[\nu(y - x), \mu(y)], \ y \in S \} \]

I from a t-conorm \( T \) or by residuation from the t-norm \( t \)

• Opening and closing by composition

• Similar properties as in classical mathematical morphology
Example of fuzzy dilation and erosion
Example of fuzzy dilation and erosion

Initial 2D fuzzy set

Fuzzy dilation

2D fuzzy structuring element

Fuzzy erosion
Application to vascular reconstruction under imprecision
Application to vascular reconstruction under imprecision
Application to vascular reconstruction under imprecision

TRAJECTOIRE DU CATHÉTER LORS DE L'ACQUISITION DES ÉCHOGRAPHIES

ÉCHOGRAPHIES
Application to vascular reconstruction under imprecision

Imprecisions in rotation and translation:

\[ \mu_{V'}(x) = \sup\{ \nu^y_1(x) \mid y \in V_{bin} \} \]

\[ V_f = \bigcup \left\{ D_{D\nu_2(\nu_1^x)}(\{x\}) \mid x \in V_{bin} \right\} = D_{\nu_2}(V'_{f}) \]
Application to vascular reconstruction under imprecision

Result after fusion and decision:
Fuzzy spatial relations

Fuzzy sets → relations become a matter of degree

- Set theoretical relations
- Topology: connectivity, connected components, neighborhood, boundaries, adjacency
- Distances
- Relative direction
- More complex relations: between, along, parallel, around...

Most of them can be defined from mathematical morphology.
Distances between fuzzy sets

Comparison between membership functions

• functional approach: distance from a $L_p$ norm

$$d_p(\mu, \nu) = \left[ \sum_{x \in S} |\mu(x) - \nu(x)|^p \right]^{1/p}$$

$$d_\infty(\mu, \nu) = \max_{x \in S} |\mu(x) - \nu(x)|$$

• set theoretical approach

$$d(\mu, \nu) = 1 - \frac{\sum_{x \in S} \min[\mu(x), \nu(x)]}{\sum_{x \in S} \max[\mu(x), \nu(x)]}$$

• ...

• adapted to cases where the fuzzy sets to be compared represent the same structure or a structure and a model of it
  • model-based object recognition
  • case-based reasoning
Distances between fuzzy sets

Taking the spatial distance $d_E$ into account

- geometrical approach
  - space of dimension $n + 1$
  - fuzzification: $d(\mu, \nu) = \int_0^1 D(\mu_\alpha, \nu_\alpha) d\alpha$
- weighting
  \[
d(\mu, \nu) = \frac{\sum_{x \in S} \sum_{y \in S} d_E(x, y) \min[\mu(x), \nu(y)]}{\sum_{x \in S} \sum_{y \in S} \min[\mu(x), \nu(y)]}
  \]
- fuzzy number
  \[
d(\mu, \nu)(r) = \sup_{x, y, d_E(x, y) \leq r} \min[\mu(x), \nu(y)]
  \]
- morphological approach
Distances between fuzzy sets: morphological approach

Expression of distances (minimum, Hausdorff...) in morphological (i.e. algebraic) terms ⇒ easy translation to the fuzzy case
**Minimum (nearest point) distance distribution**

\[ d_N(X, Y) = \inf \{ n \in \mathbb{N}, X \cap D^n(Y) \neq \emptyset \} = \inf \{ n \in \mathbb{N}, Y \cap D^n(X) \neq \emptyset \} \]

Degree to which the distance between \( \mu \) and \( \mu' \) is less than \( n \) (distance distribution):

\[ \Delta_N(\mu, \mu')(n) = f[\sup_{x \in S} t[\mu(x), D^n_{\mu'}(\mu')(x)], \sup_{x \in S} t[\mu'(x), D^n_{\mu}(\mu)(x)]] \]

Hausdorff distance: similar equations
Minimum (nearest point) distance density

\[ d_N(X, Y) = n \iff D^n(X) \cap Y \neq \emptyset \text{ and } D^{n-1}(X) \cap Y = \emptyset \]

\[ d_N(X, Y) = 0 \iff X \cap Y \neq \emptyset \]

Degree to which the distance between \( \mu \) and \( \mu' \) is equal to \( n \) (distance density):

\[
\delta_N(\mu, \mu')(n) = t \left[ \sup_{x \in S} t[\mu'(x), D^n_\nu(\mu)(x)], c \left[ \sup_{x \in S} t[\mu'(x), D^{n-1}_\nu(\mu)(x)] \right] \right]
\]

\[
\delta_N(\mu, \mu')(0) = \sup_{x \in S} t[\mu(x), \mu'(x)]
\]

Hausdorff distance: similar equations
Properties of fuzzy morphological distances

- fuzzy numbers
- positive: support included in $\mathbb{R}^+$
- symmetrical with respect to $\mu$ and $\mu'$
- if $\mu$ is normalized $\delta_N(\mu, \mu')(0) = 1$ and $\delta_N(\mu, \mu')(n) = 0$ for $n > 1$
- $\delta_H(\mu, \mu')(0) = 1$ implies $\mu = \mu'$ for $T$ being the bounded sum $(T(a, b) = \min(1, a + b))$, while it implies $\mu$ and $\mu'$ crisp and equal for $T = \max$
- triangular inequality not satisfied in general
**Fuzzy distance: example**

Distance between a model of v2 and:

<table>
<thead>
<tr>
<th></th>
<th>cn2</th>
<th>v2</th>
<th>v1</th>
<th>cn1</th>
<th>p1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
<td><img src="image5.png" alt="Graph" /></td>
</tr>
</tbody>
</table>
**Spatial representation of knowledge about distance**

- **Crisp case:** $B$ should lay between a distance $n_1$ and a distance $n_2$ of $A$ ⇒ region of interest for $B$: $D^{n_2}(A) \setminus D^{n_1-1}(A)$

- **Fuzzy case:** approximate distance as a fuzzy interval ⇒ two structuring elements:

$$
\nu_1(x) = 1 - \mu_n(d_E(x, 0)) \text{ if } d_E(x, 0) \leq n_1, \quad 0 \text{ else}
$$

$$
\nu_2(x) = 1 \text{ if } d_E(x, 0) \leq n_2, \quad \mu_n(d_E(x, 0)) \text{ else}
$$

**Fuzzy region of interest:**

$$
\mu_{distance} = t[D_{\nu_2}(\mu), 1 - D_{\nu_1}(\mu)]
$$
Spatial representation of knowledge about distance: example
What is relative direction?

- Object B
- Reference object (R)

- pi
- 0

- Spatial Reasoning – p.56/80
Directional relative position

• fuzzy “landscape” around the reference object $R$ as a fuzzy set such that the membership value of each point corresponds to the degree of satisfaction of the spatial relation: dilation by a fuzzy structuring element

$$\mu_\alpha(R) = D_\nu(\mu_R)$$

“to the right of”

• evaluation of the relation of another object $A$ to $R$ using fuzzy pattern matching (or average value):

$$\Pi^R_\alpha(A) = \sup_{x \in \mathcal{S}} t[\mu_\alpha(R)(x), \mu_A(x)] \quad N^R_\alpha(A) = \inf_{x \in \mathcal{S}} T[\mu_\alpha(R)(x), 1 - \mu_A(x)]$$
Directional relative position: properties

- evaluation in the spatial domain, and with richer information (compared to other fuzzy methods)
- the possibility has a symmetry property
- invariance with respect to translation, rotation and scaling, for 2D and 3D objects (crisp and fuzzy)
- when the distance between the objects increases, the objects are seen as points
- nice behavior in case of concavities
Directional relative position: example
Example of a complex relation: the heart is "between" the lungs
Mathematical morphology: a unifying framework

Mathematical morphology = unified framework for

- spatial knowledge representation
- spatial reasoning

in different settings:

- quantitative
- semi-quantitative / semi-qualitative (fuzzy sets, rough sets)
- qualitative and symbolic (logics)

by providing links between:

- mathematical structures
- and linguistic descriptions
Mathematical morphology in a nutshell

- Theory of space
- Strong algebraic structure
- Topology and metrics
- A lot of developments in image processing, image interpretation, structural recognition...
- Provides tools for the two main components of spatial reasoning:
  - knowledge representation about spatial entities and spatial relationships
  - reasoning on spatial entities and spatial relationships
- Three important features:
  - axiomatization
  - expressiveness
  - associated algorithms (digital spaces)
Reasoning with mathematical morphology

- Chaining operations (image interpretation, recognition)
- Fusion of spatial relations (ex: structural recognition)
- Links with logics
  - propositional logics:
    - elegant tools for revision, fusion, abduction
    - links with mereotology, "egg-yolk" structures, logics of distances, nearness logics, linear logics, logics of convexity...
  - modal logics:
    - $(\diamond, \square) = (\delta, \varepsilon)$
    - symbolic and qualitative representations of spatial relations
  - fuzzy logic
Example: dilation and erosion of a formula

Structuring element $B$: relation between worlds

Dilation:

$$Mod(D_B(\varphi)) = \{ \omega \in \Omega \mid B(\omega) \cap Mod(\varphi) \neq \emptyset \}$$
Example: dilation and erosion of a formula

Structuring element $B$: relation between worlds

Dilation:

$\text{Mod}(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap \text{Mod}(\varphi) \neq \emptyset\}$
Example: dilation and erosion of a formula

Structuring element $B$: relation between worlds

Dilation:

$$Mod(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap Mod(\varphi) \neq \emptyset\}$$

Erosion:

$$Mod(E_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \models \varphi\}$$
Dilation and erosion as modal operators

Structuring element $B$: accessibility relation $R(\omega, \omega')$ iff $\omega' \in B(\omega)$

$$\mathcal{M}, \omega \models \Box \varphi \iff \forall \omega' \in \Omega, R(\omega, \omega') \Rightarrow \mathcal{M}, \omega' \models \varphi$$

$$\iff \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \models \varphi$$

$$\iff B(\omega) \models \varphi$$

$$\mathcal{M}, \omega \models \Diamond \varphi \iff \exists \omega' \in \Omega, R(\omega, \omega') \text{ et } \mathcal{M}, \omega' \models \varphi$$

$$\iff \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \cap \text{Mod}(\varphi) \neq \emptyset$$

$$\iff B(\omega) \cap \text{Mod}(\varphi) \neq \emptyset$$

$$\Box \varphi \equiv E_B(\varphi) \quad \Diamond \varphi \equiv D_B(\varphi)$$

Spatial interpretation: restriction or necessary region / extension or possible region
Example: topological relations

Tangential part: \( \varphi \rightarrow \psi \) and \( \Diamond \varphi \land \neg \psi \nrightarrow \bot \) (or \( \varphi \rightarrow \psi \) and \( \varphi \land \neg \Box \psi \nrightarrow \bot \))

Non tangential part: \( \varphi \rightarrow \psi \) and \( \Diamond \varphi \rightarrow \psi \) (or \( \varphi \rightarrow \psi \) and \( \varphi \rightarrow \Box \psi \))

Adjacency: \( \varphi \land \phi \rightarrow \bot \) and \( \Diamond \varphi \land \psi \nrightarrow \bot \) and \( \varphi \land \Diamond \psi \nrightarrow \bot \)

Links with mereotopology: for instance, a proper tangential part can be defined as

\[
TPP(X, Y) = P(X, Y) \land \neg P(Y, X) \land \neg P(D(X), Y)
\]
Model based image understanding

Models of various types:

- acquisition properties (geometry, noise statistics...)
- shape
- appearance
- spatial relations
- ...

Important

- to use available knowledge
- to guide the image exploration, for segmentation, recognition, scene understanding
- to solve ambiguities
- to deal with imprecision
- ...

Issues:

- semantic gap
- imprecisions and uncertainties
- pathological cases
- algorithms
Semantic gap and symbol grounding

- **Symbol grounding** = How is symbol meaning to be grounded in something other than just more meaningless symbols? (Harnad)

- **Anchoring** = creating and maintaining the correspondence between symbols and sensor data that refer to the same physical object (Saffiotti & Coradeschi)

- **Semantic gap** = lack of coincidence between the information that one can extract from the visual data and the interpretation of these data by a user in a given situation (Smeulders)

⇒ fuzzy representations in concrete domains (parameter space, image space...)
Two main questions in structural recognition in images

• given two objects (possibly fuzzy), assess the degree to which a relation is satisfied

![Diagram of two objects](object.png)

Object B

Reference object (R)

• given one reference object, define the area of the space in which a relation to this reference is satisfied (to some degree)

![Diagram of spatial reasoning](spatial.png)
**Example in brain imaging**

- **Concepts:**
  - **brain**: part of the central nervous system located in the head
  - **caudate nucleus**: a deep gray nucleus of the telencephalon involved with control of voluntary movement
  - **glioma**: tumor of the central nervous system that arises from glial cells
  - ...

- **Spatial organization:**
  - the **left caudate nucleus** is **inside** the **left hemisphere**
  - it is **close** to the **lateral ventricle**
  - it is **outside (left of)** the **left lateral ventricle**
  - it is **above** the **thalamus**, etc.
  - ...

- **Pathologies**: relations are quite stable, but more flexibility should be allowed in their semantics
Integration of ontologies, spatial relations and fuzzy models

Symbolic knowledge

- Generic knowledge
  - Brain anatomy ontology + brain structural description
- Knowledge of specific cases
  - Brain tumor ontology

Ontology-based segmented image database

- Healthy cases
- Pathological cases
  - Infiltrating tumors
  - Circumscribed tumors

Graph based representation of the generic model

- Fuzzy modeling of spatial relations

Learning procedure

- Step 1: learning spatial relations (adjacency, distance, orientation) of the generic model using healthy cases
- Step 2: deducing stable relations for each class of pathologies

Dealing with a specific case

- Generic model adaptation using knowledge of specific case and results of the learning procedure
- Graph based propagation process to update the graph and to represent the tumor impact on the surrounding structures

Enrichment of the database
Ontology of the anatomy (FMA) enriched with an ontology of spatial relations
Learning spatial relations

caudate nucleus
putamen
thalamus
ventricles
tumor

\[ f(d) \]

0 \quad m \quad m + 2\sigma \quad \text{distances}

Spatial Reasoning – p.73/80
Learning spatial relations

- Caudate nucleus
- Lateral ventricle
- Thalamus

- Normal cases
- Pathological cases

- Distances (mm)

- A normal case
- A case of a class with low impact on the internal structures
- A case of a class with strong impact
Segmentation and recognition of some internal structures on a normal case (O. Colliot et al.):

- fusion of spatial relations (given by the model) to previously recognized objects
- deformable model constrained by spatial relations
Examples in pathological cases (H. Khotanlou, J. Atif, et al.)

- putamen (3)
- tumor (1)
- thalamus (2)
- caudate nucleus (3)
- tumor (1)
- lateral ventricles (2)
Best segmentation path (G. Fouquier et al.)

Generic Knowledge

Reference Structures

Model Graph

Specialized Graph

Image to segment

Saliency Map

: already segmented

: to segment

A priori knowledge

Visual information

Results

1st level

1st level

Spatial Reasoning – p.76/80
Best segmentation path (G. Fouquier et al.)

Evaluation and backtracking

Graph at step $i$:
- Node 1: Representable
- Node 2: Not representable
- Node 3: Segmented
- Node 4: Not segmented

Generation of localizations:
- Localization of 3 (from 1 and 2)
- Localization of 4 (from 1)

Segmentation and update:
- Node 1: Representable
- Node 2: Not representable
- Node 3: Segmented
- Node 4: Not segmented

Evaluation of spatial consistency:
- Evaluation consistent
- Evaluation not consistent

Localization of 1 (from 3)
Localization of 2 (from 3)
Best segmentation path (G. Fouquier et al.)

Some results
Global approach based in CSP (O. Nempont et al.)

Brain MRI

- Inhomogeneities correction
- Brain surface extraction

Initial domains

- Brain
- Inhomogeneities correction
- Brain surface extraction
- Initial domains

Structural model

Constraint propagation

- Update domains
- until convergence
- Select a constraint and compute the associated operator

Final segmentation

- Metric definition
- Domains obtained by the propagation algorithm
- Minimal surface extraction
- Final segmentation
Global approach based in CSP (O. Nempont et al.)

Constraint Satisfaction Problem (CSP):

- Constraint network = \((\chi, D, C)\)
- \(\chi\) = variables
- \(D\) = set of associated domains
- \(C\) = constraints involving variables of \(\chi\), relations on the variable domains
- Propagation of constraints:
  - Locally consistent constraint if all values of the domains can satisfy the constraint.
  - Suppression of inconsistent values: \((\chi, D, C) \rightarrow (\chi, D', C)\)
  - Propagator = operator reducing the domains according to a constraint.
Global approach based in CSP (O. Nempont et al.)

- Variables = anatomical structures.
- Domain of a variable = interval of fuzzy sets $[A, \overline{A}]$.
- Example of constraint (1): inclusion

\[
C_{A,B}^{\text{in}} : \mathcal{D}(A) \times \mathcal{D}(B) \rightarrow \{0, 1\}
\]

\[
(\mu_1, \mu_2) \mapsto \begin{cases} 
1 & \text{if } \mu_1 \leq \mu_2, \\
0 & \text{otherwise.}
\end{cases}
\]

- Associated propagator:

\[
\langle A, B; (A, \overline{A}), (B, \overline{B}); C_{A,B}^{\text{in}} \rangle
\]

\[
\langle A, B; (A, \overline{A} \land \overline{B}), (B \lor A, \overline{B}); C_{A,B}^{\text{in}} \rangle
\]
Global approach based in CSP (O. Nempont et al.)

- Example of constraint (2): directional relation

\[ C_{A,B}^{\text{dir} \nu} : \mathcal{D}(A) \times \mathcal{D}(B) \to \{0, 1\} \]

\[ (\mu_1, \mu_2) \mapsto \begin{cases} 
1 & \text{if } \mu_2 \leq \delta_\nu(\mu_1), \\
0 & \text{otherwise.} 
\end{cases} \]

- Associated propagator:

\[
\langle A, B; (A, \overline{A}), (B, \overline{B}); C_{A,B}^{\text{dir} \nu} \rangle \\
\langle A, B; (A, \overline{A}), (B, \overline{B} \land \delta_\nu(A)); C_{A,B}^{\text{dir} \nu} \rangle 
\]

- Other constraints: distance, partition, connectivity, adjacency, volume, contrast...

- Ordering of the propagators and iteration application.
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Result: example
Examples in remote sensing
Examples in remote sensing

(a) Example image.

(b) Labeled image: The blue regions represent the sea, the red and orange represent ships or boats and the yellow regions represent the docks.

(c) Concept hierarchy $T_C$ in the context of harbors.

(d) Conceptual graph representing the spatial organization of some elements of Figure 5.8(b).