Digital Representations

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Content

- Tessellations
- Digital topology
- Representation of geometrical entities
- Distance function
Digital image:
- representation as an array
- sampling from \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) into \( \mathbb{Z}^2 \) or \( \mathbb{Z}^3 \)

Two possible approaches to process points in \( \mathbb{Z}^n \):
- embed \( \mathbb{Z}^n \) in \( \mathbb{R}^n \), then apply operations and transformations in the continuous space
- definition of operations and transformations directly in the digital space
  - definitions?
  - preservation of the expected effects?
  - preservation of the properties?
A simple example: rotation
\[(x', y') = R(x, y)\]

Issues:
- \((x, y) = \text{integer coordinates} \Rightarrow (x', y')?\)
- Computation?
- Properties?

\(\pi/4 \text{ rotation: } x' = (x - y)\sqrt{2}/2 \quad y' = (x + y)\sqrt{2}/2\)

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- Direct transformation
- Inverse transformation (closest point interpolation)
Topology and resolution

Curve digitization

Tessellations

Tessellation = partition of the continuous space $\mathbb{R}^n$ into elementary cells

Constraints:
- physical sensors (regularity)
- usage of the representation (regularity, simplicity)

1. **Tessellation from point distribution**
   - distribution of points $P$
     - regular $\Rightarrow$ classical grids
     - irregular $\Rightarrow$ Voronoï diagram
   - attribution of a cell $V_P$ to each point

2. **Tessellation from cell juxtaposition**
   - prior definition of a cell model $V_P$
   - juxtaposition of $V_P$ so as to build a partition
   - constraints:
     - $V_P$ convex and regular
     - vertices in contact with other vertices only
Regular distributions
Irregular distributions: Voronoï diagram
An excluded configuration
Regular tessellations of the plane

triangular  square  hexagonal
Semi-regular tessellations
Examples

Admissible:

Non-admissible:
Admissible semi-regular tessellations
A complex tessellations (Escher)…
Duality between tessellations and mesh (or grid)
Digital topology

- Classical topology in a countable set of points:
  - every point is an open set of the topology
  - not well adapted to the representation of connected sets
- Direct definition of a topological basis
  - possible on triangular and square tessellations, not on hexagonal tessellations
  - depends on point localization
  - does not satisfy Jordan theorem
- Direct definition of elementary neighborhood
  - digital connectivity
  - image = graph
Direct definition of a topological basis: examples
Elementary neighborhood
4-connectivity graph

- vertices of the graph
- edges defining the neighbors and the connectivity
Different grids and associated connectivities
Neighbor coordinates on a hexagonal grid

for $j$ even: $(i - 1, j - 1), (i, j - 1), (i - 1, j), (i + 1, j), (i - 1, j + 1), (i, j + 1),$

for $j$ odd: $(i, j - 1), (i + 1, j - 1), (i - 1, j), (i + 1, j), (i, j + 1), (i + 1, j + 1)$.
Elementary neighborhood on a 3D cubic grid
**Path and connected component**

- **4-connected path** = sequence of points $(i_k, j_k)_{1 \leq k \leq n}$ such that:

  $$\forall k, 1 \leq k < n, \ |i_k - i_{k+1}| + |j_k - j_{k+1}| \leq 1$$

- **8-connected path** = sequence of points $(i_k, j_k)_{1 \leq k \leq n}$ such that:

  $$\forall k, 1 \leq k < n, \ \max(|i_k - i_{k+1}|, |j_k - j_{k+1}|) \leq 1$$

- **4-connected component** = set of points $S$ such that for any $(P, Q)$ in $S$, there exists a 4-connected path from $P$ to $Q$, included in $S$, and maximal for this property.

- **8-connected component** = set of points $S$ such that for any $(P, Q)$ in $S$, there exists a 8-connected path from $P$ to $Q$, included in $S$, and maximal for this property.
4-connected path
8-connected path

- - - - - -

one 8-connected component
two 4-connected components

○ background
● objects
two 4-connected components
one 8-connected component
Topological paradox
Jordan theorem

- Continuous case: any simple and closed curve divides the space into two connected components, one inside the curve and one outside.
- Digital case: duality between 4-connectivity and 8-connectivity on a square grid
  - 4-connected curve ⇔ 8-connected background,
  - 8-connected curve ⇔ 4-connected background.
- Digital case on a hexagonal grid: 6-connectivity for both objects and background (no topological problem).
- Extension to 3D.
Some definitions in the digital case

- **Simple and closed 4-connected path**: 4-connected path \((A_0, ..., A_n)\) such that \(n \geq 4\), \(A_i = A_j\) iff \(i = j\), and \(A_i\) 4-neighbor of \(A_j\) iff \(i = j \pm 1 [n + 1]\)

- **Horizontal half-line from** \(M = (a, b)\) :

  \[ H_M = \{(a + k, b), k = 0, 1, 2...\} \]

- **Inside** \(A\): set of points \(M\) such that \(H_M\) crosses \(A\) an odd number of times.

- **Outside** \(A\): set of points \(M\) such that \(H_M\) crosses \(A\) an even number of times.

⇒ proof of the digital version of Jordan theorem.
Cellular complexes

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Connected component labeling

**Diagram:**

- Initial labels:
  - 0
  - 1
  - 2
  - 3
  - 4
  - 5
  - 6

- Final labels:
  - 2
  - 2
  - 2
  - 2
  - 2
  - 2

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Example of topological characteristic: Euler number

- Number of connected components \( N_{cc} \)
- Number of holes \( N_t \)
- Euler number \( E = N_{cc} - N_t \)

8-connected objects and 4-connected holes: \( N_{cc} = 1 \) and \( N_t = 2 \), hence \( E = -1 \)

4-connected objects and 8-connected holes: \( N_{cc} = 1 \) and \( N_t = 1 \), hence \( E = 0 \)
8-connected objects and 4-connected holes:

\[ E = v - e - d + t - q \]

4-connected objects and 8-connected holes:

\[ E = v - e + q \]
Digital geometry

- How to go from the continuous domain to the digital one, and vice-versa?
- How to represent a geometric entity on a digital grid, while preserving its properties?
- Which are the continuous representations of a discrete one?
- Which are the exact intersections of a continuous representation and the digital grid?

Example of straight lines or segments
Digitization of a continuous line

- Semi-open square
- Digital representation of the continuous line
Digital representation of the continuous line
Cf Bresenham algorithm

Digital representation of the continuous line
Cord property
$S$ satisfies the cord property iff:

$$\forall (P, Q) \in S, \forall R \in [P, Q], \exists T \in S, d_\infty(T, R) < 1$$

with $d_\infty((x, y), (x', y')) = \max(|x - x'|, |y - y'|)$
Syntactic characterization

- only two “neighbor” directions
- for one direction: sections of length 1
- for the other direction: sections of length $n$ or $n + 1$
Analytical digital straight lines

\[ y = ax + b \]

Intersections with the grid?

- **Condition for non-empty intersection:**
  \[ a = \frac{p}{q} \]
  where \( p \) and \( q \) are integers, co-prime, and:
  \[ p \leq q \leq N \]

- **Farey sequence:**
  - Image of size \( N \times N \) and slope less than 1
  - \( \Rightarrow \) possible slopes = Farey sequence of order \( N \): \( F(N) \) (cardinality approximately \( 3N^2/\pi^2 \))
  - Recursive construction (\( \frac{m+m'}{n+n'} \) between \( \frac{m}{n} \) and \( \frac{m'}{n'} \))
Example for $N = 4 \ (a \leq 1) : \ F(N) = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$
Length of a digital straight segment

\[ a^2 + b^2 = L \]

with \( a \) and \( b \) integer
Voronoï diagram

- Useful representation for shapes, image structures...
- Seeds\(\{P_1, P_2, \ldots, P_n\}\)
- Voronoï cells:

\[
V(P_i) = \{ P \in \mathbb{R}^2 \mid \forall j, 1 \leq j \leq n, \ d(P, P_i) \leq d(P, P_j) \}
\]

- For the Euclidean distance: \(V(P_i) = \text{convex polygon}\)
Delaunay triangulation
Duality
Properties:

- if there are no 4 co-circular points, every Voronoi vertex is equidistant of exactly 3 seeds
- any Voronoi vertex is the center of a circle (called Delaunay circle) passing through 3 seeds and containing no other seed
- $V(P_i)$ is non-bounded iff $P_i$ belongs to the convex hull of the $P_j$s
Incremental construction

× new seed
----- new Voronoï edges
A few geometric applications:

- minimal distance between two sets of points
- triangulation such that the circle circumscribed to each triangle is empty
- convex hull of a set of points
- ...

Discrete distances

- $\mathcal{P} = \{\vec{p}_1, \ldots, \vec{p}_m\}$ set of vectors generating a graph
- Associated length $d_i$
- Conditions:
  - $\vec{p}_i \in \mathcal{P} \implies -\vec{p}_i \in \mathcal{P}$
  - $\vec{p}_i \in \mathcal{P}, \lambda \vec{p}_i \in \mathcal{P} \implies \lambda = \pm 1$
  - $||\vec{p}_i|| = ||\vec{p}_j|| \implies d_i = d_j$

Distance between to vertices / points $x$ and $y$:

$$d(x, y) = \frac{1}{s} \min \left\{ \sum_{i=1}^{m} n_i d_i \mid n_i \in \mathbb{N}, \sum_{i=1}^{m} n_i \vec{p}_i = \vec{x}\vec{y} \right\}$$

$s$: scale factor
Binary image with objects $O \rightarrow$ distance map image where the value at $x$ is $d(x, O) = \min_{y \in O} d(x, y)$

- global concept $\Rightarrow$ local computation by propagating local distances
- requirements:
  - good approximation of the Euclidean distance
  - fast algorithms
Masks representing local distances

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 4 & 3 & 4 & 11 & 11 \\
1 & 0 & 1 & 3 & 0 & 3 & 5 & 0 & 5 \\
1 & 1 & 1 & 4 & 3 & 4 & 11 & 7 & 5 & 7 & 11 \\
\end{array}
\]

\[
a \quad b \quad c \quad d
\]
Parallel algorithm

- $f^k$: image at iteration $k$
- $g$: mask
- $f^0$: points of objects set to 0, points of the background set to $+\infty$

$$f^k(x) = \min\{f^{k-1}(y - x) + g(y), \ y \in \text{support}(g)\}$$

- number of iterations: depends on image size, object size, shape...
- two images in memory
- can be adapted for any grid (2D or 3D) and any mask
- can be parallelized
Sequential algorithm

- two scans of the image, in opposite directions
- masks $g_1$ and $g_2$ containing the points already examined according to the scan direction (+ origin)
- $f^0$: points of objects set to 0, points of the background set to $+\infty$
- for $k = 1, 2$

$$f^k(x) = \min\{f^{k-1}(x), f^k(y-x) + g_k(y), \ y \in \text{support}(g_k)\}$$

- fast algorithm
- only one image in memory
- can be adapted for any grid (2D or 3D) and any mask
- recursive
Algorithms based on object contours

- Using chains:
  - contour chaining
  - point displacement and rewriting rules
  - adjustments

- Using queues
  - FIFO initialized with contour points
  - for each point of the queue: computation of the neighbors, distance value increment, and neighbors added in the queue
  - applies in 3D as well
Distance map (4-connectivity mask)
Distance map (8-connectivity mask)
Distance map (6-connectivity mask)
Example on a binarized biological image
Example on a coffee bean image
Voronoi diagram from a discrete distance
Applications

- Distance computation (e.g. model-based object recognition, scene understanding)
- Registration
- Mathematical morphology operations on binary images
Some references


