Tomographic Reconstruction

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Content

- Principle of tomography
- Backprojection
- Analytical methods
- Algebraic methods
- Regularization
- Extensions
CT acquisition systems

1 detector

a few detectors

n detectors

N detectors (fixed)
Principle of X-ray tomography

**Attenuation** for a monochromatic X-ray beam:

\[
I = I_0 \exp\left( - \int_{-\infty}^{+\infty} f dv \right)
\]

\(f(x, y) = \text{attenuation at point } (x, y) = \text{function to be reconstructed}\)

**Acquisition of projections**
Other modalities

- nuclear imaging (SPECT, PET)
- electric impedance tomography
- ...

Different physical principles - Similar reconstruction problems.
Radon transform

\[ R[f](u, \theta) = p_{\theta}(u) \]
\[ = \int_{D_\theta} f(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta) dv \]

Note that \( p_{\theta}(u) = p_{\theta + \pi}(-u) \)

Reconstruction:

\[ \{ p_{\theta}(u), \theta \in [0, \pi[, u \in \mathbb{R} \} \rightarrow \{ f(x, y), (x, y) \in \mathbb{R}^2 \} \]
Backprojection

- of a projection:

\[ h_\theta(x, y) = p_\theta(x \cos \theta + y \sin \theta) \]

(value at \((x, y)\) of the projection of angle \(\theta\) at point on which \((x, y)\) projects)

- of all projections:

\[ B[p](x, y) = \int_{0}^{\pi} p_\theta(x \cos \theta + y \sin \theta) d\theta \]
Inversion - 1

Projection theorem

\[ FT[p_\theta](U) = FT[f](U \cos \theta, U \sin \theta) \]

\((FT = \text{Fourier transform})\)

\(\Rightarrow\) Reconstruction scheme:

\[
\{p_\theta(u)\} \\
\downarrow \\
\{FT[p_\theta](U)\} \\
\downarrow \\
FT[f](X, Y) \\
\downarrow \\
f \text{ using inverse FT}
\]

\(= \text{Direct inversion (1D FT + 2D IFT)}\)
Backprojection theorem

\[ B[p](x, y) = (f \ast h)(x, y) \]

with \( h(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \)

⇒ reconstruction using deconvolution:

\[ f = IFT\left[ FT(B[p]) \cdot \rho \right] \]

with \( \rho(X, Y) = \sqrt{X^2 + Y^2} \)

(2D filtering and FT)
Filtered backprojection

\[ f = B[\tilde{p}] \]

with \( \tilde{p}_\theta = \text{IFT} [ FT[p_\theta](U) \cdot |U| ] \)

⇒ reconstruction scheme:

- filtering of projections (1D)
- \( \downarrow \)
- backprojection of filtered projections

In practice: filtering using \( H(U) = |U| \cdot W(U) \)

\( W(U) \): low-pass filter

⇒ compromise spatial resolution / noise
**Ideal continuous and infinite case:**
- domain $\mathbb{R}^2$
- continuous function $f$
- continuous $p_\theta$, known $\forall \theta \in [0, \pi[$

**In practice:**
- $p_\theta$ for a finite number of $\theta_k$ (acquisition system)
- $p_{\theta_k}$ known at discrete points $u_l$ (detectors)
- reconstruction of $f$ at a finite number of points (algorithms and computation)

**reconstruction:**
\[
\{p_{\theta_k}(u_l), 0 \leq l < NP, 0 \leq k < M\} 
\rightarrow \{f(x_i, y_j), 0 \leq i < N, 0 \leq j < N\}
\]

with:
\[
\theta_k = k\Delta\theta, \quad \Delta\theta = \frac{\pi}{M}, \quad u_l = ld
\]
\[
x_i = i\Delta x, \quad y_j = j\Delta y
\]
Two classes of methods in the discrete case

- **Analytical methods:**
  - discrete operators
  - digitization of inversion formulas

- **Algebraic methods:**
  - digitization of projection equation
  - solving a linear system of equations
Discrete analytical methods

Discrete operators

- **DFT:**

\[
F_k = \sum_{l=0}^{N-1} f_l \exp\left(\frac{-2\pi}{N} l k\right)
\]

spectrum overlap issue ⇒ Shannon
⇒ hypothesis of limited spectrum

- **Discrete backprojection:**

\[
B[p](x_i, y_j) = \frac{\pi}{M} \sum_{k=0}^{M-1} p_{\theta_k}(x_i \cos \theta_k + y_j \sin \theta_k)
\]

\[x_i \cos \theta_k + y_j \sin \theta_k \neq u_l\]

↓

interpolation

or pre-interpolation of \(p_{\theta}\)
Reconstruction using direct inversion

\[ DFT[p_{\theta_k}](U_l) = DFT[f](U_l \cos \theta_k, U_l \sin \theta_k) \]

⇒ reconstruction scheme:

\[
\begin{align*}
\{ p_{\theta_k}(u_l) \} \quad &\quad \Downarrow \quad \quad \Downarrow \quad \quad \Downarrow \\
\{ DFT[p_{\theta}](U_l) \} \quad &\quad \Downarrow \quad \quad \Downarrow \\
estimation of \ FT[f] \text{ in polar coordinates} \quad &\quad \Downarrow \\
interpolation polar / Cartesian coordinates \quad &\quad \Downarrow \\
f \text{ using IDFT} \quad &\end{align*}
\]
Sampling

Projections: \( d \Rightarrow B = \frac{1}{2d} \)

Fourier domain:

- radial: \( \rho = \frac{2B}{NP} = \frac{1}{dNP} \)
- azimuthal: \( \varepsilon = \rho \Rightarrow \Delta \theta = \frac{2}{NP} \)
- or: \( \varepsilon' = \rho \frac{3B}{4} = \frac{3}{4} B \Delta \theta = \frac{2B}{NP} \Rightarrow \Delta \theta = \frac{8}{3NP} \)
  \( \Rightarrow M \) (number of projections)
Reconstruction using 2D deconvolution

- discrete backprojection of all projections
- deconvolution using DFT
  - on a larger image (to avoid aliasing)
  - filter + window (to cope with noisy data)
Reconstruction using discrete filtered backprojection

Filtering of projections:

\[ B = \frac{1}{2d} \]

\[ \downarrow \]

\[ FT(k)(U) = \begin{cases} |U| & \text{if } |U| < B \\ 0 & \text{otherwise} \end{cases} \]

Ramachandran and Lakshminarayanan:

\[ FT(\hat{k})(U) = |U| \text{Rect}_B(U) \]

\[ \Rightarrow \hat{k}(u) = 2B^2 \left( \frac{\sin(2\pi Bu)}{2\pi Bu} \right) - B^2 \left( \frac{\sin(\pi Bu)}{\pi Bu} \right)^2 \]

\[ \Rightarrow k\left( \frac{m}{2B} \right) = \begin{cases} B^2 & \text{if } m = 0 \\ 0 & \text{if } m \text{ even and } \neq 0 \\ -\frac{4B^2}{m^2\pi^2} & \text{if } m \text{ odd} \end{cases} \]
Shepp and Logan:

\[ FT(\hat{k})(U) = |U| \text{Rect}_B(U) \frac{\sin(\frac{\pi U}{2B})}{\frac{\pi U}{2B}} \]

\[ \Rightarrow k(\frac{m}{2B}) = \frac{-4B^2}{\pi^2(4m^2 - 1)} \]

Other windows: cosinus, Hamming, etc.

Implementation:
- discrete convolution
- or in the Fourier domain (using FFT)

Advantages:
- 1D computations
- every projection can be processed as soon as it is acquired
Algebraic reconstruction methods

$f$ written as:

\[
f(x, y) = \sum_{i=1}^{n} f_i \varphi_i(x, y)
\]

Most used basis: pixel basis

\[
\varphi_i(x, y) = \begin{cases} 
1 & \text{if } (x, y) = \text{pixel } i \\
0 & \text{otherwise}
\end{cases}
\]

\[
p_j = \sum_{i=1}^{n} R_{ji} f_i
\]

\[
p = Rf
\]

with \( p_j = p_{\theta_k}(u_l) \) and \( R_{ji} = \int \varphi_i(u_l \cos \theta_k - v \sin \theta_k, u_l \sin \theta_k + v \cos \theta_k) dv \)
- $p$: measurement vector (all projection values)
  
  size $m = M \times NP = \text{number of projections} \times \text{number of points} / \text{projection}$

- $f$: vectorized image values (to be computed)

  size $n = N \times N = \text{number of pixels}$

- $R$: projection matrix

  size $m \times n$

  depends only on the acquisition design

  
  $$R_{ji} = \begin{cases} 1 \text{ if ray } j \text{ meets pixel } i \\ 0 \text{ otherwise} \end{cases}$$

  or:

  $$R_{ji} \propto \text{overlap between ray } j \text{ and pixel } i$$
Problems with direct inversion:

- Size of the matrix (at least $250000 \times 250000$)
- A lot of 0
- Noise

⇒ Iterative methods

- ART: correction of $f_i$ by using one projection at each iteration
- SIRT: correction of $f_i$ by using all rays passing through $x_i$
\[ f_i^{(k)} = f_i^{(k-1)} + \frac{p_j - R_j f^{(k-1)}_j}{\| R_j \|^2} \]

\[ j = k[m] + 1 \]
Noisy case

⇒ oscillations
\[ f_i^{(k)} = f_i^{(k-1)} + \frac{\sum_j p_j}{\sum_j \sum_i R_{ji}} - \frac{\sum_j R_j f^{(k-1)}_i}{\sum_j \| R_j \|^2} \]
Limitations

- **Physics:**
  - non-monochromatic, non infinitely thin rays
  - beam hardening
  - scattering
  - patient’s movements

- **Incomplete data:**
  - low number of projections (e.g. cardiac imaging)
  - noisy data

⇒ ill-posed problem
Well-posed problem (Hadamard)

- at least one solution for each data set
- uniqueness of the solution
- the solution is a continuous function of the data

Here, for tomography: **ill-posed problem**

⇒ Regularization
Least square solution

\[ Rf = p \]

but \( R^{-1} \) may not exist, may be ill-conditioned...

Approximation:

\[ \min C(RF, p) \]

\[ C: \text{dissimilarity criterion} \]

Least square solution:

\[ f = (R^t R)^{-1} R^t p \]

if \( \text{Rank}(R) = n \)

otherwise infinite set of solutions

\[ \Rightarrow \text{minimal norm solution} \]

But can be instable / ill-conditioned
Stability analysis

$\sigma^2_k$: eigenvalues of $R^t R$ and of $R R^t$ ($\sigma_1 > \sigma_2 > ... \geq 0$)

$$RR^t p_k = \sigma_k^2 p_k, \quad R^t R f_k = \sigma_k^2 f_k$$

for $\sigma_k \neq 0$: $p_k = \sigma_k^{-1} R f_k$, $f_k = \sigma_k^{-1} R^t p_k$

$$f = (R^t R)^{-1} R^t p = (R^t R)^{-1} R^t \left( \sum_k < p . p_k > p_k \right)$$

$$= (R^t R)^{-1} \left( \sum_k < p . p_k > \sigma_k f_k \right) = \sum_k < p . p_k > \sigma_k^{-1} f_k$$

Noisy data $\Rightarrow$ measures $p + b$

$$f = \sum_k < p . p_k > \sigma_k^{-1} f_k + \sum_k < b . p_k > \sigma_k^{-1} f_k$$

High frequency noise $\Rightarrow$ large coefficients for the small eigenvalues (large values $\sigma_k^{-1}$) $\Rightarrow$ cf. restoration

$\Rightarrow$ instability
Regularization

- troncate the decomposition (cf. restoration using SVD)
- weakening small eigenvalues:

\[ f = \sum_k w_k \sigma_k^{-1} < p.p_k > f_k \]

- stable solution + regularity constraints

\[ \min J(f) = \| Rf - p \|^2 + \gamma \Gamma(f) \]

e.g. \( \Gamma(f) = \| f \|^2 \) \( \Rightarrow \)

\[ f = (R^t R + \gamma I)^{-1} R^t p \]

\[ \Rightarrow f = \sum_k \frac{\sigma_k}{\sigma_k^2 + \gamma} < p.p_k > f_k \]

- compromise precision / stability
- introduction of other prior information in the regularization term
Extensions

Non-parallel geometry:

- Neglect divergence and use parallel approximation
  ⇒ acceptable error if beam angle < 15 degrees
- Reorganize data into parallel projections
- Reformulate the problem:
  - projection theorem does not apply
    ⇒ no direct reconstruction
  - adaptation of backprojection theorem
    ⇒ similar algorithm
  - correction of filtered backprojection formulas
    ⇒ slightly different algorithms
  - algebraic methods: adaptation of $R$
    ⇒ the simplest method
Other methods:

- statistical / Bayesian approaches
- 3D
- structural approaches
- ...

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A few references


