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Levelings, Image Simplification Filters for Segmentation

FERNAND MEYER
Centre de Morphologie Mathématique, Ecole des Mines de Paris, 35, rue Saint Honoré, 77305 Fontainebleau

Abstract. Before segmenting an image, one has often to simplify it. In this paper we investigate a class of filters able to simplify an image without blurring or displacing its contours: the simplified image has less details, hence less contours. As the contours of the simplified image are as accurate as in the initial image, the segmentation may be done on the simplified image, without going back to the initial image. The corresponding filters are called levelings. Their properties and construction are described in the present paper.

1. Introduction

Segmenting an image into a meaningful partition is often not possible without a preliminary filtering step, in order to suppress noise but also small meaningless details and textures. Morphological segmentation is mainly based on the detection of edges as watershed lines of a gradient image. But gradient operators are extremely sensitive to noise and give a high response in textured areas. Hence the image has to be smoothed before differentiation. Linear smoothing by convolving the image by Gaussian kernels were the first to be used [7]. Many authors searched the "optimal edge detector" [2]. The drawback of linear smoothing is the blurring of the contours and the smoothing across object boundaries. For this reason non linear smoothing techniques have emerged [11] with the goal to avoid smoothing across object boundaries. The problem is to find the good trade-off between smoothing and good localization of the contours: a large smoothing simplifies the detection but creates poorly localized contours whereas a reduced smoothing does not suppress enough noise. To circumvent this problem Berghom [1] proposed to detect edges at coarser scales and to follow these to finer scales using edge focusing.

In the present paper we investigate the possibility to construct filters which do not suffer from this drawback: combine a perfect localization of the contours with an efficient suppression of details. Such a filtered image may then be segmented without any need to go back to the initial image and the contours which are found match perfectly with contours already present in the initial image. Such filters are called levelings. The simplest of them are known since a long time: they are reconstruction closing and openings [3, 16], which extend to grey tone images the classical particle reconstruction. Another class has been introduced by Luc Vincent, area openings, in which each peak is clipped until the plateau forming the new maximum reaches an area above a given threshold [15]. Openings operate only on peaks and closings on valleys; they may be applied iteratively as alternate sequential filters [14]. All these transforms are particular levelings and particular cases of the more general connected operators introduced by Salembier and Serra [12]. All these operators are flat, enlarge the flat zones and produce new flat zones. Levelings have been introduced by Meyer [9] and extensively studied by Matheron [8]. Their scale space properties and PDE formulation studied in [6, 10]. Binary levelings have been studied by Serra [13].

Basically a leveling completely suppresses or attenuates a number of contours in an image. In a first part we give a general definition of contours; zones without inside contours will be called smooth zones. They serve as seeds for region growing techniques during the segmentation, whereas the contours will be detected in regions of strong transition.

In a second step we characterize the family of images which have "less contours" than a given reference image: they are levelings of this image. We then study the sub-family of levelings which are "between" the function f and a "marker function" g, and particularly its maximal element when it exists. This maximal leveling inherits properties from both its "parent functions" f and g: we call this function the leveling of f constrained by g.

Finally we will illustrate the use of levelings in practice, for the construction of simplified images before segmentation. Levelings also may be applied on gradient images, yielding families of nested partitions.

2. Up, Down and Smooth Transitions in an Image

2.1. *Representation of Dilations and Erosions with Pulse Functions*

We follow the presentation of Henk Heijmans in [5], pp. 124–126.

Let $\mathcal{T}$ be some complete totally ordered lattice, and let $\mathcal{D}, \mathcal{E}$ be arbitrary sets (in the continuous or discrete space). We call O the smallest element and Ω the largest element of $\mathcal{T}$. Fun$(\mathcal{D}, \mathcal{T})$ represents the image defined on the support $\mathcal{D}$ with value in $\mathcal{T}$. For $h, x \in \mathcal{D}$ and $t \in \mathcal{T}$ we define the up-pulse function

$$\uparrow_h^t (x) = \left\{ \begin{array}{ll} t & \text{if } x = h \\ O & \text{if } x \neq h \end{array} \right\}$$

and the down-pulse function

$$\downarrow_h^t (x) = \left\{ \begin{array}{ll} t & \text{if } x = h \\ \Omega & \text{if } x \neq h \end{array} \right\}$$

As shown by Fig. 1 up-pulse functions form a sup-generating family and down-pulse functions an inf-generating family in

$$\text{Fun}(\mathcal{D}, \mathcal{T}) : f = \bigvee_{x \in \mathcal{D}} \uparrow_x^{f(x)} = \bigwedge_{x \in \mathcal{D}} \downarrow_x^{f(x)} .$$

We consider an arbitrary dilation α: Fun$(\mathcal{D}, \mathcal{T}) \to$ Fun$(\mathcal{E}, \mathcal{T})$ and its adjunct erosion β: Fun$(\mathcal{E}, \mathcal{T}) \to$ Fun$(\mathcal{D}, \mathcal{T})$. We define $\alpha_{x,y}(t) = \alpha(\uparrow_x^t)(y)$ and $\beta_{y,x}(s) = \beta(\downarrow_y^s)(x)$. $\alpha_{x,y}$ is a dilation on $\mathcal{T}$ and $\beta_{y,x}$ is its adjunct erosion : $f_y < \alpha_{x,y}(f_x) \Leftrightarrow \beta_{y,x}(f_y) < f_x$ (see Fig. 2).

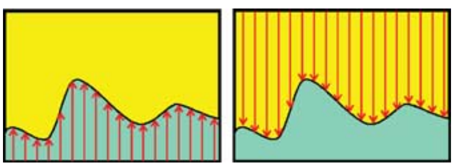


Figure 1. $f = \bigvee_{x \in \mathcal{D}} \uparrow_x^{f(x)} = \bigwedge_{x \in \mathcal{D}} \downarrow_x^{f(x)}$.

Figure 2. $f_y = s$ is lower than $f_x = t$ as $\{f_y < \alpha_{x,y}(f_x) \Leftrightarrow \beta_{y,x}(f_y) < f_x\}$.

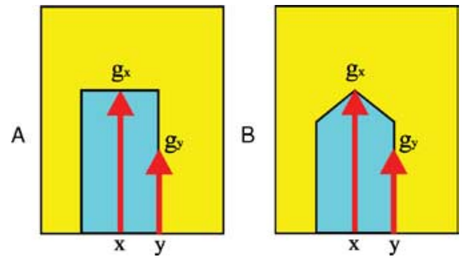


Figure 3. For x and y neighbors on a digital grid, $g_y \sqsubset g_x$ In part A, we have the ordinary dilation δ and $g_y \sqsubset g_x$ means $g_y < g_x$. In part B, as we dilate g_x by a cone $g_y \sqsubset g_x$ means $g_y < g_x - 1$.

2.2. *Up and Down Transitions*

Let us consider Fig. 3 where a function g defined on a digital grid takes two distinct values g_x and g_y indicating that a contours passes between the neighboring points x and y. Starting from this example we want to define a general mechanism for expressing that x and y are neighbors and that $g_y < g_x$. We dilate the pulse function $\uparrow_x^{g_x}$ by the elementary flat dilation δ and observe the value $\delta_{x,y}(g_x)$ taken at pixel y: since $\delta_{x,y}(g_x) > O$, we say that x and y are comparable neighbors and since $\delta_{x,y}(g_x) > g_y$ we say that g_y is lower than g_x. Replacing dilation δ by another dilation will give a new meaning to "be neighbor" and "to be lower". For instance, in Fig. 3B we dilate g_x by a cone ($\hat{\delta} = Id \vee (\delta - 1)$, where Id is the identity) . The pixels x and y are still neighbors, but g_y is lower than g_x in the sense $g_y < g_x - 1$.

More generally, let us consider an adjunct pair (α, β) of dilation and erosion verifying for $k, l \in \mathcal{T}$ and $O \leq k \leq \Omega$: $\{\alpha_{x,y}(k) \leq k \leq \beta_{x,y}(k)\}$ and $\{O < \alpha_{x,x}(k) = k = \beta_{x,x}(k) < \Omega\}$, which implies that α is extensive and β antiextensive.

Let us consider now the values taken by a function f at two pixels x and y. We suppose that $O < f_x, f_y < \Omega$.

Definition 1. f_x and f_y are comparable if and only if $\alpha_{x,y}(f_x) > O$ and $\beta_{y,x}(f_y) < \Omega$.

The relation "to be comparable" is not a symmetrical relation: we may have $\alpha_{x,y}(f_x) > O$ and $\alpha_{y,x}(f_y) = O$.

Definition 2. f_y is lower than f_x (we write $f_y \sqsubset f_x$) if and only if $f_y < \alpha_{x,y}(f_x)$ (which is equivalent to $\beta_{y,x}(f_y) < f_x$) (see Fig. 2).

Notice that $f_y \sqsubset f_x$ is not a preorder relation as it is not transitive: if (f_x, f_y) are comparable, (f_y, f_z) are comparable, then (f_x, f_z) are not necessarily comparable. Negating the relation $\{f_y \sqsubset f_x\}$ for comparable pixels permits to define the relation "greater or equal":

Definition 3. f_y is greater or equal than f_x and we write $f_y \sqsupseteq f_x$ if and only if $f_y \geq \alpha_{x,y}(f_x) > O$ (which is equivalent to $f_x \leq \beta_{y,x}(f_y) < \Omega$).

Let f and g be two functions of $\mathrm{Fun}(\mathcal{D}, \mathcal{T})$ and (p, q) two pixels of $\mathcal{T}$. The following algebraic relations will be useful:

a) $$\left|\begin{matrix} g_q \sqsubset g_p \\ f_q \sqsubset g_p \end{matrix}\right| \Rightarrow (g_q \vee f_q) \sqsubset g_p$$

b) $$\left|\begin{matrix} g_p \sqsubset g_q \\ g_p \sqsubset f_q \end{matrix}\right| \Rightarrow g_p \sqsubset (g_q \wedge f_q)$$

c) $$g_q \sqsubset g_p \leq f_p \Rightarrow g_q \sqsubset f_p$$

Proof: For instance let us prove (a).

$$\left|\begin{matrix} g_q \sqsubset g_p \\ f_q \sqsubset g_p \end{matrix}\right| \Rightarrow \left|\begin{matrix} O < g_q < \alpha_{p,q}(g_p) \\ O < f_q < \alpha_{p,q}(g_p) \end{matrix}\right|$$
$$\Rightarrow O < g_q \vee f_q < \alpha_{p,q}(g_p) \Rightarrow (g_q \vee f_q) \sqsubset g_p \qquad \square$$

Let us consider the case where $\mathcal{T} = \mathcal{R}$ or $\mathcal{Z}$. Each function f has then a complementary function $-f$. Let us suppose furthermore that the erosion β and dilation α commute with the addition of a constant $k \in \mathcal{T}$: $\alpha(f+k) = \alpha(f)+k$. Then $\{f_y \sqsubset f_x \Leftrightarrow -f_x \sqsubset -f_y\}$ if and only if α and β are symmetrical, $\beta_{y,x} = \beta_{x,y}$.

2.3. *Similar Values of a Function and Smooth Zones*

Let f be a function of $\mathrm{Fun}(\mathcal{D}, \mathcal{T})$. Combining the relations $\{f_y \sqsupseteq f_x\}$ and $\{f_x \sqsupseteq f_y\}$ yields a symmetrical relation, expressing that there is a smooth transition between f_y and f_x.

Definition 4. We define the similarity of f_y and f_x by: $\{f_x \simeq f_y\} \Leftrightarrow \{O < \alpha_{x,y}(f_x) \leq f_y \leq \beta_{x,y}(f_x) < \Omega\} \Leftrightarrow \{O < \alpha_{y,x}(f_y) \leq f_x \leq \beta_{y,x}(f_y) < \Omega\}$. We say that f_x and f_y are at level.

As α and β verify $\{\alpha_{x,y}(k) \leq k \leq \beta_{x,y}(k)\}$ we conclude that if $f_x = f_y = k$ at two comparable pixels x and y, then $f_x \simeq f_y$. Furthermore, as $\{O < \alpha_{x,x}(k) = k = \beta_{x,x}(k) < \Omega\}$ for $k \notin \{O, \Omega\}$ we have $f_x \simeq f_x$.

Figure 4(A) presents examples of a dilation of a pulse function $\uparrow_x^t$: compared to (x, t), all pixels below are in dark grey, above or equal in light grey, not comparable in white. Figure 4(B) presents the adjunct erosion of the down-pulse function $\downarrow_x^t$: compared to (x, t), all pixels above are in dark grey, below or equal in light grey, not comparable in white. Figure 4(C) presents in light grey the pixels which are at level with (x, t); it is the set of pixels which are neither above nor below (x, t); they are obtained by intersection of the light grey domains of Fig. 4(A) and (B); this domain is symmetrical with respect to (x, t).

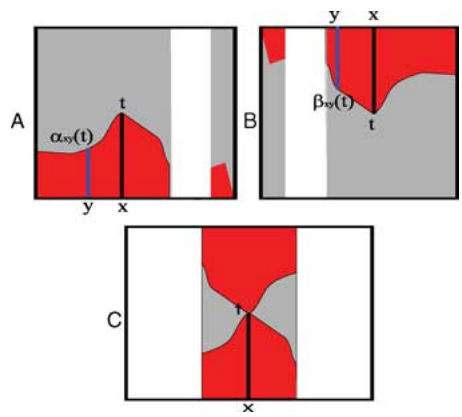


Figure 4. (A) dilation of a pulse function $\uparrow_x^t$: compared to (x, t), all pixels below are in dark grey, above or equal in light grey, not comparable in white. (B) adjunct erosion of the down-pulse function $\downarrow_x^t$: compared to (x, t), all pixels above are in dark grey, below or equal in light grey, not comparable in white. (C) the intersection of the light grey zones give contains the pixels at level with (x, t); the union of the white zones represents the pixels not comparable with (x, t).

We are now able to define smooth zones based on arcwise connectivity.

Definition 5. We say that two values f_x and f_y are smoothly linked and we write $f_x \bowtie f_y$ if there exists a series of pixels $\{x_0 = x, x_1, x_2, \dots x_n = y\}$ such that $f_{x_i} \leftrightsquigarrow f_{x_{i+1}}$.

Definition 6. A set X is a smooth zone of an image f if and only if $f_x \bowtie f_y$ for any two pixels x and y in X.

The relation $\bowtie$ is an equivalence relation. The associated equivalence classes are the maximal smooth zones. It is easy to verify that the smooth zones of f form a connection of $\mathcal{D}$ [13]. For the pair of elementary dilation and erosion (δ, ε), one obtains ordinary flat zones.

Regional minima and maxima are easily defined:

Definition 7. A set X is a regional minimum of an image f if and only if it is a smooth zone of f and for any two pixels x and y, $x \in X$ and $y \notin X$, such that f_x and f_y are comparable, we have $f_x \sqsubset f_y$.

2.4. *Uniformly Smooth Zones*

Definition 8. A set X is uniformly smooth if $f_x \leftrightsquigarrow f_y$ for any two mutually comparable pixels x and y in X: $\{f_x \leftrightsquigarrow f_y\} \Leftrightarrow \{O < \alpha_{x,y}(f_x) \le f_y \le \beta_{x,y}(f_x) < \Omega\}$.

But for non comparable pixels we have $O = \alpha_{x,y}(f_x)$ and $\beta_{x,y}(f_x) = \Omega$ and $\alpha_{x,y}(f_x) \le f_y \le \beta_{x,y}(f_x)$ is also verified

Hence if X is an $\alpha\beta$-smooth zone of f and $y \in X$, we have

$$\bigvee_{x \in X, x \neq y} \alpha_{x,y}(f_x) \le f_y \le \bigwedge_{x \in X, x \neq y} \beta_{x,y}(f_x).$$

But since

$$\alpha_{yy} f_y = \beta_{yy} f_y = f_y,$$

we also have

$$\bigvee_{x \in X} \alpha_{x,y}(f_x) \le f_y \le \bigwedge_{x \in X} \beta_{x,y}(f_x),$$

that is

$$\alpha_X(f_y) \le f_y \le \beta_X(f_y),$$

where α_X and β_X are respectively the restrictions of α and β to X. But since α_X is extensive and β_X anti-extensive, we finally have $\alpha_X(f_y) = f_y = \beta_X(f_y)$. Inversely, it is obvious that if this last relation is true, then X is an $\alpha\beta$-smooth zone of f. In fact $\{\alpha_X(f_y) = f_y\} \Leftrightarrow \{f_y = \beta_X(f_y)\}$ as $\alpha_X(f_y) = f_y$ implies $\alpha_X(f_y) \le f_y$, which by adjunction implies $f_y \le \beta_X(f_y)$; but since β_X is anti-extensive, we obtain $f_y = \beta_X(f_y)$.

Proposition 9. *A set X is a uniformly smooth zone of f if and only if $\{\alpha_X(f) = f\}$ or equivalently $\{f = \beta_X(f)\}$.*

Uniformly smooth zones of f do not form a connection: if X and Y are uniformly smooth zones of f, such that $X \wedge Y \neq \infty$, then $X \vee Y$ is not necessarily a uniformly smooth zone of f. For the cone dilation $\delta_1(g) = g \vee (\delta g - 1)$, $X = \begin{smallmatrix} 3 & 2 \\ 4 & 1 \end{smallmatrix}$ is a smooth zone since there exists a path with a slope smaller or equal to 1 between any couple of pixels. However, there exist within X a sharp transition between values 1 and 4, hence X is not a uniformly smooth zone. And the sets $X_1 = \begin{smallmatrix} 3 & 2 \\ 4 & \end{smallmatrix}$ and $X_2 = \begin{smallmatrix} 3 & 2 \\ & 1 \end{smallmatrix}$ are each uniformly smooth, have a non empty intersection; however, their union X is smooth but not uniformly smooth.

3. Levelings, Razings, Floodings and Flattenings

3.1. *Definition of Levelings and Flattenings*

Being able to compare the values of "neighboring pixels", we may now define a particular class of images with less contours than a given image f, called levelings of f.

Definition 10. A function g is a leveling of the function f if and only if for any pair of comparable pixels $(p, q) : g_q \sqsubset g_p \Rightarrow f_q \le g_q$ and $g_p \le f_p$.

Each transition $g_q \sqsubset g_p$ of g for two neighboring pixels p and q induces a pointwise relation between f and g on each pixel p and q. Levelings are called monotone planings [9] as $g_q \sqsubset g_p$ implies $f_q \sqsubset f_p$. The inequality $g_q \sqsubset g_p$ means $g_q < \alpha_{p,q}(g_p)$ and implies by definition of levelings $f_q \le g_q$ and $g_p \le f_p$. On the other hand, $\alpha_{p,q}$ being increasing, $g_p \le f_p$ implies $\alpha_{p,q}(g_p) \le \alpha_{p,q}(f_p)$, hence $f_q \le g_q < \alpha_{p,q}(g_p) \le \alpha_{p,q}(f_p)$, yielding $f_q \sqsubset f_p$.

Levelings enlarge smooth zones: $g_q \sqsubset g_p \Rightarrow f_q \sqsubset f_p$ is equivalent with $f_q \sqsupset f_p \Rightarrow g_q \sqsupset g_p$ from which we derive $f_q \bowtie f_p \Rightarrow g_q \bowtie g_p$ this last relation shows that any smooth (resp. uniformly smooth) zone for f is also a smooth zone (resp. uniformly smooth) for g. Serra and Salembier called connected operators which enlarge flat zones in [12]. Levelings preserve or enlarge smooth zones : they are a generalization of connected operators.

Levelings create smooth zones: The implication $[g_q \sqsubset g_p \Rightarrow f_q \le g_q$ and $g_p \le f_p]$ is equivalent to $[f_q > g_q$ or $g_p > f_p \Rightarrow g_q \sqsupseteq g_p]$. Now, if two comparable pixels (p, q) verify $f_q > g_q$ and $f_p > g_p$, we conclude that simultaneously $g_q \sqsupseteq g_p$ and $g_p \sqsupseteq g_q$, that is g_p and g_q are at level. That means that on $\{g > f\}$ (resp. $\{g < f\}$) any two mutually compatible pixels are at level. Any smooth zone of $\{g > f\}$ (resp. $\{g < f\}$) is uniformly smooth.

3.2. *Characterization of Levelings*

In order to characterize the levelings, it is useful to consider two subclasses, the upper and lower levelings.

3.2.1. Upper Levelings.

Definition 11. A function g is an upper-leveling of the function f if and only if for any pair of comparable pixels (p, q): $g_q \sqsubset g_p \Rightarrow g_p \le f_p$.

The meaning of $g_q \sqsubset g_p$ being $\beta_{q,p} g_q < g_p$, the implication $g_p > \beta_{q,p}(g_q) \Rightarrow g_p \le f_p$ may be interpreted as $[g_p \le \beta_{q,p}(g_q)$ *or* $g_p \le f_p]$ that is $[g_p \le f_p \vee \beta_{q,p}(g_q)]$.

This criterion should be satisfied for any pair of comparable pixels (p, q). However since for non comparable pixels we have $\beta_{q,p}(g_q) = \Omega$, the criterion will be fulfilled for any couple of pixels:

Criterion up-lev2: *A function g is a upper-leveling of the function f if and only if for any pair of pixels* (*p*,*q*) *the following criterion holds*:

$$g_p \le f_p \vee \beta_{q,p}(g_q).$$

Fixing the central pixel p repeating the criterion for all pixels comparable with p yields *Criterion up-lev3: A function g is a upper-leveling of the function f if and only if* $g \le f \vee \beta g$.

Algebraic properties: If g and h are both upper-levelings of the function f, then $f \vee h$, $f \wedge h$, $g \vee h$, $g \wedge h$ and the morphological center $[f \wedge (g \vee h)] \vee (g \wedge h)$ also are upper levelings of f.

Lets for instance prove that $g \vee h$ and $g \wedge h$ are upper-levelings: by criterion up-lev3 $g \le f \vee \beta g$ and $h \le f \vee \beta h$ which implies $g \vee h \le f \vee \beta g \vee f \vee \beta h \le f \vee \beta(g \vee h)$, since β is increasing. On the other hand $g \wedge h \le (f \vee \beta g) \wedge (f \vee \beta h) = f \vee (\beta g \wedge \beta h) = f \vee \beta(g \wedge h)$, since β is an erosion, i.e commutes with the infimum.

3.2.2. Floodings.

Floodings are anti-extensive upper levelings. Flat floodings are known as reconstruction closings and are largely used for segmenting images: they allow to suppress minima in gradient images before the construction of the watershed line [16].

Definition 12. A function g is a flooding of the function f if and only if $g \ge f$ and g is an upper-leveling of the function f.

But then for *any pair of* comparable *pixels* (p, q), *we have* $g_q \sqsubset g_p \Rightarrow g_p \le f_p$ as g is an upper leveling of f, but also $f_q \le g_q$ as $g \ge f$. This shows that a flooding is in fact a particular leveling.

The following characterization of floodings are all equivalent

– *Flood1: $g \ge f$ and for any pair of* comparable *pixels*

$$(p, q) : g_q \sqsubset g_p \Rightarrow g_p = f_p$$

– *Flood2:* $g = f \vee \beta g$

Criterion *Flood1* has an obvious physical meaning. Fig. 5(A) and (B) represent respectively a possible and an impossible flooding g of a relief f: if for two comparable pixels a lake verifies $g_q \sqsubset g_p$, then the highest pixel is necessarily at ground level ($g_p = f_p$), otherwise the lake presents an unconstrained wall of water as in Fig. 5(B).

The relation $g = f \vee \beta g$ may be interpreted as an algorithm on a digital grid: applied to a couple of

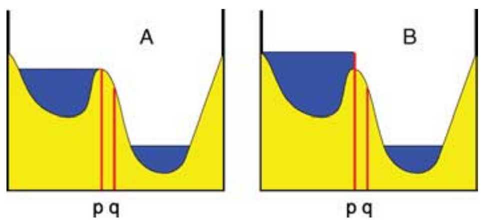


Figure 5. $g_q \sqsubset g_p \Rightarrow g_p = f_p$.

functions f and g until stability, one recognizes the algorithm for constructing reconstruction closings.

Algebraic properties: If g and h are both floodings of the function f, then $g \vee h$, $g \wedge h$ also are floodings of f.

If h is a flooding of the function f and $f \leq g \leq h$, then h also is an flooding of the function g.

Order relation between floodings: The relation "to be a flooding of" is a reflexive, antisymmetric and transitive relation: it is an order relation. The family of all floodings of a function f form a complete lattice. The order relation is the ordinary order relation $>$. Infimum and supremum also are the ordinary infimum and supremum of images. In the binary case, successive floodings fill holes. In the grey tone case, they fill lakes. Figures 6 and 7 present successive floodings respectively of a binary image and of a grey tone function. In this case $\alpha = \delta$.

3.2.3. *Lower Levelings and Razings.* Lower levelings and razings are the dual counterpart of respectively

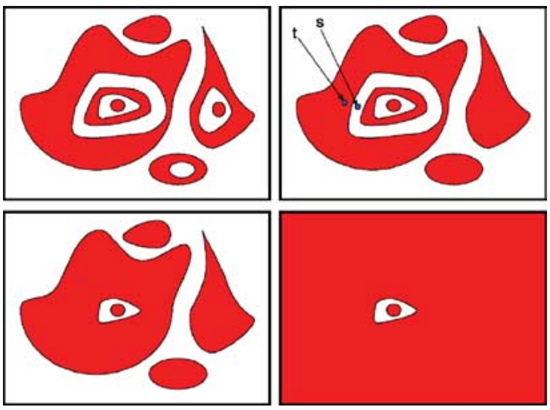


Figure 6. Successive floodings.

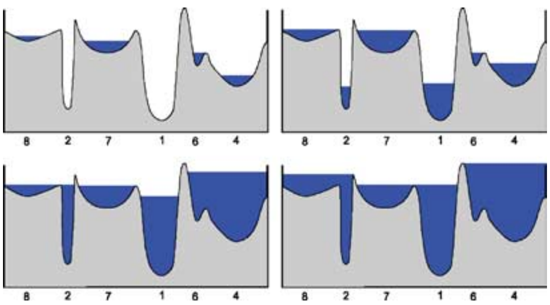


Figure 7. Successive floodings.

upper-levelings and floodings. We summarize here the definitions and criteria

Lower levelings: *A function g is a lower-leveling of the function f if and only if*:

– *Low-lev1: for any pair of* comparable *pixels*

$$(p, q1) : g_q \sqsubset g_p \Rightarrow f_q \leq g_q$$

– *Low-lev2: for any pair of* comparable *pixels*

$$(p, q) : f_p \wedge \alpha_{q,p}(g_q) \leq g_p$$

– *Low-lev3:* $f \wedge \alpha g \leq g$

Razings: *A function g is a razing of the function f if and only if*:

– *Raz1: g is a lower-leveling of the function f and* $g \leq f$
– *Raz2:* $g \leq f$ *and for any pair of* comparable *pixels* (p, q):

$$g_q \sqsubset g_p \Rightarrow f_q = g_q$$

– *Raz3: for any pair (p, q) of comparable pixels*

$$f_p \wedge \alpha_{q,p}(g_q) \leq g_p \leq f_p$$

– *Raz4:* $g = f \wedge \alpha g$.

3.2.4. *Levelings.*

Definition 13. A function g is a leveling of the function f if and only if g is both an upper and a lower leveling.

Levelings are characterized by a number of criteria [8].

A function g is a leveling of the function f if and only if:

– Lev1: for any pair of comparable pixels

$$(p, q) : g_q \sqsubset g_p \Rightarrow f_q \leq g_q \text{ and } g_p \leq f_p$$

– Lev2: *for any pair of pixels*

$$(p, q) : f_p \wedge \alpha_{q,p}(g_q) \leq g_p \leq f_p \vee \beta_{q,p}(g_q)$$

– Lev3:

$$f \wedge \alpha g \le g \le f \vee \beta g$$

– Lev4:

$$\left| \begin{array}{l} g_p > \beta g(p) \Rightarrow g_p \le f_p \\ g_p < \alpha g(p) \Rightarrow f_p \le g_p \end{array} \right|$$

– Lev5:

$$\left| \begin{array}{l} f \vee g = f \vee \beta g \\ f \wedge g = f \wedge \alpha g \end{array} \right|$$

– Lev6:

$$\left| \begin{array}{ll} \text{On}\{f \le g\} & g = f \vee \beta g \\ \text{On}\{f \ge g\} & g = f \wedge \alpha g \end{array} \right|$$

– Lev7:

$$g = (f \vee \beta g) \wedge \alpha g = (f \wedge \alpha g) \vee \beta g$$

Order relation between levelings: The relation {to be a leveling of} is a preorder relation. Obviously reflexive, it is also transitive. Suppose that {h leveling of g} and {g leveling of f}, let us show that {h leveling of f}. Since h is a leveling of g, we have for any couple of comparable pixels x and y : $h_y \sqsubset h_x \Rightarrow g_y \le h_y$ and $h_x \le g_x$. But as seen earlier $h_y \sqsubset h_x$ implies $g_y \sqsubset g_x$. And g being a leveling of f, $g_y \sqsubset g_x \Rightarrow f_y \le g_y$ and $g_x \le f_x$. Finally we get $f_y \le g_y \le h_y$. The other inequality $h_x \le f_x$ is obtained by duality.

Figures 8–10 present successive levelings respectively of a binary image, of a grey tone function. and of a grey tone image. In this case $\alpha = \delta$.

3.2.5. Levelings and Regional Minima. If (α, β) are flat operators, i.e. for comparable pixels p and q, we have $\alpha_{p,q}(t) = t = \beta_{p,q}(t)$, implying that $g_q \sqsupset g_p \Rightarrow g_q > g_p$, then the leveling based on (α, β) does not create regional minima or maxima. More precisely if g is a leveling of f, and X a regional minimum of g, then there exists a set $Z \subset X$, which is a regional minimum for f. The idea of the proof is as follows : if x is a pixel for which f is minimal within X, then the flat zone of f containing x is necessarily contained in X and is a regional minimum of f. However, this is not true if (α, β) are not flat operators, as shows the following

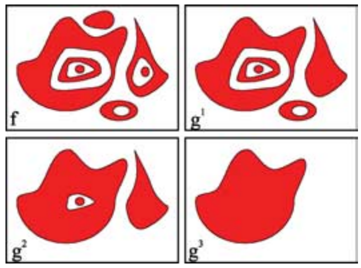


Figure 8. Sequential leveling : each figure levels all preceding ones.

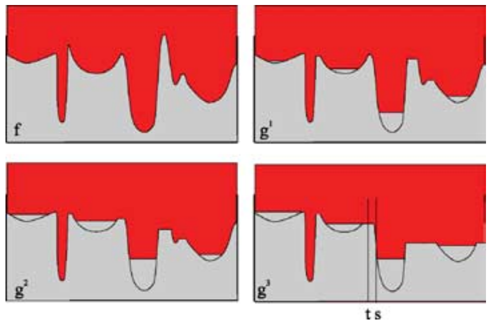


Figure 9. Sequential leveling : each figure levels all preceding ones.

Figure 10. Initial image and 3 increasing levelings.

counter-example.

$$f = \begin{matrix} 9 & 9 & 9 & 9 & 9 \\ 9 & 3 & 1 & 2 & 9 \\ 9 & 2 & 4 & 3 & 9 \\ 9 & 9 & 9 & 9 & 9 \end{matrix} \quad \text{and} \quad g = \begin{matrix} 9 & 9 & 9 & 9 & 9 \\ 9 & 4 & 4 & 4 & 9 \\ 9 & 4 & 4 & 4 & 9 \\ 9 & 9 & 9 & 9 & 9 \end{matrix}.$$

Obviously, g is a leveling of f for cone dilation and erosion. The pixels with value 4 form a regional minimum of g but do not contain a regional minimum for f.

4. Construction of Floodings, Razings and Levelings

The largest flooding of an image is a flooding covering everything. A maximal leveling or razing also is completely flat. But completely flat images are useless; therefore one has to find extremal levelings of f which satisfy some additional constraint, so that they are not completely flat. Let us start with the simplest flat floodings, razings and levelings, those associated to the elementary couple of dilation and erosion (δ, ε). Since the associated smooth zones are flat, there exist simple means for constructing levelings. The simplest is the threshold

$$Th_\lambda^+ : f_x \to g_x = \begin{matrix} f_x & \text{if } f_x \leq \lambda \\ \lambda & \text{if } f_x > \lambda \end{matrix}$$

transforms the image f into a razing. It replaces all zones where $f > \lambda$ into a flat zone of level λ. By duality one obtains floodings. A razing followed by a flooding produces a leveling.

As the threshold operates uniformly on the whole image, it is often of limited interest and more refined operations are needed. Luc Vincent has introduced "area filters", Corinne Vachier "volume filters". The idea here is to threshold the function locally, starting from each maximum and clipping the corresponding peak until some condition is met. Area filters introduced by Luc Vincent stop clipping a peak as soon a flat zone with a minimal area is produced. Corinne Vachier stops at the highest flat zone for which the volume of the removed part of the peak reaches a given threshold [14].

In the next section, we will go back to the general levelings associated to (α, β) and will constrain the levelings of f by a constraining function g. We will construct the largest flooding of f below g, the highest razing above g, and the largest leveling in some sense between f and g.

4.1. *The f-Activity Lattice*

This section sets the framework in which we will work and introduces the f-activity lattice. It closely follows [8].

Definition 14. For $g, h, f \in T^E$, we say h separates g and f, and we write $(g\ h\ f)$ or equivalently $(f\ h\ g)$ if and only if for any $x \in E$, the series $(g_x\ h_x\ f_x)$ is monotonous: $\forall x \in E : g_x \leq h_x \leq f_x$ or $g_x \geq h_x \geq f_x$.

Definition 15. We will call Inter(g, f) the class of functions $h \in T^E$, which separate g and f.

Obviously $h \in \text{Inter}(g, f) \Leftrightarrow g \wedge f \leq h \leq g \vee f$.

In particular, $g \wedge f$ is the smallest element of Inter(g, f) and $g \vee f$ its largest element.

4.2. *The Order $g >_f h$*

Definition 16. We say that g is more far away from f than h, or that g is bigger than h in the order f and we write $g >_f h$ if and only if h separates g and f: $g >_f h \Leftrightarrow (g\ h\ f)$.

Proposition 17. *$>_f$ is an order relation on T^E, and moreover $g >_f h \Leftrightarrow f >_g h$.*

Proposition 18. *For $a, f \subset T^E$,* Inter(a, f) *is a complete lattice for the order f. The function a is then the highest element. For any family h_i of* Inter(a, f):

$$\bigwedge_f h_i = \left| \begin{matrix} \vee h_i \text{ on } \{a \leq f\} \\ \wedge h_i \text{ on } \{a \geq f\} \end{matrix} \right|;$$

$$\bigvee_f h_i = \left| \begin{matrix} \wedge h_i \text{ on } \{a \leq f\} \\ \vee h_i \text{ on } \{a \geq f\} \end{matrix} \right|$$

Considering a pair of functions f and h we will study the family of floodings, razings and levelings of f within Inter(f, h). We will search for maximal elements in this family for the order $>_f$. Such maximal elements exist for floodings and for razings; they exist for levelings only if h verifies some additional condition.

4.3. *Construction of Floodings and Razings*

Given two functions f and h, we search the largest flooding of f in Inter(f, h). The floodings of f form a lattice: if (g_i) is a family of floodings of f, then $\bigvee g_i$

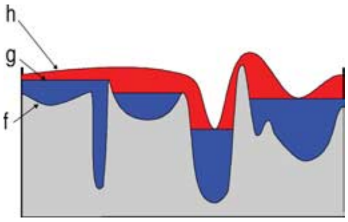


Figure 11. If (g_i) is a family of floodings of f, then $\bigvee g_i$ also is a flooding of f. Hence the supremum (for both order relations $>$ and $>_f$) of the family of floodings of f belonging to Inter (f, h) is the largest flooding in Inter(f, h) and we will write Fl(f, h).

also is a flooding of f. Hence the supremum (for both order relations $>$ and $>_f$) of the family of floodings of f belonging to Inter (f, h) is the largest flooding in Inter (f, h) and we will write Fl(f, h) (see Fig. 11. Any flooding g of f included in Inter (f, h) verifies $f \le g \le f \vee \beta g$ as a flooding and $f \wedge h \le g \le f \vee h$ as element of Inter (f, h).

Now, how can we construct it? If we write $h_0 = f \vee h$, as β is increasing, $g \le h_0$ implies $\beta g \le \beta h_0$. Putting everything together we obtain the inequalities: $f \le g \le f \vee \beta g \le f \vee \beta h_0$. Defining the recurrence $h_n = f \vee \beta h_{n-1}$, the same arguments produce $f \le g \le h_n$. But then $f \le g \le \bigwedge h_n.\beta$ being an anti-extensive operator, the sequence h_n is decreasing and bounded by f. Its limit is equal to $h_\infty = \bigwedge h_n$; and this limit, verifying $f \le h_\infty \le f \vee \beta h_\infty$ is itself a flooding of f; so it is necessarily equal to Fl(f, h), the largest flooding of f within Inter (f, h) . For finite digital images, the limit is obtained by finite iteration until stability of $h_n = f \vee \beta h_{n-1}$, with $h_0 = f \vee h$. We recognize the usual reconstruction closing if $\beta = \varepsilon$ [16].

Similarly the largest razing of f for the order relation $>_f$ in Inter (f, h), which is also the smallest razing for the order relation $>$ is equal to $\bigvee h_n$, where $h_n = f \wedge \alpha h_{n-1}$, with $h_0 = f \wedge h$; we write Rz(f, h). In the case of finite digital images, this supremum is obtained by finite iteration until $h_{n+1} = h_n$.

***4.3.1. Properties of* Rz(f, h) *and* Fl(f, h).** For the order relation $>_f$ and considered as functions of h, Rz(f, h) and Fl(f, h) are increasing, anti-extensive and idempotent: they are openings.

4.4. Construction of Levelings

Given two functions f and h, we search the largest (for $>_f$) leveling of f in Inter (f, h), if it exists (which it does not always). If g is a leveling of f, then $f \vee g$ is an extensive leveling of f, that is a flooding and $f \wedge g$ an anti-extensive leveling of f, that is a razing. Hence, it may seem only natural to combine the largest flooding of f below $f \vee h$, that is Fl(f, h) with the smallest razing of f for the order relation $>$, above $f \wedge h$, that is Rz(f, h). The supremum Fl$(f, h) \vee_f$ Rz(f, h) should produce the largest leveling in Inter (f, h). Unfortunately, the supremum $\vee_f$ of two levelings is not necessarily a leveling but a flattening, as shows Fig. 12 where g_1 and g_2 are both levelings of f in Inter (f, h), but $g_1 \vee_f g_2$ which equals h itself is not a leveling of f. As a matter of fact, levelings are particular flattenings and the supremum $\vee_f$ of two flattenings is a flattening, as is $g_1 \vee_f g_2$.

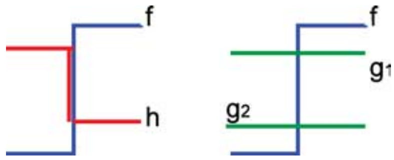


Figure 12. g_1 and g_2 are both levelings of f in Inter(f, h), but $g_1 \vee_f g_2 = h$ is not a leveling of f but only a flattening.

As levelings, flattenings attenuate or suppress contours, but contrarily to levelings, they may transform an upwards transition into a downwards transition and inversely:

Definition 19. A function g is a flattening of the function f if and only for any pair of comparable pixels

$$(p, q) : g_q \sqsubset g_p \Rightarrow \left| \begin{array}{ccc} f_q \le g_q & \text{and} & g_p \le f_p \\ & \text{or} & \\ f_q \sqsupseteq g_p & \text{and} & g_q \sqsupseteq f_p \end{array} \right|$$

We will be able to construct maximal levelings thanks to the following lemma.

Lemma 20. *It h is of the form $\alpha k \wedge_f \beta k$ for some function k, then all flattenings in* Inter (f, h) *are levelings.*

Hence we will define the leveling of f constrained by h and write $\Lambda(f, h)$ as the largest flattening contained in Inter$(f, \alpha h \wedge_f \beta h)$. Defining k as $k = \alpha h \wedge_f \beta h$, we obtain $\Lambda(f, h) = $ Fl$(f, k) \vee_f$ Rz(f, k).

4.4.1. Levelings are Strong Filters. If we write $g_+^0 = f \vee \beta h$, we define $\Lambda_h^+ f = \bigwedge g_+^n$, where g_+^n is defined by the recurrence $g_+^n = f \vee \beta g_+^{n-1}$ and $\Lambda_h^- f = \bigvee g_-^n$, where g_-^n is defined by the recurrence $g_-^n = f \wedge \alpha g_-^{n-1}$,

$g_-^0 = f \wedge \alpha h$. Then the levelings are obtained by

$$\left| \begin{array}{l} \text{on } \{h \geq f\} : \Lambda_h^+ f \\ \text{on } \{h \leq f\} : \Lambda_h^- f \end{array} \right| .$$

It is possible to prove that they are also obtained by the commutative product of the opening Λ_h^- and the closing $\Lambda_h^+ : \Lambda_h = \Lambda_h^+ \Lambda_h^- = \Lambda_h^- \Lambda_h^+$, which shows that levelings are strong filters (see also [13]).

5. Floodings, Razings and Levelings in Practice

We are now able to associate to each couple (α, β) of adjunct extensive dilation and anti-extensive erosion a type of contours, of smooth zones and a series of transformations which extend these smooth zones. Furthermore, we are able to construct levelings of an image constrained by another image. The last part of this paper presents how these tools may be used for image simplification and segmentation. We first discuss the choice of the couple (α, β), according to the desired type of simplification. We then discuss the choice of markers, in particular for constructing a hierarchy of simplifications. We conclude by presenting two strategies for segmenting an image : the first based on floodings a gradient image, the second on detecting smooth zones in the image to segment and expanding them with the watershed transform.

5.1. Choice of the Leveling Type

The problem here is to chose the right couple (α, β) for a given task and to compare the effect of various choices. In the case where two couples (α_1, β_1) and (α_2, β_2) verify $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, it is possible to predict their effect on the contours, smooth zones and levelings:

- $f_y \sqsubset f_x \Leftrightarrow f_y < \alpha_{x,y}(f_x)$: then obviously $f_y \sqsubset_1 f_x \Rightarrow f_y \sqsubset_2 f_x$
- on smooth zones which are the complementary part of transition, we have the opposite inclusion: $f_x \Bumpeq_2 f_y \Rightarrow f_y \Bumpeq_1 f_x$
- Levelings being characterized by $f \wedge \alpha g \leq g \leq f \vee \beta g$ when g is a leveling of f; we conclude that any leveling of f for (α_2, β_2) is also a leveling of f for (α_1, β_1) as $f \wedge \alpha_2 g \leq g \leq f \vee \beta_2 g \Rightarrow f \wedge \alpha_1 g \leq g \leq f \vee \beta_1 g$.

We will now compare two couples of levelings: based on flat and cone dilations on one hand, based on connected and non connected structuring elements on the other hand.

5.1.1. Comparison Between a Flat and a Cone Dilation. We compare here the elementary dilation δ and the dilation by a cone $\hat{\delta}$ defined earlier and illustrated in Fig. 3. We call flat zones the smooth zones associated to (ε, δ) and slope zones the smooth zones associated to $(\hat{\varepsilon}, \hat{\delta})$. The associated levelings will be called respectively flat and slope levelings. We detect the smooth zones on the same image (see Fig. 13) for both (ε, δ) and $(\hat{\varepsilon}, \hat{\delta})$: as expected, the number of flat zones (41854) is much higher than the number of slope zones (22261). The flat zones are composed exclusively of small particles, whereas some slope zones already appear as larger areas in uniform regions. We then construct a marker image by an alternate sequential filter based on increasing openings and closings by disks. Associated to this marker image we construct the slope leveling of the initial image: a number of small details, as the eye brows disappear, and the contours, which were displaced in the marker image are restored to their original position. The pattern of the slope zones detected after slope leveling is characteristic: their number again drastically reduced (12532 against 22261 before leveling); more important is their distribution: large zones inside the object separated by a large number of small slope zones in the transitions zones, along the contours.

5.1.2. Comparison Between a Dilation by a Connected and a non Connected Structuring Element. We compare now two levelings on the same reference image f (see Fig. 15_left) and marker image h (not illustrated here: it is completely black with a white dot on the hand holding the telephone). The first leveling is associated to the dilation δ^{++} and its adjunct erosion ε^{--} and is illustrated in Fig. 15_center; the structuring element of the dilation δ^{++} and the erosion ε^{--} is composed by a hexagon and two pixels at a distance of 4 pixels apart on each side (see Fig. 14). The central part cares for the normal connectivity reconstructions whereas the couple of added pixels permits jumps from one zone to another. Indeed the ordinary leveling based on (ε, δ) illustrated by Fig. 15_right is unable to reconstruct some parts of the image, although it uses the same marker; it is unable to jump from one book to the next on the shelve in the background. As expected, since $\delta^{++} > \delta$, the (ε, δ) leveling has larger flat zones than the $(\varepsilon^{--}, \delta^{++})$ leveling.

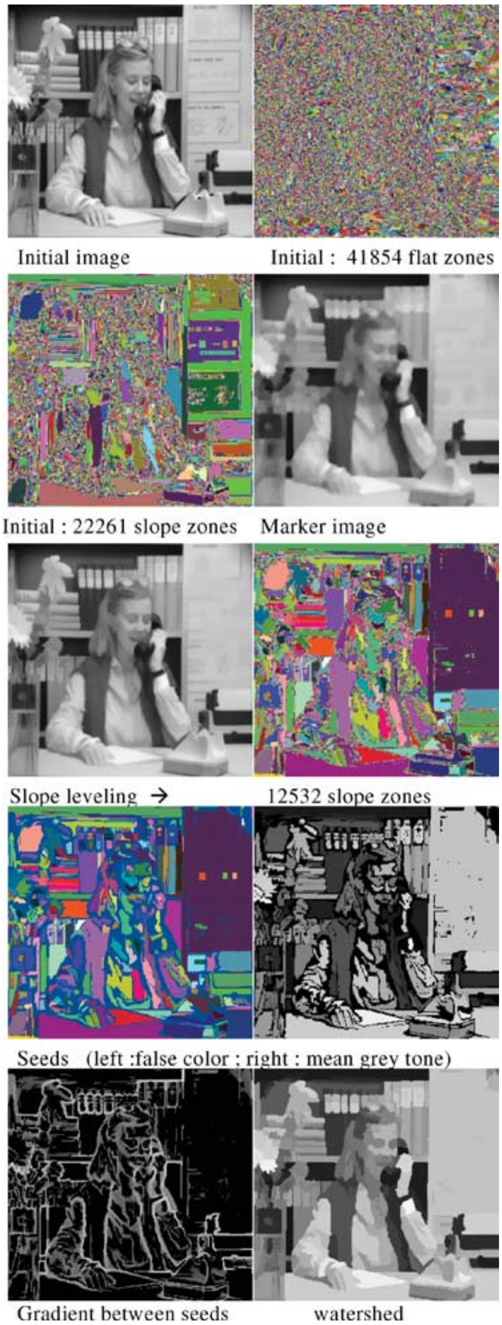


Figure 13. Comparison of flat zones and smooth zones on the same image, before and after leveling. Selection of the largest smooth zones as markers for a watershed segmentation.

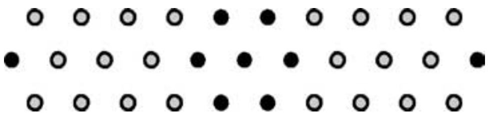
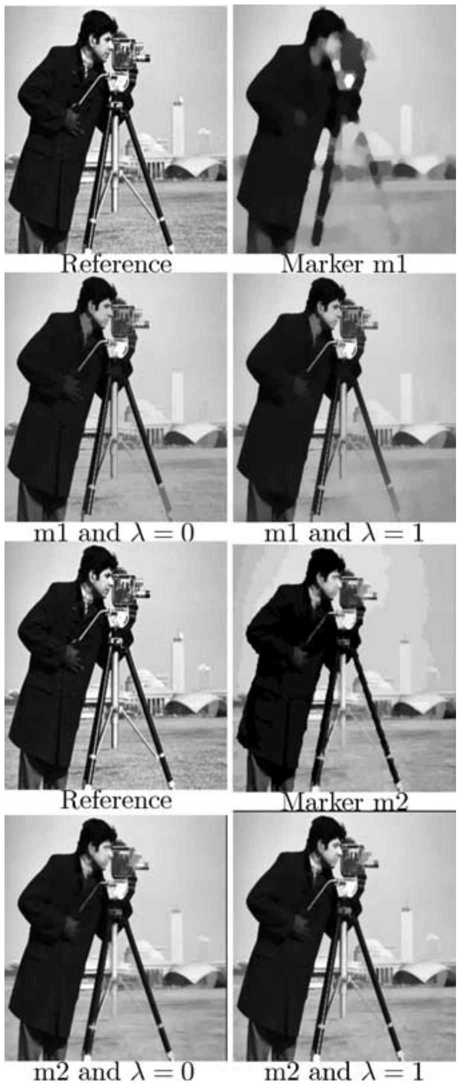


Figure 14. Non connected structuring element.

5.2. *Choice of Markers*

The series of "cameraman" images illustrate the importance of the choice of the marker and of the leveling. The first marker, $m1$, which is used is alternative sequential filter, with openings and closings by disks. The second marker, $m2$, preserves more details; it is also an alternative sequential filter, where the opening is a supremum by openings by segment and the closing its dual. For each of the markers, one compares the effect of a flat and of a slope opening.



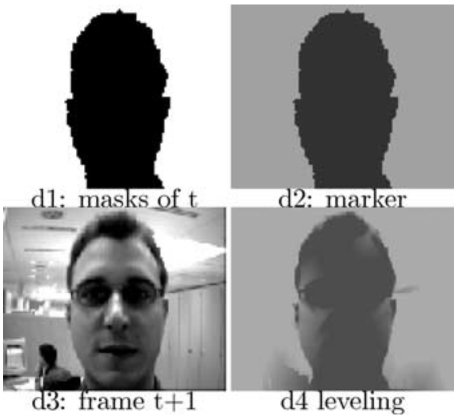
5.2.1. Markers for the Construction of a Hierarchy of Levelings. As the relation "to be a leveling" is transitive, it is interesting to produce a sequence of levelings of a function f associated to a series of markers

$m1, m2, \ldots$:

$$f_1 = f$$
$$f_2 = \Lambda(f_1, m_1)$$
$$f_3 = \Lambda(f_2, m_2) \text{ etc.}$$

One produces like that a series of simpler and simpler images, with less and less smooth zones. Figure 10 presents an example where the markers used were alternate sequential filters applied on the initial image. Alternatively, one may use as markers larger and larger blurrings by increasing gaussians.

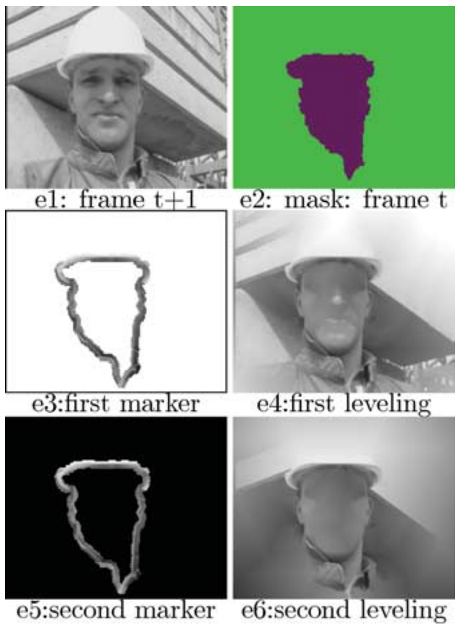
5.2.2. Markers for Tracking Objects in Sequences. The design of markers may take advantage of the knowledge we already have from the scene. In case of the segmentation of sequences, this knowledge is particularly high. The following two examples illustrate the problem of tracking a face of interest in a sequence. In steady state of the segmentation, one knows the mask of interest for the frame $t-1$ and is interested in producing the new mask in frame t.



In this first example the lighting conditions of the sequence are such that the face in the foreground is darker than the background. For this reason an interesting marker $d2$ is produced by computing in each region of the mask $d1$ computed in frame $t-1$ its mean value. Taking this image as marker and the image of frame $t+1$ (image d3) as reference image yields the slope leveling d4, where almost only the outside contour of the face is contrasted, and all other details, whether in the face or in the background have disappeared. This new image is now extremely easy to segment in order to produce the mask of the face for frame t.

In the next example the lighting conditions are more balanced and we will construct a new marker. In the next series of figures, the image e1 represents frame $t+1$ in which we have to track the zone of interest; image e2 represents the zone of interest detected in frame t. We will produce a leveling which suppresses almost all information in frame $t+1$ except the contours of the new region of interest. For this we construct a first marker in the following manner. We produce a ribbon like mask around the boundary of the mask in image e2: we take all pixels which are within a distance ρ of this boundary.

Figure 15. Left: $f =$ original image. The marker image h is completely black with a white dot on the left hand of the girl. Center: leveling $\Lambda_h f$ associatet to the dilation δ^{++} and erosion ε^{--}; Left: leveling $\Lambda_h f$ associated to the dilation δ and erosion ε; without jumps, the reconstruction is much less complete (see for instance the books).



The parameter ρ is chosen in such a way that the ribbon contains the contours of the zone of interest of frame $t+1$. We now construct a composite image, by cutting out the content of frame $t+1$ within the ribbon and the white (255) outside the ribbon. This produces the first marker image represented as image e3. The slope flooding of frame $t+1$ with this marker image is represented in image e4. It is already much simpler than the original image. We then take advantage from another interesting feature of levelings, namely that a the concatenation of levelings still is a leveling. So we will construct a new leveling, in fact a razing for which the reference image is the result of the first leveling and the marker is the same ribbon as before, but on a black background (image e5). The result of this second leveling is illustrated in image e6, where only the contour of the object of interest appears with its original strength, whereas the contours of all other objects of the scene have vanished or their contrast has been drastically reduced.

5.3. *Levelings for Segmentation*

5.3.1. Increasing Floodings. The watershed transform is the tool of choice for detecting the contours; generally it is used on a gradient image, associated to a set of markers. In some cases, one is interested in producing a hierarchy of segmentations, that is a series of nested partitions. To this effect one may flood the gradient image and take as segmentations the catchment basins of a series of increasing floodings. As the flooding increases, adjacent basins progressively merge producing coarser and coarser segmentations. Depending on the law governing the progression of the flooding, one obtains different results. Size oriented flooding [4, 14] is produced by placing sources at each minimum and flooding the surface in such a way that all lakes share some common measure (height, volume or area of the surface). As the flooding proceeds, the level of some lakes cannot grow any further, as the level of the lowest path point has been reached. In the Fig. 16, a flooding starts from all minima in such a way that all lakes always have uniform depth. Size oriented flooding permits to produce hierarchical segmentation with good psychovisual properties. The depth criterion ranks the region according to their contrast, the area according their size and the volume offers a nice balance between size and contrast as shows the example of the segmentation of the cameraman, where 3 segmentations are compared with the same number of regions: the first based on the depth, the second on the area and the last on the volume of the lakes.

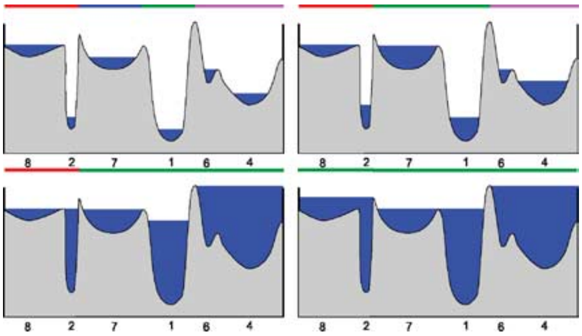


Figure 16. Example of a height synchronous flooding. Four levels of flooding are illustrated; each of them is topped by a figuration of the corresponding catchment basins.

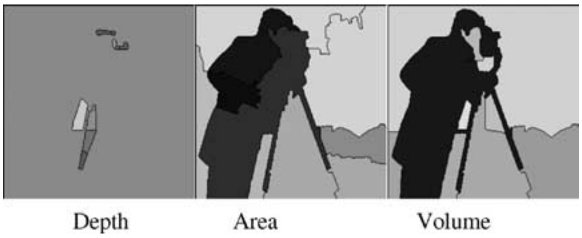


Figure 17. Comparison of 3 segmentations based resp. on depth, area and volume based flooding of a gradient image.

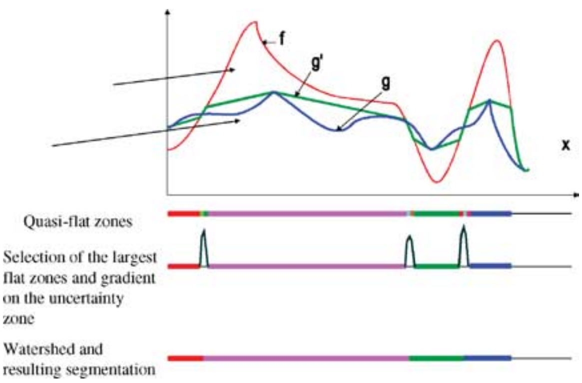


Figure 18. The major steps for constructing a segmentation: leveling, detection of flat zones, extraction of seeds and final watershed.

5.3.2. Segmentation Based on the Detection of Smooth Zones and Watershed. The Fig. 18 shows another way to use levelings in the segmentation process, based on the extraction and growing of smooth zones. The image f has to be segmented. To this purpose a marker image g is created and the leveling g' of f constrained by g offers a simplified representation of f, sharing with f the same steep transition and departing from f through large flat zones. The detection of the flat zones produces some large flat zones and a manifold of tiny flat zones in the transition zones. The smaller flat zones are replaced by the modulus of the gradient of the image and the larger flat zones are used as markers for the watershed transform applied on this gradient. The result is a tesselation in which each large flat zone gave rise to a region.

On the result of the slope leveling (in Fig. 13), there are still 12532 quasi flat zones, but they obviously are of two different natures: tiny quasi flat zones within the transition zones, and larger quasi flat zones within the objects of interest. The smaller flat zones are replaced by the modulus of the gradient of the image and the larger flat zones are used as markers for the watershed transform applied on this gradient. The result is a tesselation in which each large flat zone gave rise to a region.

6. Conclusion

We have proposed a general family of filters able to simplify images without blurring their contours. Each filter is associated to a particular definition of transitions and smooth zones, and precisely enlarges existing smooth zones and creates new ones: they are as numerous as there are couples of adjunct extensive dilations and anti-extensive erosions. We have concluded the paper by presenting a few examples of applications.

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Fernand Meyer got an engineer degree from the Ecole des Mines de Paris in 1975. He works since 1975 at the Centre de Morphologie Mathematique (CMM) of the Ecole des Mines de Paris, where he is currently director. His first research area was "Early and automatic detection of cervical cancer on cytological smears," subject of his PhD thesis, obtained in 1979. He participated actively to the development of mathematical morphology: particle reconstruction, top-hat transform, the morphological segmentation paradigm based on the watershed transform and markers, the theory of digital skeleton, the introduction of hierarchical queues for high speed watershed computations, morphological interpolations, the theory of levelings, multiscale segmentation.