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Grey-level hit-or-miss transforms—Part I: Unified theory

Benoît Naegel[a,*], Nicolas Passat[b,c], Christian Ronse[b]

[a]EIG-HES (École d'Ingénieurs de Genève), 4 rue de la Prairie, CH-1202 Genève, Switzerland
[b]LSIIT, UMR 7005 CNRS-ULP (Laboratoire des Sciences de l'Image, de l'Informatique et de la Télédétection), Bd S. Brant, BP 10413, F-67412 Illkirch Cedex, France
[c]Institut Gaspard Monge, Laboratoire A2SI (Algorithmique et Architecture des Systèmes Informatiques), Groupe ESIEE, Cité Descartes, BP 99, F-93162 Noisy-le-Grand Cedex, France

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Abstract

The *hit-or-miss transform* (HMT) is a fundamental operation on binary images, widely used since 40 years. As it is not increasing, its extension to grey-level images is not straightforward, and very few authors have considered it. Moreover, despite its potential usefulness, very few applications of the grey-level HMT have been proposed until now. Part I of this paper, developed hereafter, is devoted to the description of a theory leading to a unification of the main definitions of the grey-level HMT, mainly proposed by Ronse and Soille, respectively (part II will deal with the applicative potential of the grey-level HMT, which will be illustrated by its use for vessel segmentation from 3D angiographic data). In this first part, we review the previous approaches to the grey-level HMT, especially the *supremal* one of Ronse, and the *integral* one of Soille; the latter was defined only for flat structuring elements (SEs), but it can be generalized to non-flat ones. We present a unified theory of the grey-level HMT, which is decomposed into two steps. First a *fitting* associates to each point the set of grey-levels for which the SEs can be fitted to the image; as in Soille's approach, this fitting step can be *constrained*. Next, a *valuation* associates a final grey-level value to each point; we propose three valuations: *supremal* (as in Ronse), *integral* (as in Soille) and *binary*.

Keywords: Mathematical morphology; Hit-or-miss transform; Grey-level interval operator; Morphological probing

1. Introduction

Consider a Euclidean or digital space E ($E = \mathbf{R}^n$ or $\mathbf{Z}^n$). For $X \in \mathscr{P}(E)$, write $X^c = E \setminus X$ (the complement of X), $\check{X} = \{-x \mid x \in X\}$ (the symmetrical of X), and for $p \in E$, $X_p = \{x + p \mid x \in X\}$ (the translate of X by p). Then the Minkowski addition $\oplus$ and subtraction $\ominus$ are defined by setting for $X, B \in \mathscr{P}(E)$:

$$X \oplus B = \bigcup_{b \in B} X_b \quad \text{and} \quad X \ominus B = \bigcap_{b \in B} X_{-b}.$$

This leads to the operators $\delta_B : X \mapsto X \oplus B$ (dilation by B) and $\varepsilon_B : X \mapsto X \ominus B$ (erosion by B); here B is considered as a structuring element (SE) that acts on the binary image X. (NB: Our terminology follows [1,2], in accordance with the algebraic theory of dilations and erosions; it is slightly different from that of Refs. [3,4], in the sense that for some operations, the SE B is replaced by its symmetrical $\check{B}$, see Refs. [2,5] for a more detailed discussion.)

The *hit-or-miss transform* (in brief, HMT) uses a pair (A, B) of SEs, and looks for all positions where A can be fitted within a figure X, and B within the background X^c, in other words it is defined by

$$X \circledast (A, B) = \{p \in E \mid A_p \subseteq X \text{ and } B_p \subseteq X^c\}$$
$$= (X \ominus A) \cap (X^c \ominus B). \quad (1)$$

One assumes that $A \cap B = \emptyset$, otherwise we have always $X \circledast (A, B) = \emptyset$. One calls A and B, respectively, the *foreground* and *background* SE. In practice, one often uses bounded SEs A and B.

* Corresponding author. Tel.: +41 022 338 05 66.
E-mail address: benoit.naegel@hesge.ch (B. Naegel).

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This operation was devised by Matheron and Serra in the mid-sixties [3,6], and has been widely used since. It represents the morphological expression of the notion of template matching.

The binary HMT is often applied in shape recognition, for example in document analysis [7–9]. Hardware implementations with optical correlators have been studied in Refs. [10–15]. These implementations seem interesting, since computational time is independent from the size of the SE used, which is obviously not the case with software ones.

A recurrent issue consists in determining the SEs in order to cope with the noise and the variability of the patterns to be recognized.

Zhao and Daut [16] propose a method to match imperfect shapes in an image. They start with a set of shapes to be recognized, then smooth each element of this set by some kind of opening. The boundaries of these smoothed sets are then used as SEs for the HMT.

Doh et al. [17] discuss the choice of SEs for the recognition of a class of various objects. They start from two sets: a set of hit SEs (i.e., SEs that fit the objects to be recognized) and a set of miss SEs (SEs that fit the background). Their conclusion is to use a synthetic hit SE composed of the intersection of all hit SEs and a synthetic miss SE composed of the union of all miss SEs.

Bloomberg et al. [8,9] introduce a blur HMT which consists in dilating both set X and complement X^c. They also propose to subsample the SEs by imposing a regular grid. This allows the HMT to be less sensitive to noise while preserving the global characteristics of the shape.

The operator $X \mapsto (X \circledast (A, B)) \oplus A$ has been considered in Ref. [18] (it was suggested to the author by Heijmans), and later in Ref. [4, p. 149], where it was called *hit-or-miss opening*. It is idempotent and anti-extensive, like an opening, but not increasing.

Although the HMT is widely used in binary image processing, there are only a few authors who considered its possible extension to grey-level images (we review the main works in the next section). The main difficulty resides in the fact that this operator uses both the set X and its complement X^c, and is thus neither increasing nor decreasing. Let us explain how to remove X^c from definition (1).

Let $A, B \in \mathscr{P}(E)$ such that $A \subseteq B$. Consider the *interval*

$$[A, B] = \{C \in \mathscr{P}(E) \mid A \subseteq C \subseteq B\}.$$

Then we define $\eta_{[A,B]}$, the *interval operator by* $[A, B]$, by setting for every $X \in \mathscr{P}(E)$:

$$\begin{aligned}\eta_{[A,B]}(X) &= \{p \in E \mid X_{-p} \in [A, B]\} \\ &= \{p \in E \mid A_p \subseteq X \subseteq B_p\}. \qquad (2)\end{aligned}$$

Heijmans and Serra [19] were the first to consider such an operation, but they wrote it $X \oslash (A, B)$ instead of $\eta_{[A,B]}(X)$. Clearly $\eta_{[A,B]}(X) = X \circledast (A, B^c)$. Here the inclusion constraint $A \subseteq B$ (without which we always get $\eta_{[A,B]}(X) = \emptyset$) corresponds to the disjointness condition $A \cap B^c = \emptyset$ of the corresponding HMT $X \circledast (A, B^c)$. In practice, one usually chooses A and the complement B^c of B to be bounded.

This variant formulation was fruitful. First it allowed to give a very short proof of the theorem of Banon and Barrera [20], namely that every translation-invariant operator is a union of HMTs. More precisely, given a *translation-invariant* operator $\psi : \mathscr{P}(E) \to \mathscr{P}(E)$, Matheron's *kernel* [6] is the set

$$\mathscr{V}(\psi) = \{A \in \mathscr{P}(E) \mid 0 \in \psi(A)\}, \qquad (3)$$

and indeed Matheron showed that *if ψ is increasing*, we have

$$\psi(X) = \bigcup_{A \in \mathscr{V}(\psi)} X \ominus A \qquad (4)$$

for every $X \in \mathscr{P}(E)$, in other words ψ is a union of erosions. Consider now the *bi-kernel* [19]

$$\mathscr{W}(\psi) = \{(A, B) \in \mathscr{P}(E)^2 \mid A \subseteq B, [A, B] \subseteq \mathscr{V}(\psi)\}, \qquad (5)$$

then an elegant proof in Ref. [19] shows that for every $X \in \mathscr{P}(E)$ we have

$$\psi(X) = \bigcup_{(A,B) \in \mathscr{W}(\psi)} \eta_{[A,B]}(X), \qquad (6)$$

in other words ψ is a union of interval operators (equivalently, of HMTs).

However, the main advantage of considering an interval operator (2) instead of a HMT (1), is that it gave way to the first theory (by Ronse [18]) of interval operators on grey-level images and more generally on complete lattices, in particular the operators $\delta_A \eta_{[A,B]}$ are part of a very interesting family of idempotent and anti-extensive operators, called in Ref. [18] *open-over-condensations*.

A few years later, Soille [4,21] gave independently another definition of a HMT for grey-level images. His framework was restricted to the use of flat SEs and of discrete grey-levels. However, as we will see in the next section, it can easily be generalized to non-flat structuring functions and to images with arbitrary grey-levels (continuous or discrete). Moreover, he introduced the possibility of constraining the HMT; as we will see later on, this constraining of the HMT can also be applied to Ronse's version.

When it is extended to non-flat SEs, the unconstrained version of Soille's HMT has some resemblance with Ronse's interval operator [18], and is also very similar to an operation introduced by Barat et al. [22–24] under the name of *morphological probing*.

The authors have successfully applied grey-level HMTs to the detection of blood vessels in 3D angiographic images [25–28]. In fact, we used both Ronses and Soille's unconstrained versions, but also some new variants. Therefore we have felt that it would be useful to make a review of the different grey-level HMTs found in the literature, and to give a unified theory containing each one as a particular case.

The paper is organized as follows. In Section 2 we review the various approaches to the grey-level HMT found in the literature, mainly the ones of Ronse [18], Soille [4,21] and Barat et al. [22–24]; we generalize Soille's approach to arbitrary (not necessarily flat) SEs and arbitrary (not necessarily discrete) grey-levels. We will see that these HMTs can be better understood by expressing them as grey-level extensions of the interval operator $\eta_{[A,B]}$ (2).

In Section 3 we give a unified theory of grey-level interval operators. Such an operator can be decomposed into two steps:

(i) a *fitting* which extracts from a grey-level image and a pair of structuring functions, a set of pairs (p,t) (p a point, t a grey-level); we have two versions (following the approaches of Ronse and Soille), and each one can optionally be *constrained* as in Soille's approach;
(ii) a *valuation* which constructs from this set of pairs (p,t) the resulting grey-level image; we have three versions: a *supremal* one (following Ronse), an *integral* one (following Soille), and a *binary* one (which produces a binary image).

This gives thus in theory a set of six unconstrained grey-level HMTs, and six constrained ones (however, there is some redundancy in this set).

The Conclusion summarizes our findings and gives some perspectives for further research. In particular, we have not extended our theory to the general framework of complete lattices, nor have we analysed the operators obtained by composition of the HMT and the dilation by the foreground SE (both things were done in Ref. [18] for one version of the HMT).

Part II of this paper [29] will provide a review of our work on the application of grey-level HMTs to the detection and enhancement of blood vessels in 3D angiographic images, but also algorithmic remarks about grey-level HMT, still valid for more general applications.

2. Existing approaches to the grey-level HMT

We will review the various forms given in the literature for the grey-level HMT. But let us beforehand recall the basics from grey-level morphology [1,30].

We consider a space E of points, which can in general be an arbitrary set. However, in order to define translation-invariant operators (like the dilation and erosion by a SE), we need to add and subtract points, so in this case we assume E to be the digital space $\mathbf{Z}^n$ or the Euclidean space $\mathbf{R}^n$, for which the addition and subtraction of vectors are well-defined.

We have a set T of grey-levels, which is part of the extended real line $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty, -\infty\}$. We require T to be closed under non-void infimum and supremum operations (equivalently, T is a topologically closed subset of $\overline{\mathbf{R}}$); for example we can take $T = \overline{\mathbf{R}}$, $T = \overline{\mathbf{Z}} = \mathbf{Z} \cup \{+\infty, -\infty\}$, $T = [a,b]$ ($a, b \in \mathbf{R}$, $a < b$) or $T = [a \ldots b] = [a,b] \cap \mathbf{Z}$ ($a, b \in \mathbf{Z}$, $a < b$). Then T is a *complete lattice* [1] w.r.t. the numerical order $\leqslant$. Write $\top$ and $\bot$, respectively, for the greatest and least elements of T.

Grey-level images are numerical functions $E \to T$, they are generally written by capital letters $F, G, H, \ldots$ The set T^E of such functions is a complete lattice for the componentwise ordering defined by

$$F \leqslant G \iff \forall p \in E, \quad F(p) \leqslant G(p),$$

with the componentwise supremum and infimum operations:

$$\bigvee_{i \in I} F_i : E \to V : p \mapsto \sup_{i \in I} F_i(p)$$

and

$$\bigwedge_{i \in I} F_i : E \to V : p \mapsto \inf_{i \in I} F_i(p).$$

Let us now introduce some notation. Given $F, G \in T^E$, we write $G \gg F$ (or equivalently, $F \ll G$) if there is some $h > 0$ such that for every $p \in E$ we have $G(p) \geqslant F(p) + h$. For $F \in T^E$ and $p \in E$, the *translate* of F by p is the function $F_p : E \to T : x \mapsto F(x-p)$. The *support* of a function F is the set $\mathrm{supp}(F)$ of points of E having grey-level $F(p)$ strictly above the least value $\bot$:

$$\mathrm{supp}(F) = \{p \in E \mid F(p) > \bot\}, \tag{7}$$

and the *dual support* of F is the set $\mathrm{supp}^*(F)$ of points of E having grey-level $F(p)$ strictly below the greatest value $\top$:

$$\mathrm{supp}^*(F) = \{p \in E \mid F(p) < \top\}. \tag{8}$$

For every $t \in T$, write C_t for the function $E \to T$ with constant value t: $\forall p \in E$, $C_t(p) = t$. We see in particular that the least and greatest elements of the lattice T^E of numerical functions are the constant functions $C_\bot$ and $C_\top$, respectively. For any $B \subseteq E$ and $t \in T$, the *cylinder of base B and level t* is the function $C_{B,t}$ defined by

$$\forall p \in E, \quad C_{B,t}(p) = \begin{cases} t & \text{if } p \in B, \\ \bot & \text{if } p \notin B. \end{cases} \tag{9}$$

Note in particular that $C_t = C_{E,t}$. Also, for $h \in E$ and $t \in T$, the *impulse* $i_{h,t}$ is the cylinder $C_{\{h\},t}$, thus

$$\forall p \in E, \quad i_{h,t}(p) = \begin{cases} t & \text{if } p = h, \\ \bot & \text{if } p \neq h. \end{cases} \tag{10}$$

For $F \in T^E$, we have $i_{h,t} \leqslant F$ iff $t \leqslant F(h)$, and

$$F = \bigvee \{i_{h,t} \mid h \in E,\ t \in T,\ t \leqslant F(h)\},$$

in other words every function is a supremum of the impulses below it.

The *dual cylinder of base B and level t* is the function $C^*_{B,t}$ defined by

$$\forall p \in E, \quad C^*_{B,t}(p) = \begin{cases} t & \text{if } p \in B, \\ \top & \text{if } p \notin B. \end{cases} \tag{11}$$

For $V, W \in T^E$ with $V \leqslant W$, we have the *interval* $[V, W] = \{F \in T^E \mid V \leqslant F \leqslant W\}$.

Every *increasing* operator $\psi : \mathscr{P}(E) \to \mathscr{P}(E)$ on sets extends to a *flat* operator $\psi^T : T^E \to T^E$ on grey-level images [31]. For every $F \in T^E$ and $t \in T$ we define the *threshold set* [1]

$$X_t(F) = \{p \in E \mid F(p) \geqslant t\}.$$

Clearly $X_t(F)$ is decreasing with respect to t. Now ψ^T is defined by applying ψ to each threshold set and stacking the results. Formally:

$$\psi^T(F) = \bigvee_{t \in T} C_{\psi(X_t(F)),t}, \tag{12}$$

so that for every point $p \in E$ we have

$$\psi^T(F)(p) = \bigvee \{t \in T \mid p \in \psi(X_t(F))\}. \tag{13}$$

In particular, when $E = \mathbf{R}^n$ or $\mathbf{Z}^n$, the dilation δ_B and erosion ε_B by a SE B extend as follows:

$$\delta^T_B(F) = \bigvee_{b \in B} F_b \quad \text{and} \quad \varepsilon^T_B(F) = \bigwedge_{b \in B} F_{-b}, \tag{14}$$

so that for every point $p \in E$ we have

$$\delta^T_B(F)(p) = \bigvee_{b \in B} F(p-b)$$

and

$$\varepsilon^T_B(F)(p) = \bigwedge_{b \in B} F(p+b). \tag{15}$$

We will also write $F \oplus B$ and $F \ominus B$ for $\delta^T_B(F)$ and $\varepsilon^T_B(F)$, respectively.

Let us now consider morphological operations with SEs that are functions instead of sets. Here grey-levels will be added and subtracted in formulas, thus in order to avoid grey-level overflow in computations, T must necessarily be unbounded (so $\top = +\infty$ and $\bot = -\infty$), in fact we assume that $T = \overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$ (however, $T = \overline{a\mathbf{Z}} = \{az \mid z \in \mathbf{Z}\} \cup \{+\infty, -\infty\}$, where $a > 0$, is also possible). Let $T' = T \setminus \{+\infty, -\infty\}$, the set of finite grey-levels. We saw above that a function F can be translated by a point $p \in E$, this is a horizontal translation; now there is also a vertical translation, namely by a finite grey-level $t \in T'$, transforming F into $F + t$. Combining both, we get the translation by (p, t), the translate of F by (p, t) is $F_{(p,t)} = F_p + t : x \mapsto F(x-p) + t$. We consider impulses $i_{h,t}$ only for $t \in T'$. The *umbra* of a function $F \in T^E$ is the set

$$U(F) = \{(h, t) \mid h \in E, t \in T', t \leqslant F(h)\}. \tag{16}$$

Note that for an impulse $i_{h,t}$, we have $i_{h,t} \leqslant F$ iff $(h, t) \in U(F)$, and

$$F = \bigvee \{i_{h,t} \mid (h, t) \in U(F)\}. \tag{17}$$

For $F, G \in T^E$, we can define the Minkowski addition $F \oplus G$ and subtraction $F \ominus G$ as follows:

$$F \oplus G = \bigvee_{(h,t) \in U(G)} F_{(h,t)}$$

and

$$F \ominus G = \bigwedge_{(h,t) \in U(G)} F_{(-h,-t)}. \tag{18}$$

At every point $p \in E$ we have

$$\begin{aligned}(F \oplus G)(p) &= \sup_{h \in E} (F(p-h) + G(h)) \\ &= \sup_{h \in \mathrm{supp}(G)} (F(p-h) + G(h))\end{aligned}$$

and

$$\begin{aligned}(F \ominus G)(p) &= \inf_{h \in E} (F(p+h) - G(h)) \\ &= \inf_{h \in \mathrm{supp}(G)} (F(p+h) - G(h)).\end{aligned}$$

Since $T = \overline{\mathbf{R}}$ or $\overline{\mathbf{Z}}$, the terms $F(p-h)$, $F(p+h)$, and $G(h)$ can have an infinite value, so the expressions $F(p-h) + G(h)$ and $F(p+h) - G(h)$ can take the form $+\infty - \infty$ or $-\infty + \infty$, which are arithmetically undefined; then their evaluation is achieved by the following rules:

- In the formula for $(F \oplus G)(p)$, we consider that $+\infty = \bigvee T'$ and $-\infty = \bigvee \emptyset$, so $+\infty - \infty = \bigvee_{t \in T'} \bigvee_{t' \in \emptyset} (t + t') = \bigvee \emptyset = -\infty$, in other words expressions of the form $+\infty - \infty$ or $-\infty + \infty$ must be evaluated as $-\infty$.
- Dually, in the formula for $(F \ominus G)$, we consider that $+\infty = \bigwedge \emptyset$ and $-\infty = \bigwedge T'$, so expressions of the form $+\infty - \infty$ or $-\infty + \infty$ must be evaluated as $+\infty$.

We obtain thus the dilation and erosion by G, namely $\delta_G : F \mapsto F \oplus G$ and $\varepsilon_G : F \mapsto F \ominus G$. These two operations form an *adjunction* [1]:

$$\forall F_1, F_2 \in T^E, \quad F_1 \oplus G \leqslant F_2 \iff F_1 \leqslant F_2 \ominus G. \tag{19}$$

Consider the *symmetrical* $\check{G}$ of G defined by $\check{G}(x) = G(-x)$, and the *grey-level inversion* $T \to T : t \mapsto -t$, which extends to functions by transforming F into $-F : x \mapsto -F(x)$. From Eq. (18) is easily seen that

$$-(F \oplus G) = (-F) \ominus \check{G}$$

and

$$-(F \ominus G) = (-F) \oplus \check{G}, \tag{20}$$

in other words, erosion is the *dual under grey-level inversion* of the dilation with the symmetrical structuring function. Let us define the *dual* of G as $G^* = -\check{G} : x \mapsto -G(-x)$.

Taking a set $B \in \mathscr{P}(E)$ as SE, the flat dilation and erosion by B seen in Eqs. (14) and (15) are a particular case of dilation and erosion by a grey-level function, since we have

$$F \oplus B = F \oplus C_{B,0} \quad \text{and} \quad F \ominus B = F \ominus C_{B,0}. \tag{21}$$

More generally, for $t \in T'$, we have

$$F \oplus C_{B,t} = (F \oplus B) + t$$

and

$$F \ominus C_{B,t} = (F \ominus B) - t. \tag{22}$$

Structuring functions of the form $C_{B,0}$ are also called *flat SEs*.

Grey-level Minkowski operations do not always preserve the bounds of image grey-levels:

Lemma 1. *Let $F, G \in T^E$ such that $F(p) \in [a, b]$ for all $p \in E$, and let $g = \sup_{p \in E} G(p)$. Then for all $p \in E$ we have $(F \oplus G)(p) \in [a+g, b+g]$ and $(F \ominus G)(p) \in [a-g, b-g]$.*

Proof. From Eq. (18) we check easily that for any $t \in T$, $C_t \oplus G = C_{t+g}$ and $C_t \ominus G = C_{t-g}$. The fact that $\forall p \in E$, $F(p) \in [a, b]$, means that $C_a \leqslant F \leqslant C_b$. Hence we get $C_{a+g} = C_a \oplus G \leqslant F \oplus G \leqslant C_b \oplus G = C_{b+g}$ and $C_{a-g} = C_a \ominus G \leqslant F \ominus G \leqslant C_b \ominus G = C_{b-g}$, that is $\forall p \in E$, $(F \oplus G)(p) \in [a+g, b+g]$ and $(F \ominus G)(p) \in [a-g, b-g]$. □

The next result is fundamental for our analysis:

Proposition 2. *Let $F, V, W \in T^E$, $p \in E$ and $t \in T'$. Then*:

(i) $V_{(p,t)} \leqslant F$ *iff* $(F \ominus V)(p) \geqslant t$.
(ii) $V_{(p,t)} \lll F$ *iff* $(F \ominus V)(p) > t$.
(iii) $F \leqslant W_{(p,t)}$ *iff* $(F \oplus W^*)(p) \leqslant t$.
(iv) $F \lll W_{(p,t)}$ *iff* $(F \oplus W^*)(p) < t$.

Proof. (1) $(F \ominus V)(p) \geqslant t$ means $i_{(p,t)} \leqslant F \ominus V$, and by adjunction (19) this is equivalent to $i_{(p,t)} \oplus V \leqslant F$; but $i_{(p,t)} \oplus V = V_{p,t}$, so the result follows.

(2) $V_{(p,t)} \lll F$ iff there is some $h > 0$ with $V_{(p,t)} \leqslant F - h$; by item 1, this is equivalent to $((F - h) \ominus V)(p) \geqslant t$, in other words $(F \ominus V)(p) - h \geqslant t$ for some $h > 0$, that is $(F \ominus V)(p) > t$.

(3) By grey-level inversion, $F \leqslant W_{(p,t)}$ iff $-F \geqslant -(W_{(p,t)}) = (-W)_{(p,-t)}$. Applying item 1, this is equivalent to $((-F) \ominus (-W))(p) \geqslant -t$. Inverting again, this means $-((-F) \ominus (-W))(p) \leqslant t$; by duality (20), $-((-F) \ominus (-W)) = -(-F) \oplus (-W)^\vee = F \oplus W^*$, and the result follows.

(4) $F \lll W_{(p,t)}$ iff there is some $h > 0$ with $F + h \leqslant W_{(p,t)}$; by item 3, this is equivalent to $((F + h) \oplus W^*))(p) \leqslant t$, in other words $(F \oplus W^*)(p) + h \leqslant t$ for some $h > 0$, that is $(F \oplus W^*)(p) < t$. □

Let us apply this result to the case where $(F \ominus V)(p)$ or $(F \oplus W^*)(p)$ has an infinite value:

Corollary 3. *Let $F, V, W \in T^E$ and $p \in E$. Then*:

(i) $(F \ominus V)(p) = +\infty$ *iff* $(\forall t \in T',\ V_{(p,t)} \leqslant F)$ *iff* $\bigvee_{t \in T'} V_{(p,t)} \leqslant F$.
(ii) $(F \ominus V)(p) = -\infty$ *iff* $(\forall t \in T',\ V_{(p,t)} \not\lll F)$.
(iii) $(F \oplus W^*)(p) = +\infty$ *iff* $(\forall t \in T',\ F \not\lll W_{(p,t)})$.
(iv) $(F \oplus W^*)(p) = -\infty$ *iff* $(\forall t \in T',\ F \leqslant W_{(p,t)})$ *iff* $F \leqslant \bigwedge_{t \in T'} W_{(p,t)}$.

Proof. Items 1 and 2 follow from item 1 of Proposition 2, and the fact that $+\infty$ is the only value $\geqslant t$ for all $t \in T'$, while $-\infty$ is the only one $\not> t$ for all $t \in T'$. Items 3 and 4 follow from item 3 of Proposition 2, and the fact that $+\infty$ is the only value $\not< t$ for all $t \in T'$, while $-\infty$ is the only one $\leqslant t$ for all $t \in T'$. □

Note that if F has all its values in an interval $[t_0, t_1] \subset \mathbf{R}$, and $\sup_{p \in E} V(p) = v \in \mathbf{R}$ and $\inf_{p \in E} W(p) = w \in \mathbf{R}$, then by Lemma 1, $F \ominus V$ and $F \oplus W^*$ will have all their values in the intervals $[t_0 - v, t_1 - v]$ and $[t_0 - w, t_1 - w]$, respectively, hence infinite values do not occur in such a case.

2.1. Ronse's supremal interval operator

The basic ideas in Ronse's approach [18] are to start from the interval operator (2) instead of the HMT, and to consider the fact that a grey-level image is a supremum of impulses (17) as the parallel of the fact that a set is a union of singletons. We still assume that $T = \overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$. We define thus for $V, W \in T^E$ such that $V \leqslant W$ the *supremal interval operator* $\eta^S_{[V,W]}$ by setting for every $F \in T^E$:

$$\begin{aligned}\eta^S_{[V,W]}(F) &= \bigvee \{ i_{(p,t)} \mid (p,t) \in E \times T',\ F_{(-p,-t)} \in [V, W] \} \\ &= \bigvee \{ i_{(p,t)} \mid (p,t) \in E \times T',\ V_{(p,t)} \leqslant F \leqslant W_{(p,t)} \}. \end{aligned} \tag{23}$$

Note that following Ref. [19], Ronse wrote $F \circledS (V, W)$ for $\eta^S_{[V,W]}(F)$. By Proposition 2, for $t \in T'$, $V_{(p,t)} \leqslant F \leqslant W_{(p,t)}$ iff $(F \oplus W^*)(p) \leqslant t \leqslant (F \ominus V)(p)$. Hence for every $p \in E$,

$$\eta^S_{[V,W]}(F)(p) = \sup \{ t \in T' \mid V_{(p,t)} \leqslant F \leqslant W_{(p,t)} \}.$$

Now for $a \leqslant b$, we have $b = \sup\{ t \in T' \mid a \leqslant t \leqslant b \}$, except if $a = b = +\infty$, in which case we get the empty supremum, that is $-\infty$. We obtain thus:

$$\eta^S_{[V,W]}(F)(p) = \begin{cases} (F \ominus V)(p) & \text{if } (F \ominus V)(p) \geqslant (F \oplus W^*)(p) \neq +\infty, \\ -\infty & \text{otherwise.} \end{cases} \tag{24}$$

Note that if $(F \ominus V)(p) = (F \oplus W^*)(p) = +\infty$, by Corollary 3 we have $F \not\lll W_{(p,t)}$ for all $t \in T'$, so that $\eta^S_{[V,W]}(F)(p) = -\infty$, and not $+\infty$.

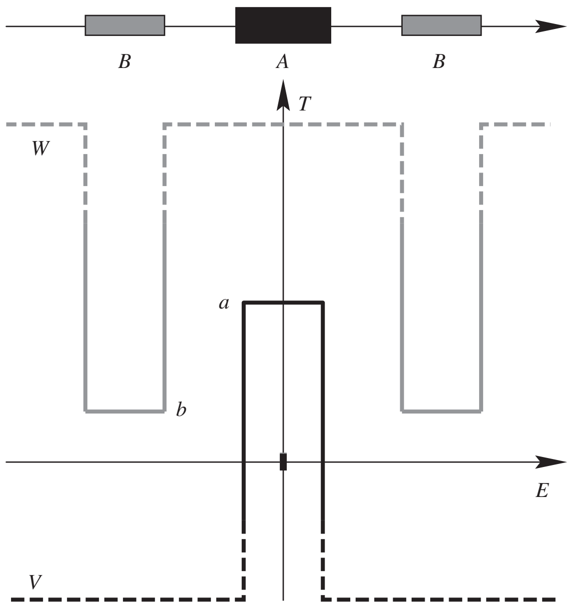

Fig. 1. Top: The two structuring elements A (in black) and B (in grey). Bottom: the cylinder $V = C_{A,a}$ (in black) has support A and the dual cylinder $W = C^*_{B,b}$ (in grey) has dual support B.

In practice, one usually chooses V with bounded support, and W with bounded dual support (i.e., W^* has bounded support). For example, we can take $V = C_{A,a}$ and $W = C^*_{B,b}$, see Fig. 1; then $W^* = C_{\check{B},-b}$ and by Eq. (22), $F \ominus V = (F \ominus A) - a$ and $F \oplus W^* = (F \oplus \check{B}) - b$, so that Eq. (24) gives here:

$$\eta^S_{[V,W]}(F)(p) = \begin{cases} (F \ominus A)(p) - a & \text{if } (F \ominus A)(p) \\ & \quad \geqslant (F \oplus \check{B})(p) + a - b \neq +\infty, \\ -\infty & \text{otherwise.} \end{cases}$$

For A, B and a fixed, $\eta^S_{[V,W]}(F)$ increases with b, as more and more points will get the value $(F \ominus A)(p) - a$ instead of $-\infty$. We illustrate this in Fig. 2.

The operator $\delta_V \eta^S_{[V,W]}$ maps $F \in T^E$ on

$$\bigvee \{V_{(p,t)} \mid (p,t) \in E \times T',\, V_{(p,t)} \leqslant F \leqslant W_{(p,t)}\}.$$

It is idempotent and anti-extensive like an opening [18], but not increasing. It is part of a family of operators called *open-over-condensations*.

In Ref. [18] the theorem of Banon–Barrera, Refs. (5) and (6) was also extended to grey-level images (and more generally, in a complete lattice where Minkowski operations are properly defined [2]): every translation-invariant operator is a supremum of supremal interval operators.

2.2. *Soille's integral HMT*

Soille [4,21] assumes discrete grey-levels (an interval in $\mathbf{Z}$) and flat SEs. If we return to formula (13) for the

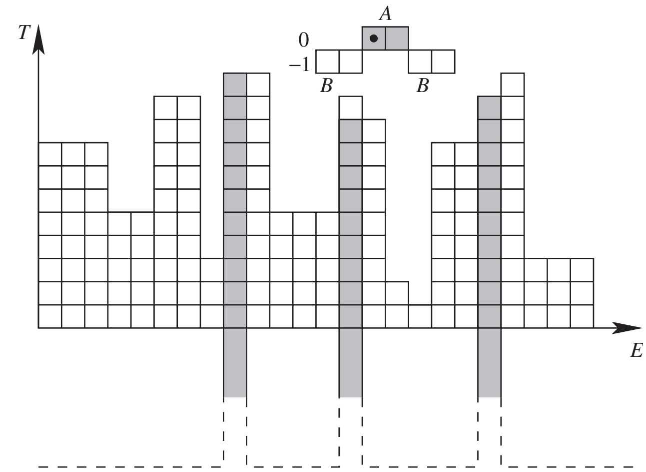

Fig. 2. Here $E = \mathbf{Z}$ and $T = \overline{\mathbf{Z}}$. On top we show the two structuring elements A and B (the origin being the left pixel of A), with the associated levels $a = 0$ and $b = -1$ (thus $V = C_{A,0}$ and $W = C^*_{B,-1}$). Below we show a function F, and in grey we have $\eta^S_{[V,W]}(F)$, forming three peaks. The left peak would disappear for $b \leqslant -2$, and the right one for $b \leqslant -3$.

construction of the flat operator ψ^T from an *increasing* set operator ψ, the set of all $t \in T$ such that $p \in \psi(X_t(F))$ is a closed interval $[\bot, b]$, where b gives the value $\psi^T(F)(p)$ (NB: this holds because we have discrete grey-levels; otherwise we could have the half-open interval $[\bot, b[$). This is no longer valid if ψ is not increasing; in particular, if ψ is a HMT, we will see below that it is an interval, but generally not containing $\bot$. The idea in Refs. [4,21] is to take as value of the grey-level HMT the length of that interval.

Let $A, B \in \mathscr{P}(E)$ be disjoint SEs, and consider the finite grey-level set $\hat{T} = [t_0 \ldots t_1] \subset \mathbf{Z}$. Soille's (unconstrained) HMT on grey-level images, written $UHMT_{A,B}$, is defined [4, Eq. (5.3), p. 143] by setting for every $F \in \hat{T}^E$ and $p \in E$:

$$UHMT_{A,B}(F)(p) = card\{t \in T \mid p \in X_t(F) \circledast (A,B)\}. \tag{25}$$

Note that the resulting grey-level values will be non-negative, in fact they belong to the interval $[0, t_1 - t_0]$. We illustrate it in Fig. 3 (to be compared with Fig. 2).

In order to analyse Soille's operator, we embed the grey-level set $\hat{T}$ into $\overline{\mathbf{Z}}$:

Proposition 4. *Let* $A, B \in \mathscr{P}(E)$, $T = \overline{\mathbf{Z}}$ *and* $\hat{T} = [t_0 \ldots t_1] \subset \mathbf{Z}$. *For every* $t \in T$, $F \in \hat{T}^E$ *and* $p \in E$, *we have* $p \in X_t(F) \circledast (A,B)$ *iff*

$$(C_{A,0})_{(p,t)} = C_{A_p,t} \leqslant F \lll C^*_{B_p,t} = (C^*_{B,0})_{(p,t)},$$

iff $(F \oplus \check{B})(p) < t \leqslant (F \ominus A)(p)$.

Proof. Recall that $q \in X_t(F)$ iff $F(q) \geqslant t$. The condition $p \in X_t(F) \circledast (A,B)$ means that $A_p \subseteq X_t(F)$ and $B_p \subseteq X_t(F)^c$. The first part $A_p \subseteq X_t(F)$ translates as: for every $q \in A_p$, $F(q) \geqslant t$; on the other hand, for $q \notin A_p$, we have

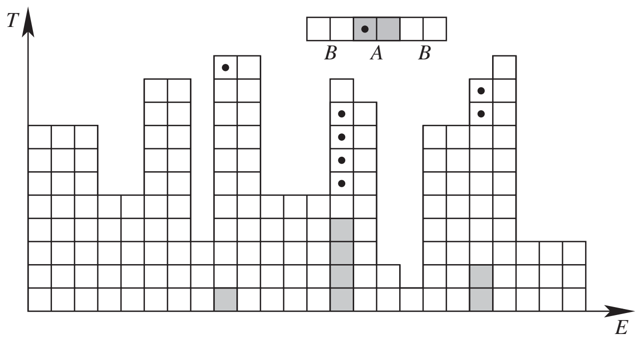

Fig. 3. Here $E=\mathbf{Z}$ and $T=[0\ldots t_1]\subset\mathbf{Z}$. On top we show the two structuring elements A and B (the origin being the left pixel of A). Below we show a function F (the same as in Fig. 2); the dots indicate the pairs (p,t) with $p\in X_t(F)\circledast(A,B)$, and in grey we have $UHMT_{A,B}(F)$.

always $F(q)\geqslant-\infty$; hence $A_p\subseteq X_t(F)\Leftrightarrow C_{A_p,t}\leqslant F$. The second part $B_p\subseteq X_t(F)^c$ translates as: for every $q\in B_p$, $F(q)<t$, that is $F(q)+1\leqslant t$; on the other hand, for $q\notin B_p$, we have always $F(q)+1\leqslant t_1+1\leqslant+\infty$; hence $B_p\subseteq X_t(F)^c\Leftrightarrow F\ll C^*_{B_p,t}$. Therefore $p\in X_t(F)\circledast(A,B)$ iff $C_{A_p,t}\leqslant F\ll C^*_{B_p,t}$. Now clearly $C_{A_p,t}=(C_{A,0})_{(p,t)}$ and $C^*_{B_p,t}=(C^*_{B,0})_{(p,t)}$. Applying Proposition 2, and the fact that $(C^*_{B,0})^*=C_{\check{B},0}$, the condition $(C_{A,0})_{(p,t)}\leqslant F\ll(C^*_{B,0})_{(p,t)}$ is equivalent to $(F\oplus C_{\check{B},0})(p)<t\leqslant(F\ominus C_{A,0})(p)$, in other words by (22), $(F\oplus\check{B})(p)<t\leqslant(F\ominus A)(p)$. □

We get thus:

$$UHMT_{A,B}(F)(p) = \max\{(F\ominus A)(p)-(F\oplus\check{B})(p),0\}, \qquad (26)$$

in other words [4, Eq. (5.4), p. 143] it has value

$$(F\ominus A)(p)-(F\oplus\check{B})(p)$$

if $(F\ominus A)(p)>(F\oplus\check{B})(p)$, and 0 otherwise.

From Proposition 4, we see that Soille's grey-level HMT is not restricted to flat SEs; the two sets A and B correspond implicitly to the cylinder $C_{A,0}$ and the dual cylinder $C^*_{B,0}$. Also, it does not require discrete grey-levels; we have simply to measure at each point p the half-open interval $](F\oplus\check{B})(p),(F\ominus A)(p)]$. Now the Lebesgue measure in $\mathbf{R}$ and the discrete measure (cardinal) in $\mathbf{Z}$, when applied to a half-open interval $]a,b]$, both give its length $b-a$.

Assume thus $T=\overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$. Let *mes* be the measure used on T' (Lebesgue's for $T'=\mathbf{R}$ and discrete for $T'=\mathbf{Z}$). For $V,W\in T^E$ such that $V\leqslant W$, we define the *integral interval operator* $\eta^I_{[V,W]}$ by setting for every $F\in T^E$ and $p\in E$:

$$\begin{aligned}\eta^I_{[V,W]}(F)(p) &= mes(\{t\in T' \mid V_{(p,t)}\leqslant F\ll W_{(p,t)}\})\\ &= mes(\{t\in T' \mid (F\oplus W^*)(p)<t\leqslant(F\ominus V)(p)\})\\ &= \max\{(F\ominus V)(p)-(F\oplus W^*)(p),0\}. \qquad (27)\end{aligned}$$

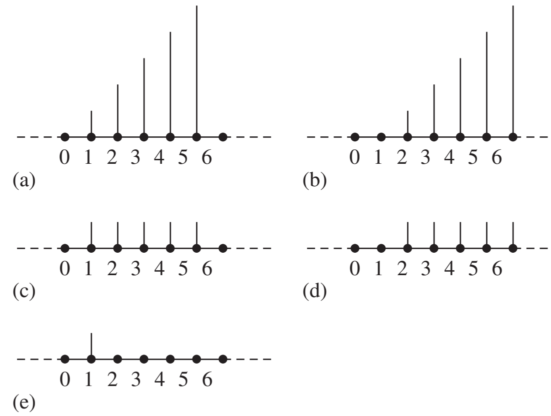

Fig. 4. (a) The function F; we have $F\ominus V=F$ (with $V=C_{\{0\},0}$). (b) The function $F'=F\oplus W^*$ (with $W=C^*_{\{-1\},0}$) is the translate of F by +1. (c) $G=\eta^I_{[V,W]}(F)$ is given by $G(p)=\max\{F(p)-F'(p),0\}$; we have $G=\delta_V(G)=\delta_V\eta^I_{[V,W]}(F)$, and $G=G\ominus V$. (d) $G'=G\oplus W^*$ is the translate of G by +1. (e) $H=\eta^I_{[V,W]}(G)$ is given by $H(p)=\max\{G(p)-G'(p),0\}$; we have $H=\delta_V(H)=\delta_V\eta^I_{[V,W]}(G)=(\delta_V\eta^I_{[V,W]})^2(F)$.

In the third line of the equation, an expression of the form $+\infty-\infty$ or $-\infty+\infty$ for $(F\ominus V)(p)-(F\oplus W^*)(p)$ must lead to the value 0. Indeed, if $(F\ominus V)(p)=(F\oplus W^*)(p)=+\infty$, Corollary 3 gives $F\not\ll W_{(p,t)}$ for all $t\in T'$, while if $(F\ominus V)(p)=(F\oplus W^*)(p)=-\infty$, Corollary 3 gives $V_{(p,t)}\not\leqslant F$ for all $t\in T'$; in both cases the second line of the equation gives $mes(\emptyset)=0$.

We can take, as above with Ronse's operator, $V=C_{A,a}$ and $W=C^*_{B,b}$. For A, B and a fixed, increasing b increases $\eta^I_{[V,W]}(F)$ by the same amount on all points having nonzero value. For flat SEs ($V=C_{A,0}$ and $W=C^*_{B,0}$), we obtain Soille's original operator $UHMT_{A,B}$.

As can be seen with Figs. 2 and 3, the two interval operators $\eta^S_{[V,W]}$ and $\eta^I_{[V,W]}$ can be used to detect in an image all locations p where the grey-level on A_p is higher than that on B_p by at least some height h: here we take $V=C_{A,a}$ and $W=C^*_{B,b}$ with $h=a-b$. While $\eta^S_{[V,W]}$ behaves as the erosion ε_V at such locations, $\eta^I_{[V,W]}$ measures the effective difference between the grey-levels in A_p and B_p.

Note that, contrarily to $\delta_V\eta^S_{[V,W]}$, the operator $\delta_V\eta^I_{[V,W]}$ is not necessarily idempotent. Take for example $E=\mathbf{Z}$, the flat SEs $A=\{0\}$ and $B=\{-1\}$ (thus $V=C_{\{0\},0}$ and $W=C^*_{\{-1\},0}$). Then $\delta_V=\varepsilon_V$ is the identity, while δ_{W^*} is the translation by +1. We illustrate in Fig. 4 the construction of $\delta_V\eta^I_{[V,W]}(F)$ and $(\delta_V\eta^I_{[V,W]})^2(F)$ for F given by $F(z)=z$ for $z=1\ldots5$ and $F(z)=0$ otherwise.

Soille introduced a constrained variant $CMHT_{A,B}$ of his HMT. Here we assume that one of the two SEs A and B contains the origin o. If $o\in A$, in Eq. (25) we require that $p\in X_t(F)\circledast(A,B)$ for $t=F(p)$, which means that $(F\oplus\check{B})(p)<F(p)\leqslant(F\ominus A)(p)$; if the requirement is not met, the result is 0. As $o\in A$, we always have

$(F \ominus A)(p) \leqslant F(p)$, hence we get

$$CHMT_{A,B}(F)(p) = card\{t \in T \mid (F \oplus \check{B})(p) < t \leqslant (F \ominus A)(p) = F(p)\},$$

in other words it is equal to

$$\begin{cases} (F \ominus A)(p) - (F \oplus \check{B})(p) & \text{if } F(p) = (F \ominus A)(p) > (F \oplus \check{B})(p), \\ 0 & \text{otherwise.} \end{cases}$$

If $o \in B$, in Eq. (25) we require that $p \in X_t(F) \circledast (A, B)$ for $t = F(p) + 1$, which means that $(F \oplus \check{B})(p) < F(p) + 1 \leqslant (F \ominus A)(p)$ that is $(F \oplus \check{B})(p) \leqslant F(p) < (F \ominus A)(p)$, and the result is 0 if this condition fails. As $o \in B$, we always have $(F \oplus \check{B})(p) \geqslant F(p)$, so we get

$$CHMT_{A,B}(F)(p) = card\{t \in T \mid F(p) = (F \oplus \check{B})(p) < t \leqslant (F \ominus A)(p)\},$$

in other words it is equal to

$$\begin{cases} (F \ominus A)(p) - (F \oplus \check{B})(p) & \text{if } (F \ominus A)(p) > (F \oplus \check{B})(p) = F(p), \\ 0 & \text{otherwise.} \end{cases}$$

In order to generalize this to arbitrary structuring functions, we can forget the requirement that A or B contains the origin, but keep only the constraint that $F(p) = (F \ominus A)(p)$ or $F(p) = (F \oplus \check{B})(p)$. Thus we obtain, for $V, W \in T^E$ such that $V \leqslant W$, the *constrained integral interval operator* $\eta^C_{[V,W]}$, which gives for every $F \in T^E$ and $p \in E$:

$$\eta^C_{[V,W]}(F)(p) = \begin{cases} \eta^I_{[V,W]}(F)(p) & \text{if } F(p) = (F \ominus V)(p) \text{ or } F(p) = (F \oplus W^*)(p), \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

2.3. Barat's morphological probing

Barat et al. [22–24] introduced under the name of *morphological probing* an operation which has some similarity to the integral grey-level interval operator $\eta^I_{[V,W]}$. We consider again two structuring functions $V, W \in T^E$; the idea is to measure at each point $p \in E$ two numerical values t_v and t_w defined as follows: t_v is the greatest t such that $V_{(p,t)} \leqslant F$, while t_w is the least t such that $F \leqslant W_{(p,t)}$; then one associates to p the value $t_w - t_v$.

From Proposition 2 and Corollary 3, we have

$$(F \ominus V)(p) = \sup\{t \in T' \mid V_{(p,t)} \leqslant F\}$$

and

$$(F \oplus W^*)(p) = \inf\{t \in T' \mid F \leqslant W_{(p,t)}\}. \quad (29)$$

Moreover:

- if $(F \ominus V)(p) \neq \pm\infty$, $(F \ominus V)(p)$ is the greatest $t \in T'$ such that $V_{(p,t)} \leqslant F$;

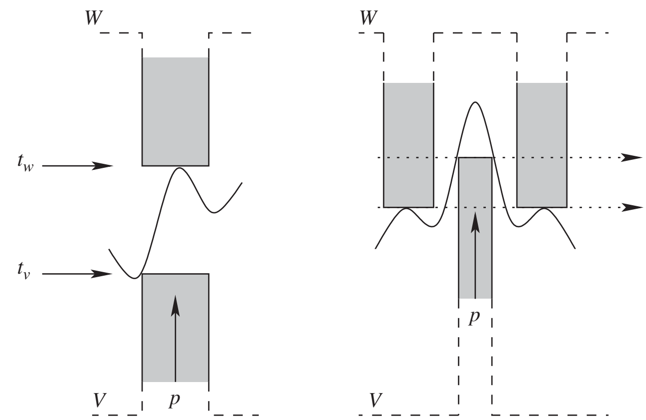

Fig. 5. Left: In morphological probing, we look for the least interval $[t_v, t_w]$ such that $V_{(p,t_v)} \leqslant F$ and $F \leqslant W_{(p,t_w)}$. Right: In the integral interval operator, we look for the greatest interval $\{t \mid V_{(p,t)} \leqslant F \leqslant W_{(p,t)}\}$.

- if $(F \oplus W^*)(p) \neq \pm\infty$, $(F \oplus W^*)(p)$ is the least $t \in T'$ such that $F \leqslant W_{(p,t)}$.

Thus Barat's morphological probing operator $MP_{V,W}$ is given by

$$MP_{V,W}(F)(p) = (F \oplus W^*)(p) - (F \ominus V)(p) \quad (30)$$

for every $F \in T^E$ and $p \in E$. We have $\eta^I_{[V,W]}(F)(p) = \max\{-MP_{V,W}(F)(p), 0\}$ by comparison to Eq. (27). We illustrate in Fig. 5 the difference between morphological probing and the integral grey-level interval operator.

Contrarily to the two interval operators seen above, here we do not require on the structuring functions V and W that $V \leqslant W$, but rather that we always have $F \oplus W^* \geqslant F \ominus V$. For example consider two functions G_v, G_w defined on a support S, such that $-\infty < G_w(p) \leqslant G_v(p) < +\infty$ for all $p \in S$, and let V, W be defined by $V(p) = G_v(p)$ and $W(p) = G_w(p)$ for $p \in S$, while $V(p) = -\infty$ and $W(p) = +\infty$ for $p \notin S$. Here we will have

$$\begin{aligned} (F \oplus W^*)(p) &= \sup_{h \in S} (F(p+h) - G_w(h)) \\ &\geqslant \inf_{h \in S} (F(p+h) - G_v(h)) \\ &= (F \ominus V)(p). \end{aligned}$$

In Refs. [22–24] the particular case where $G_v = G_w$ was considered. For instance, if $G_v = G_w$ has constant value 0 on S, we get $V = C_{S,0}$ and $W = C^*_{S,0}$, as in the left image in Fig. 5.

2.4. Other works

Khosravi and Schafer [32] use a single structuring function V and define a grey-level HMT on F as the arithmetical sum $[F \ominus V] + [(-F) \ominus (-V)]$; by duality (20), this is equal to $[F \ominus V] - [F \oplus V^*]$. This is thus the same as $\eta^I_{[V,V]}$, except that negative values are not changed into 0.

Schaefer and Casasent [13] use two structuring functions V and W, and define a grey-level HMT on F as the meet $[F \ominus V] \wedge [(-F) \ominus W]$ (however, they use a non-standard notation for expressing this).

Raducanu and Graña [33] compare the grey-level HMT (GHMT) defined by Khosravi and Schafer [32] with an operator called the level set HMT (LSHMT). This operator consists in applying a binary HMT to the successive thresholds of a function F and of a structuring function G, and keep the supremum of all results:

$$F \circledast G = \sup\{t \in T \mid p \in X_t(F) \circledast (X_t(G), X_t(G)^c)\}.$$

3. Unified theory

From the two interval operators described in Sections 2.1 and 2.2, we see that both involve two steps: first a *fitting* which associates to an image F a set of pairs $(p, t) \in E \times T'$, for which the translates $V_{(p,t)}$ and $W_{(p,t)}$ have some relation to F; it can eventually be associated to the operation of *constraining*; second a *valuation* which derives from this set of (p, t) a new grey-level image.

Assume $V, W \in T^E$ with $V \leqslant W$. The fitting used in Ronse's supremal interval operator will be written $H_{V,W}$, it is defined by

$$H_{V,W}(F) = \{(p, t) \in E \times T' \mid V_{(p,t)} \leqslant F \leqslant W_{(p,t)}\}. \quad (31)$$

Another one was used in Soille's integral interval operator, we write it $K_{V,W}$, it is defined by

$$K_{V,W}(F) = \{(p, t) \in E \times T' \mid V_{(p,t)} \leqslant F \lessdot W_{(p,t)}\}. \quad (32)$$

Next, the constraining is the operator $C_{V,W} : T^E \to \mathscr{P}(E \times T')$, associating to a function $F : E \to T$ the set

$$C_{V,W}(F) = \{p \in E \mid F(p) = (F \ominus V)(p) \text{ or } F(p) = (F \oplus W^*)(p)\} \times T'. \quad (33)$$

We get thus the two *constrained fittings* $H^C_{V,W}$ and $K^C_{V,W}$ defined by

$$H^C_{V,W}(F) = H_{V,W}(F) \cap C_{V,W}(F)$$

and

$$K^C_{V,W}(F) = K_{V,W}(F) \cap C_{V,W}(F). \quad (34)$$

The valuation must associate to any subset of $E \times T'$ a function $E \to T$. The one used in Ronse's supremal interval operator is the upper envelope operator S, associating to any $Y \in \mathscr{P}(E \times T')$ the function

$$S(Y) : E \to T : p \mapsto \sup\{t \in T' \mid (p, t) \in Y\}. \quad (35)$$

Note that S is a dilation in the algebraic sense [1], that is

$$S\left(\bigcup_{i \in I} Y_i\right) = \bigvee_{i \in I} S(Y_i); \quad (36)$$

the adjoint erosion [1] is the map associating to a function F its umbra $U(F)$, see Eq. (16).

Soille's integral interval operator uses another one, written I, associating to any $Y \in \mathscr{P}(E \times T')$ the function

$$I(Y) : E \to T : p \mapsto mes(\{t \in T' \mid (p, t) \in Y\}), \quad (37)$$

where mes means the measure (Lebesgue's for $T' = \mathbf{R}$ and discrete for $T' = \mathbf{Z}$). Following Ref. [19], for a sequence X_n of sets and a set X, we write $X_n \uparrow X$ to mean that the sequence X_n ($n \in \mathbf{N}$) is increasing (i.e., $X_n \subseteq X_{n+1}$ for all $n \in \mathbf{N}$) and converges to X (i.e., $X = \bigcup_{n \in \mathbf{N}} X_n$); similarly for a numerical sequence r_n, $r_n \uparrow r$ means that the sequence is increasing and converges to r (i.e., $r_n \leqslant r_{n+1}$ for all $n \in \mathbf{N}$, and $r = \sup_{n \in \mathbf{N}} r_n$). A well-known property of measures is that for a sequence X_n of measurable sets, $X_n \uparrow X = \Rightarrow mes(X_n) \uparrow mes(X)$ (see Theorem 1.8(c) on p. 25 of Ref. [34]). We have thus for a sequence Y_n ($n \in \mathbf{N}$) in $E \times T'$:

$$Y_n \uparrow Y \Longrightarrow I(Y_n) \uparrow I(Y), \quad (38)$$

which is weaker than being a dilation, as in Eq. (36).

We introduce a third valuation, the binary one B, which associates to any $Y \in \mathscr{P}(E \times T')$ the set

$$B(Y) = \{p \in E \mid \exists t \in T',\ (p, t) \in Y\}. \quad (39)$$

We can represent it as a function with value $+\infty$ on $B(Y)$ and $-\infty$ elsewhere, this gives thus the binary mask valuation M associating to Y the function

$$M(Y) : E \to T : \quad p \mapsto \begin{cases} +\infty & \text{if } \exists t \in T',\ (p, t) \in Y, \\ -\infty & \text{otherwise.} \end{cases} \quad (40)$$

Note that B and M are also dilations in the algebraic sense, that is

$$B\left(\bigcup_{i \in I} Y_i\right) = \bigcup_{i \in I} B(Y_i)$$

and

$$M\left(\bigcup_{i \in I} Y_i\right) = \bigvee_{i \in I} M(Y_i); \quad (41)$$

their adjoint erosions are, respectively; for B the map $\mathscr{P}(E) \to \mathscr{P}(E \times T') : X \mapsto X \times T'$, and for M the map $\{-\infty, +\infty\}^E \to \mathscr{P}(E \times T') : F \mapsto \mathrm{supp}(F) \times T' = U(F)$.

Composing one of $H_{V,W}$ or $K_{V,W}$, optionally constrained by intersection with $C_{V,W}$, by one of S, I and M, we obtain an interval operator. We have thus six unconstrained operators $SH_{V,W}$, $SK_{V,W}$, $IH_{V,W}$, $IK_{V,W}$, $MH_{V,W}$ and $MK_{V,W}$, as well as six constrained ones, $SH^C_{V,W}$, $SK^C_{V,W}$, $IH^C_{V,W}$, $IK^C_{V,W}$, $MH^C_{V,W}$ and $MK^C_{V,W}$. We see then that Ronse's supremal interval operator is $SH_{V,W}$, Soille's unconstrained integral interval operator is $IK_{V,W}$, while the constrained one is $IK^C_{V,W}$. In Refs. [27,28] we used a union of $BH_{V,W}$ for various choices of pairs (V, W), as a form of segmentation of tubular shapes, while in Refs. [25,26] we associated to an image F the image

$$F \wedge MK_{V,W}(F):\quad p \mapsto \begin{cases} F(p) & \text{if } \exists t \in T',\ \ V_{(p,t)} \leqslant F \ll W_{(p,t)}, \\ -\infty & \text{otherwise,} \end{cases}$$

which represents tubular shapes with their original grey-level.

Let us compare, for each valuation S, I or M, the interval operators according to the two fitting operators $H_{V,W}$ (31) and $K_{V,W}$ (32). The relation between the two fittings differs with the choice of $\overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$ for T:

$$\begin{aligned} &T = \overline{\mathbf{Z}}: \\ &\quad H_{V,W} = K_{V,W+1} \quad \text{and} \quad K_{V,W} = H_{V,W-1}; \\ &T = \overline{\mathbf{R}}: \\ &\quad H_{V,W} = \bigcap_{\varepsilon>0} K_{V,W+\varepsilon} \quad \text{and} \quad K_{V,W} = \bigcup_{\varepsilon>0} H_{V,W-\varepsilon}. \end{aligned} \tag{42}$$

Since intersection with the set $C_{V,W}$ distributes union and intersection, by Eq. (33) these equalities remain valid for constrained fittings, in other words if we replace H by H^C and K by K^C in each expression.

For $T = \overline{\mathbf{Z}}$, each one of the six interval operators using $H_{V,W}$ (with valuation S, I or M, with or without constraining) is equal to the corresponding operator with $K_{V,W+1}$. Consider now the case where $T = \overline{\mathbf{R}}$. As S is a dilation (36), by Eq. (42) we get

$$T = \overline{\mathbf{R}}: \quad SK_{V,W} = \bigvee_{\varepsilon>0} SH_{V,W-\varepsilon}, \tag{43}$$

and similarly for the constrained versions $SK^C_{V,W}$ and $SH^C_{V,W-\varepsilon}$. For the integral valuation I, the fact that a closed interval $[a, b]$ has the same Lebesgue measure as the corresponding half-open interval $]a, b]$ (namely, its length $b-a$), we get

$$\begin{aligned} &mes(\{t \in T' \mid (F \oplus W^*)(p) < t \leqslant (F \ominus V)(p)\}) \\ &\quad = \max\{(F \ominus V)(p) - (F \oplus W^*)(p), 0\} \\ &\quad = mes(\{t \in T' \mid (F \oplus W^*)(p) \leqslant t \leqslant (F \ominus V)(p)\}), \end{aligned}$$

so that for $T = \overline{\mathbf{R}}$, $IK_{V,W} = IH_{V,W}$ and $IK^C_{V,W} = IH^C_{V,W}$ (but this is not true for $T = \overline{\mathbf{Z}}$, where $IK_{V,W}(F)(p) = \max\{IH_{V,W}(F)(p)-1, 0\}$). Finally, as B and M are dilations (41), we get

$$T = \overline{\mathbf{R}}: \quad \text{and} \quad \begin{aligned} BK_{V,W} &= \bigcup_{\varepsilon>0} BH_{V,W-\varepsilon} \\ MK_{V,W} &= \bigvee_{\varepsilon>0} MH_{V,W-\varepsilon}, \end{aligned} \tag{44}$$

and similarly for the constrained versions.

3.1. Bounded grey-levels

As we did not make any restriction on structuring functions, we presented our operators in the framework of unbounded grey-levels, namely $T = \overline{\mathbf{Z}}$ or $\overline{\mathbf{R}}$, for which it is guaranteed that the result of an operator will not produce a grey-level overflow. In practical situations, one takes as grey-level set a finite interval $\hat{T} = [t_0, t_1] \subset \mathbf{Z}$, and we have to see how the theory adapts to this situation.

The first problem is to ascertain that the results of our operations will have their grey-levels in the interval $[t_0, t_1]$. If V and W are flat ($V = C_{A,0}$ and $W = C^*_{B,0}$), or more generally if

$$\sup_{h \in E} V(h) = \inf_{h \in E} W(h) = 0,$$

then by Lemma 1, $F \ominus V$ and $F \oplus W^*$ have their grey-levels in $[t_0, t_1]$. This shows that V and W are not necessarily in $\hat{T}^E$. In other words, *the space of grey-level images is often different from that of structuring functions*.

If we use Soille's approach, hence the integral valuation I, as we get only non-negative values in the result, we must assume that $t_0 = 0$, so $[t_0, t_1] \subset \mathbf{N}$.

With Ronse's approach, and the supremal valuation, we use the lattice-theoretical supremum operation. Now in $\hat{T} = [t_0, t_1]$, all suprema and infima are the same as in $\overline{\mathbf{Z}}$ and $\overline{\mathbf{R}}$, *except the empty ones*: $\sup \emptyset = \bot$ gives $-\infty$ in $\overline{\mathbf{Z}}$ and $\overline{\mathbf{R}}$, but t_0 in $[t_0, t_1]$, while $\inf \emptyset = \top$ gives $+\infty$ in $\overline{\mathbf{Z}}$ and $\overline{\mathbf{R}}$, but t_1 in $[t_0, t_1]$; thus the resulting value $-\infty$ in Eq. (24) or in an empty supremum returned by S, must be set to t_0 instead of $-\infty$.

Note also that the special interpretation of the case $(F \ominus V)(p) = (F \oplus W^*)(p) = +\infty$ in Eq. (24), and of the case $(F \ominus V)(p) = (F \oplus W^*)(p) = \pm\infty$ in Eq. (27), which arose because $\pm\infty \notin T'$, does not apply here for $(F \ominus V)(p) = (F \oplus W^*)(p) = t_1$ or t_0.

Finally, in the binary mask valuation M, the resulting values $+\infty$ and $-\infty$ should be replaced by t_1 and t_0.

We have thus the following guidelines for translating our theory to the case of an arbitrary complete lattice T of numerical values (with greatest element $\top$ and least element $\bot$):

(i) Choose the structuring functions V, W in such a way that the results of the interval operators will have their grey-levels in T (no overflow); in particular V and W do not necessarily have their values in T.
(ii) Let $T' = T \cap \mathbf{R}$, the set of finite values of T. All special cases given above for $(F \ominus V)(p)$ or $(F \oplus W^*)(p) = +\infty$ or $-\infty$ do not apply to $\top$ and $\bot$ when the latter are finite.
(iii) An empty supremum (in the supremal approach) must be set to $\bot$ instead of $-\infty$. The values $+\infty$ and $-\infty$ in the binary mask valuation M must be replaced by $\top$ and $\bot$.

We illustrate in Fig. 6 the application of the three unconstrained interval operators with fitting K in the case of bounded non-negative integer grey-levels.

It is interesting to see what happens for binary images, that is for $T = \{0, 1\}$. Taking two disjoint SEs A, B,

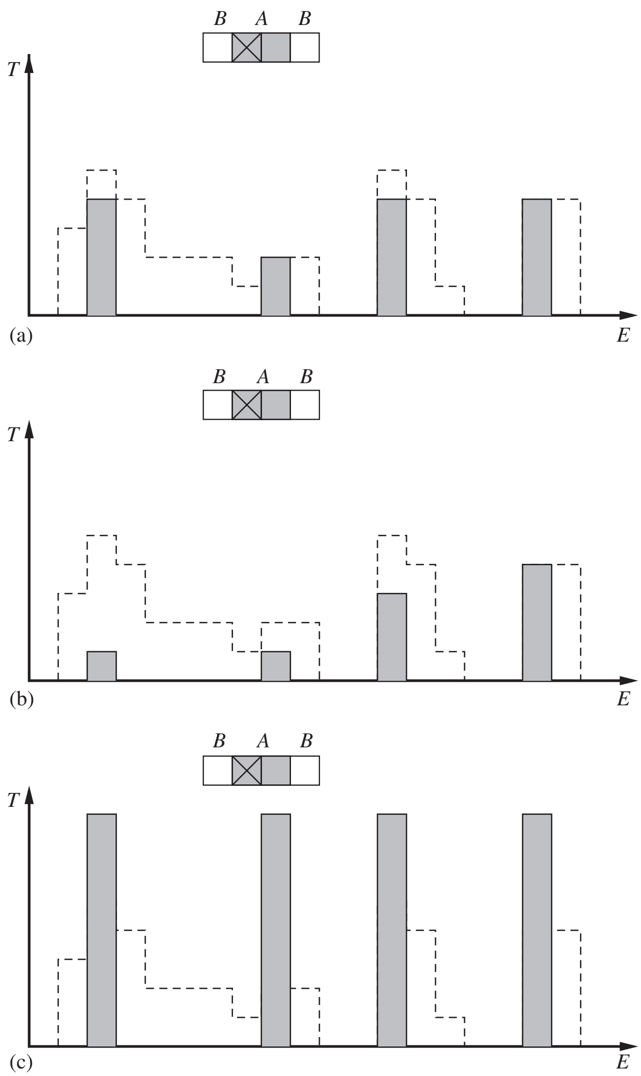

Fig. 6. Here $E = \mathbf{Z}$ and $T = [0 \cdots t_1] \subset \mathbf{N}$. We use flat structuring elements A and B (the origin being the left pixel of A), setting $V = C_{A,0}$ and $W = C^*_{B,0}$. From top to bottom, we show $SK_{V,W}(F)$, $IK_{V,W}(F)$ and $MK_{V,W}(F)$, as they are computed in the framework of bounded grey-levels; each time the result is given with F shown dashed.

the cylinder $V = C_{A,0}$ and dual cylinder $W = C^*_{B,0}$, then the three unconstrained and three constrained interval operators using $K_{V,W}$ (namely, $SK_{V,W}$, $IK_{V,W}$, $MK_{V,W}$, $SK^C_{V,W}$, $IK^C_{V,W}$ and $MK^C_{V,W}$) are all equal; in fact for $F : E \to \{0, 1\}$, $SK_{V,W}(F)$ (or anyone of the five others applied to F) has value 1 on all points $p \in E$ where $(F \ominus A)(p) = 1$ and $(F \oplus \check{B})(p) = 0$, and value 0 on other points. Now every subset X of E corresponds to its characteristic function having value 1 on X and 0 on X^c; thus if F is the characteristic function of X, then $SK_{V,W}(F)$ is the characteristic function of $(X \ominus A) \setminus (X \oplus \check{B}) = X \circledast (A, B)$. To summarize, all six interval operators with $K_{V,W}$ are equal, and correspond to the original HMT by (A, B) for sets (1).

4. Conclusion

HMT have proved to be very useful in binary image processing. However, they have seldom been considered in the case of grey-level images, the greatest obstacle being the difficulty to extend this non-increasing operator to grey-level images. This contribution provides a comprehensive theory of the various forms of HMTs for grey-level images while generalizing the previous approaches [4,18,21] and the variant of morphological probing [22–24].

Applications of morphological probing were given in Ref. [22–24,35,36]. Several applications of the grey-level HMT have been given in Ref. [4]. In Part II of this paper [29] we will present some applications of the grey-level HMT in the specific case of analysing 3D angiographic image (i.e., medical images visualizing vessels) [25–28]. This should convince the reader of its wide applicability in the field of grey-level image processing.

In the same way as the composition of dilation and erosion leads to opening and closing, it would be interesting to analyse the properties of the operators obtained by composition of an interval operator and the dilation by the first SE. For example $\delta_V \eta^S_{[V,W]}$ is idempotent, but not $\delta_V \eta^I_{[V,W]}$.

Also, a complete theory of interval operators in a complete lattice still remains to be done. Some steps in that direction were made in Ref. [18]. Let us give a further pointer. We consider a complete lattice $\mathscr{L}$ with a *sup-generating family* S, that is

$$\forall X \in \mathscr{L}, \quad X = \bigvee \{s \in S \mid s \leqslant X\}$$

(say, for $\mathscr{L} = \mathscr{P}(E)$, S consists of all singletons, for $\mathscr{L} = T^E$, S is the set of impulses). Given two algebraic dilations δ, δ' such that $\delta \leqslant \delta'$, we define the interval operator $\eta_{[\delta,\delta']}$ by

$$\eta_{[\delta,\delta']}(X) = \bigvee \{s \in S \mid \delta(s) \leqslant X \leqslant \delta'(s)\}.$$

Using the tools of Ref. [18], it can be shown that $\delta\eta_{[\delta,\delta']}$ is idempotent. It would be interesting to see under which conditions an arbitrary operator on $\mathscr{L}$ is a supremum of interval operators $\eta_{[\delta,\delta']}$. This topic will be the subject of a future paper.

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About the Author—BENOÎT NAEGEL was born in 1978. He studied computer science at the Université Louis Pasteur, Strasbourg (Licence, 1998; M.Sc., 2000; Ph.D., 2004). From 2000 to 2004 he worked on medical image segmentation and mathematical morphology at the Research Institute Against Cancer of the Digestive System (IRCAD) under the guidance of Christian Ronse and Luc Soler. He is presently in post-doctoral fellowship as a research assistant at the École d'Ingénieurs de Genève (Switzerland). His research interests include image processing and biomedical applications of mathematical morphology.

About the Author—NICOLAS PASSAT was born in 1978. He studied computer science at the Université d'Orléans (Licence, 2000) and the Université Louis Pasteur, Strasbourg (M.Sc., 2002; Ph.D., 2005), specializing in image processing. During his research practice, he worked at the LSIIT and

IPB-LNV (Université Louis Pasteur, Strasbourg), on vascular structure segmentation from 3D medical images, under the supervision of Christian Ronse and Joseph Baruthio. He is currently in post-doctoral fellowship at the A^2SI Laboratory (Université de Marne-la-Vallée). His scientific interests include image processing, mathematical morphology, and medical imaging.

About the Author—CHRISTIAN RONSE was born in 1954. He studied pure mathematics at the Université Libre de Bruxelles (Licence, 1976) and the University of Oxford (M.Sc., 1977; Ph.D., 1979), specializing in group theory. Between 1979 and 1991 he was Member of Scientific Staff at the Philips Research Laboratory Brussels, where he conducted research on combinatorics of switching circuits, feedback shift registers, discrete geometry, image processing, and mathematical morphology. During the academic year 1991–1992 he worked at the Université Bordeaux-1, where he obtained his Habilitation diploma. Since October 1992, he has been Professor of Computer Science at the Université Louis Pasteur, Strasbourg (promotion to First Class Professorship in 2001), where he contributed to the development of a research group on image analysis, and the teaching of image processing to students at various levels. His scientific interests include imaging theory, mathematical morphology, image segmentation and medical imaging.