Segmentation using Deformable Models

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Deformable models = evolution of curves / surfaces to minimize an energy function for
- finding the best image partition into homogeneous regions
- finding the contours of an object

Representation:
- parametric,
- implicit, using level sets.

Criteria on:
- contours
- region homogeneity
- other types of constraints:
  - regularity (internal)
  - external: balloon force, spatial relations, geometry...
Parametric active contours

- Principle: evolution of a curve under internal and external forces (one object, parametric representation)

\[ \mathbf{v}(s) = [x(s), y(s)]^t \quad s \in [0, 1] \]

- Energy: \( E_{\text{total}} = \int_0^1 (E_{\text{int}}(\mathbf{v}(s)) + E_{\text{image}}(\mathbf{v}(s)) + E_{\text{ext}}(\mathbf{v}(s))) \, ds \)
- Internal energy: regularization

\[ E_{\text{int}} = \alpha(s) \left( \frac{dv}{ds} \right)^2 + \beta(s) \left( \frac{d^2v}{ds^2} \right)^2 \]

control of tension (length of the curve) and of curvature (if \( s = \) curvilinear coordinate, tangent \( T = \frac{dv}{ds}, \|T\| = 1 \), and \( \frac{dT}{ds} = kN, k \) curvature).

- Energy from image information (gradient, information on contours):

\[ E_{\text{image}} = g(\|\nabla f\|) \]

- External energy: many possibilities
Euler-Lagrange equation:

\[
\frac{\partial E}{\partial v} - \frac{d}{ds}\left(\frac{\partial E}{\partial v'}\right) + \frac{d^2}{ds^2}\left(\frac{\partial E}{\partial v''}\right) = 0
\]

⇒ variational problem

\[-(\alpha v')'(s) + (\beta v'')''(s) + \nabla P(v) = 0\]

\[P(v) = E_{image}(v) + E_{ext}(v) \quad F(v) = -\nabla P(v)\]

+ limit conditions.

Discretization using finite differences

\[V^t = [v_0^t, v_1^t, v_2^t, \ldots, v_{n-1}^t]^t\]

\[
\frac{\beta}{h^2} v_{i+2} - \left(\frac{\alpha}{h} + 4 \frac{\beta}{h^2}\right) v_{i+1} + \left(\frac{2\alpha}{h} + 4 \frac{6\beta}{h^2}\right) v_i - \left(\frac{\alpha}{h} + 4 \frac{\beta}{h^2}\right) v_{i-1} + \frac{\beta}{h^2} v_{i-2} = F(v_i)
\]

\[AV = F\]

A: pentadiagonal matrix
Evolution of the curve in time:

\[ \tau \frac{\partial v}{\partial t} = -\alpha v'' + \beta v''' + F(v) \]

\[ v(t + 1) = (A + \tau I)^{-1}(F(v(t)) + \tau v(t)) \]

- initialization (crucial for convergence)
- choice of \( \tau \) (inertia, regularization of \( A \))
- matrix inversion
- constant discretization step for \( t \) (set to have a displacement of 1-2 pixels)
- stopping criterion
Types of active contours and associated matrices

Closed active contour

\[
\begin{bmatrix}
2\alpha + 6\beta & -\alpha - 4\beta & \beta & 0 & 0 & \beta. & -\alpha - 4\beta \\
-\alpha - 4\beta & 2\alpha + 6\beta & -\alpha - 4\beta & \beta & 0 & \ldots \\
\beta & -\alpha - 4\beta & 2\alpha + 6\beta & -\alpha - 4\beta & \beta \\
0 & \beta & -\alpha - 4\beta & 2\alpha + 6\beta & -\alpha - 4\beta \\
0 & 0 & \beta & -\alpha - 4\beta & 2\alpha + 6\beta \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
-\alpha - 4\beta & \beta & 0 & 0 & \beta & -\alpha - 4\beta & 2\alpha + 6\beta \\
\end{bmatrix}
\]
Free end-points

\[
\begin{bmatrix}
\beta & -\alpha & \beta & 0 & 0 & 0 & 0 & \ldots & \ldots \\
-\alpha & 2\alpha + 5\beta & -\alpha & \beta & 0 & \ldots & \ldots \\
\beta & -\alpha & 2\alpha + 6\beta & -\alpha & \beta & \ldots & \ldots \\
0 & \beta & -\alpha & 2\alpha + 6\beta & -\alpha & \ldots & \ldots \\
0 & 0 & \beta & -\alpha & 2\alpha + 6\beta & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \beta & -2 & \beta & \beta \\
\end{bmatrix}
\]

(with \(v''(0) = v''(1) = v'''(0) = v'''(1) = 0\))
**Fixed end-points**

\[
\begin{bmatrix}
2\alpha + 6\beta & -\alpha - 4\beta & \beta & 0 & 0 & 0 & 0 \\
-\alpha - 4\beta & 2\alpha + 6\beta & -\alpha - 4\beta & \beta & 0 & \ldots \\
\beta & -\alpha - 4\beta & 2\alpha + 6\beta & -\alpha - 4\beta & \beta \\
0 & \beta & -\alpha - 4\beta & 2\alpha + 6\beta & -\alpha - 4\beta \\
0 & 0 & \beta & -\alpha - 4\beta & 2\alpha + 6\beta \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \beta & -2\beta & \beta \\
\end{bmatrix}
\]
Example
Evolution example

Evolution
Problem:
- bad initialization ⇒ no attraction
- no forces ⇒ curve collapses

Solution using a pression force (balloon): \( k_1 N(s) \) (\( N(s) \) unit normal vector at \( s \)).

Initialization: inside or outside the object (not necessarily very close to the searched object)
Distance map \( D(x, y) \) ⇒ potential

\[
P_{\text{dist}}(x, y) = we^{-D(x,y)}
\]

\[
F_{\text{ext}} = -\nabla P_{\text{dist}}
\]
Objectives:
- get rid of the constraints on the initialization
- convergence towards concave regions
- Diffusion of gradients in the whole image
- For $\mathbf{v}(x, y) = (u(x, y), v(x, y))$, minimization of:

$$E = \int \int \mu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + |\nabla f(x, y)|^2 |\mathbf{v} - \nabla f(x, y)|^2 dxdy$$

$f =$ contour map
Solving and generalizing GVF

\[ \mu \nabla^2 u - (u - f_x)(f_x^2 + f_y^2) = 0 \]
\[ \mu \nabla^2 v - (v - f_y)(f_x^2 + f_y^2) = 0 \]

More general formulation:

\[ \frac{\partial v}{\partial t} = g(\|\nabla f\|)\nabla^2 v - h(\|\nabla f\|)(v - \nabla f) \]

\[ v(x, y, 0) = \nabla f(x, y) \]

Examples for functions \( g \) and \( h \):

- \( g(r) = \mu, \ h(r) = r^2 \)
- \( g(r) = \exp(-r^2/k), \ h(r) = 1 - g(r) \)
Classical active contour: illustration
Using GVF:
Using GVF:
GVF: example of evolution

Evolution
GVF: example in medical imaging

GVF field

GVF field. Detail.
3D parametric deformable models

- **Segmentation with regularization:**
  - data fidelity
  - smooth surface

- **Minimization of energy:**

\[
E(v) = \int_{\Omega} w_{10} \left\| \frac{\partial v}{\partial r} \right\|^2 + w_{01} \left\| \frac{\partial v}{\partial s} \right\|^2 + w_{20} \left\| \frac{\partial^2 v}{\partial r^2} \right\|^2 + w_{02} \left\| \frac{\partial^2 v}{\partial s^2} \right\|^2 + 2w_{11} \left\| \frac{\partial^2 v}{\partial r \partial s} \right\|^2 dr ds + \int_{\Omega} P(v) dr ds
\]

- first order: elastic membrane (curvature)
- second order: thin plate (torsion)
- \( P \): attraction potential

Similar resolution schemes as in 2D.
Geodesic active contours (Caselles, 1997)

- Principle:

\[
J_1(v) = \alpha \int_a^b |v'(s)|^2 ds + \lambda \int_a^b g(|\nabla I(v(s))|)^2 ds
\]

minimization of \( J_2(v) = 2\sqrt{\lambda \alpha} \int_a^b |v'(s)|g(|\nabla I(v(s))|)ds \)

\( \Rightarrow \) computation of geodesics according to a new metric (induced by the image)

- Evolution equation:

\[
\frac{\partial v}{\partial t} = g(I)\kappa N - (\nabla g.N)N
\]
Geodesic active contours: example

Evolution
Level sets

Implicit representation, without parametrization

- **Principle:**
  Let \( \Gamma(t) \) be a closed hypersurface (dimension \( d - 1 \))
  Let \( \psi \) (dimension \( d \)) be a function taking values in \( \mathbb{R} \) with

  \[
  \Gamma(t) = \{ x \in \mathbb{R}^d \mid \psi(x, t) = 0 \}
  \]

  propagation of \( \Gamma \) (evolution along the normal) \( \iff \) propagation of \( \psi \)

- **Example:** distance function

- **Evolution equation of \( \psi \)**

  \[
  \frac{\partial \psi}{\partial t} = -F \|
  \nabla \psi \|
  \]

  NB: normal \( N = \frac{\nabla \psi}{\| \nabla \psi \|} \), mean curvature \( k = \text{div}(\frac{\nabla \psi}{\| \nabla \psi \|}) \)
Additional dimension: example of distance
Propagation using level sets

- Propagation speed:
  \[ F = (F_A + F_G)k_I \]

  - \( F_A \) expansion or contraction, independent of geometry,
  - \( F_G \) geometric properties (curvature),
  - \( k_I \) stopping criterion (image).

  \[ F = \frac{1}{1 + \|\nabla G_\sigma \ast I\|^p} (\pm 1 + \epsilon \kappa) \]

- Advantage: potential modification of topology.
- Speeding-up by computation only in a narrow band.
Level sets and change of topology

Example
Example: bone segmentation in MRI (H. Rifai)
Simulated images:

Résolution : 1x1x1 mm$^3$ (Université McGill)
Segmentation error on simulated images with various noise levels:

Sans estimation du volume partiel.

Avec estimation du volume partiel.
Real images:

Evolution
Result
Region-based approach: Mumford and Shah (1989)

- Image $f$ on a support $\mathcal{I}$.
- Approximation by smooth functions $g(x, y)$ on regions $R_i$ limited by contours $\Gamma_j$ ($\mathcal{I} = \bigcup_i R_i \cup \Gamma$, $\Gamma = \bigcup_j \Gamma_j$):

  Functional to be minimized: 
  \[
  U(\Gamma, g, f) = 
  \lambda \iint_{\mathcal{I} \setminus \Gamma} (f(x, y) - g(x, y))^2 \, dx \, dy
  + \mu \iint_{\mathcal{I} \setminus \Gamma} \| \nabla g(x, y) \|^2 \, dx \, dy
  + \nu \int_{\Gamma} dl
  \]
g_i = constant on each \( R_i \):

\[
U_0(\Gamma, f) = \sum_i \lambda_i \int\int_{R_i} (f - g_i)^2 \, dx \, dy + \nu \int_{\Gamma} dl
\]

\[
\Rightarrow g_i = \frac{1}{s_i} \int\int_{R_i} f(x, y) \, dx \, dy
\]

where \( s_i \) is the area of \( R_i \)

\( \Rightarrow \) Partition of space into homogeneous regions, characterized by their average gray level.
Formulation using level sets

- Contour $\Gamma$
- Two regions $R_1 = \text{int}(\Gamma)$ and $R_2 = \text{ext}(\Gamma)$ with constant values $g_1$ and $g_2$

$$U(\Gamma, g, f) = \lambda_1 \iint_{R_1} (f - g_1)^2 \, dx \, dy + \lambda_2 \iint_{R_2} (f - g_2)^2 \, dx \, dy + \nu \int_{\Gamma} dl$$

Level set:

$$\phi(x, y) \begin{cases} 
= 0 & \text{on } \Gamma \\
> 0 & \text{in } R_1 = \text{int}(\Gamma) \\
< 0 & \text{in } R_2 = \text{ext}(\Gamma) 
\end{cases}$$

$$\Gamma(t) = \{ \phi(t) = 0 \}$$

Evolution (mean curvature motion):

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \nabla \left( \frac{\nabla \phi}{|\nabla \phi|} \right) \quad \phi(0) = \phi_0$$

$$(v(r, s) = (r, s, \Phi(r, s)) \Rightarrow \frac{\partial v}{\partial r} = (1, 0, \frac{\partial \Phi}{\partial r})^t, \frac{\partial v}{\partial s} = (0, 1, \frac{\partial \Phi}{\partial s})^t, \text{etc.})$$
\[ H(z) = \begin{cases} 
1 & \text{if } z \geq 0 \\
0 & \text{if } z < 0 
\end{cases} \quad H'(z) = \delta(z) \]

\[ \Rightarrow \int_{\Gamma} dl = \int_{\mathcal{I}} |\nabla H(\phi)| dxdy = \int_{\mathcal{I}} \delta(\phi)|\nabla \phi| dxdy \]

and \( U(\Gamma, g, f) = \)

\[ \lambda_1 \int_{\mathcal{I}} (f - g_1)^2 H(\phi) dxdy + \lambda_2 \int_{\mathcal{I}} (f - g_2)^2 (1 - H(\phi)) dxdy + \nu \int_{\mathcal{I}} \delta(\phi)|\nabla \phi| dxdy \]

Minimization of \( U \):

\[ g_1 = \frac{\int_{\mathcal{I}} f H(\phi) dxdy}{\int_{\mathcal{I}} H(\phi) dxdy} \quad g_2 = \frac{\int_{\mathcal{I}} f (1 - H(\phi)) dxdy}{\int_{\mathcal{I}} (1 - H(\phi)) dxdy} \]

\[ \frac{\partial \phi}{\partial t} = \delta(\phi)[\nu \nabla \left( \frac{\nabla \phi}{|\nabla \phi|} \right) - \lambda_1 (f - g_1)^2 + \lambda_2 (f - g_2)^2] \]

In practice: smooth versions of \( \delta \) and \( H \).
Level sets and multi-phase models (Chan, Vese, 2001, 2002)
Segmentation of a MRI image with two level sets
Example with junctions
1. Generate sparse texture features by nonlinear diffusion filtering

Brox, Weickert ’04, ’06
2. Mumford-Shah segmentation of vector-valued features

Brox, Weickert ’04, ’06
Region competition

Regions characterized by intensity distributions. Probabilistic setting: partition \( \mathcal{P}(\Omega) = \{\Omega_e, \Omega_i\} \) maximizing \( p(l|\mathcal{P}(\Omega))p(\mathcal{P}(\Omega)) \).

\[
p(\mathcal{P}(\Omega)) \propto \nu \exp(-\nu |C|), \quad \nu > 0
\]

\[
p(l|\mathcal{P}(\Omega)) = p(l|\Omega_e)p(l|\Omega_i) = \prod_{x \in \Omega_e} p_e(l(x), \theta_e) \prod_{x \in \Omega_i} p_i(l(x), \theta_i)
\]

\[
p(\mathcal{P}(\Omega)|l) = \nu \exp(-\nu |C|) \prod_{x \in \Omega_e} p_e(l(x), \theta_e) \prod_{x \in \Omega_i} p_i(l(x), \theta_i)
\]

Formulation as an energy minimization problem:

\[
E(\{C, \theta_e, \theta_i\}) = E_{\text{reg}}(C) + E_e(\{C, \theta_e\}) + E_i(\{C, \theta_i\})
\]

\[
\begin{align*}
E_{\text{reg}}(C) &= -\log \nu + \nu |C|, \\
E_e(\{C, \theta_e\}) &= -\int_{x \in \Omega_e} \log p_e(l(x), \theta_e) \, dx \\
E_i(\{C, \theta_i\}) &= -\int_{x \in \Omega_i} \log p_i(l(x), \theta_i) \, dx
\end{align*}
\]
Implementation using level sets

\( \phi : \Omega \rightarrow \mathbb{R}, \phi(x) > 0 \) in \( \Omega_e \), \( \phi(x) < 0 \) in \( \Omega_i \) and \( \phi(x) = 0 \) on \( C \).

\[
E(\phi, \theta_i, \theta_e) = E_{\text{reg}}(\phi) + E_e(\phi, \theta_e) + E_i(\phi, \theta_i)
\]

\[
\begin{align*}
E_{\text{reg}}(\phi) &= \nu \int_{x \in \Omega} \delta(\phi(x))|\nabla \phi(x)| dx, \\
E_e(\phi, \theta_e) &= -\int_{x \in \Omega} H(\phi(x)) \log(p_e(I(x), \theta_e)) dx \\
E_i(\phi, \theta_i) &= -\int_{x \in \Omega} (1 - H(\phi(x))) \log(p_i(I(x), \theta_i)) dx
\end{align*}
\]
Example in ultrasound imaging (Jérémie Anquez)
Learning distributions and their parameters:

(a) \(\hat{p}_A^F\) and \(p_R\)

(b) \(\hat{p}_A^{FT}\), \(p_R\), and \(p_N\)
Constraining deformable models by spatial relations (Olivier Colliot et al.)

Examples

*close to the lateral ventricle*  

⇒ *additional external force* (avoids leaking in undesired regions)
Retina imaging (ISEP and XV-XX)

\[
E(V, V_1, V_2, b) = E_{\text{Image}}(V_1) + E_{\text{Image}}(V_2) + E_{\text{Int}}(V) + R(V_1, V_2, b)
\]

\[
E_{\text{Image}}(V_i) = \int_0^1 P(V_i(s)) ds = -\int_0^1 |\nabla I(V_i(s))|^2 ds
\]

\[
E_{\text{Int}}(V) = \frac{1}{2} \int \alpha(s) \left| \frac{\partial V(s, t)}{\partial s} \right|^2 + \beta(s) \left| \frac{\partial^2 V(s, t)}{\partial s^2} \right|^2 ds
\]

\[
R(V_1, V_2, b) = \int_0^1 \varphi(s)(b'(s))^2 ds
\]
Eye fundus:
Optical coherence tomography (OCT):

Adaptive optics:
Scikit-Image - Segmentation

- `skimage.segmentation.active_contour`: parametric representation, contour-based approach.
- `skimage.segmentation.chan_vese`: implicit representation (using level sets), region-based approach.


