Temporal and Spatial Reasoning

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Knowledge representation and reasoning on spatial entities and spatial relationships

- largely developed in the artificial intelligence community
  - mainly topological relations
  - formal logics (ex: mereotopology)
  - inference
- less developed in image interpretation
  - need for imprecise knowledge representation
  - (semi-)quantitative framework (⇒ numerical evaluation)
  - examples: structural recognition in images under imprecision
- main ingredients:
  - knowledge representation (including spatial relations)
  - imprecision representation and management
  - fusion of heterogeneous information
  - reasoning and decision making
Knowledge

Formal representation

Graphs
[COLLIOT, DERUYVER, ...]

Stochastic grammars
[ZHU, MUMFORD, ...]

Ontologies
[DAMERON, HU, ...]

Data

Reasoning

Decision

I. Bloch
Symbolic AI
From Pythagoras (c. 570-495 BC) to Zeno (c. 490-430 BC): concept of space linked to the first developments in arithmetics and Pythagorian geometry - Problem of infinitely subdivision possibility.

Descartes (1596-1650): spatial extension is specific to material entities, governed by the only laws of mechanics.

Newton (1643-1727): notion of absolute space.

Hume (1711-1776): space reduced to a pure psychological function.

Leibniz (1646-1716): space cannot be an absolute reality, motion and position are real and detectable only in relation to other objects, not in relation to space itself.

Kant (1724-1804): objectivity of space.
Poincaré (1854-1912): empiricist point of view where spatial knowledge is mainly derived from motor experience. Relativity of space.

Bergson (1859-1941): a position in the space can be considered as an instantaneous cut of the movement, but the movement is more that a sum of positions in the space.

Einstein (1879-1955): geometry is linked to the sensible and perceptible space. The geometrical configuration of the world itself becomes relative.

Purely philosophical views of space developed by the phenomenologists and the existentialists.

Reichenbach (1891-1953): geometry as a theory of relations.
Rich variety of lexical terms for describing spatial location of entities.

Concern all lexical categories (nouns, verbs, adjectives, adverbs, prepositions).

- French, and other Romance languages, shows a typological preference for the lexicalization of the path in the main verb.
- In Germanic and Slavic languages, the path is rather encoded in satellites associated to the verb (particle or prefix).

Source of inspiration of many works on qualitative spatial information representation and qualitative spatial reasoning.

Asymmetry, importance of reference, of context, of functional properties of the considered physical entities

Imprecision (too precise statements can even become inefficient if they make the message too complex).
Human perception: example of distance

- Purely spatial measures, in a geometric sense, give rise to "metric distances", and are related to intrinsic properties of the objects.
- Temporal measures lead to distances expressed as travel time, and can be considered of extrinsic type, as opposed to the previous class.
- Economic measures, in terms of costs to be invested, are also of extrinsic type.
- Perceptual measures lead to distances of deictic type; they are related to an external point of view, which can be concrete or just a mental representation, which can be influenced by environmental features, by subjective considerations, leading to distances that are not necessarily symmetrical.
- Influence of other objects.
Cognitive understanding of a spatial environment is issued from two types of processes:

- route knowledge acquisition (first acquired during child development), which consists in learning from sensori-motor experience (i.e. actual navigation) and implies an order information between visited landmarks,

- survey knowledge acquisition, from symbolic sources such as maps, leading to a global view (”from above”) including global features and relationships, which is independent of the order of landmarks.
Neuro-imaging:

- A right hippocampal activation can be observed for both mental navigation and mental map tasks,
- A parahippocampal gyrus activation is additionally observed only for mental navigation, when route information and object landmarks have to be incorporated.

Internal representation of space in the brain:

- Egocentric representations,
- Allocentric representations ("map in the head").

Intensively used in several works in the modeling and conception of geographic information systems, and in mobile robotics.
Spatial reasoning formalisms

- Quantitative
- Qualitative (QSR)
- Fuzzy representations and reasoning: semi-quantitative / semi-qualitative approaches
- Spatial entities
- Spatial relations
- Real world problems: dealing with imprecision and uncertainty.

Common to several representation and reasoning frameworks, used in the next parts of the course.
Spatial entities

- Regions, fuzzy regions.
- Key points.
- Simplified regions (centroid, bounding box...).
- Abstract representations (e.g. in mereotopology, without referring to points, formulas in some logics...).
Spatial relations

- Useful... (see e.g. Freeman 1975, Kuipers 1978...).
- Structural stability (more than shape, size, absolute position).
- Different types (binary / n-ary, simple / complex, well-defined / vague).
Quantitative representations

- Precisely defined objects.
- Computation of well defined relations.
- Many limitations:
  - on the objects,
  - on the relations,
  - on the type of representations,
  - for reasoning.

But does not always match the usual way of reasoning (e.g. to the north, closer...).
Qualitative / symbolic representations

- Allen’s intervals (temporal reasoning): 13 relations.
- Rectangle calculus (Allen on each axis): 169 relations.
- Cube calculus...
- Cardinal directions: 9 positions.
- Region Connection Calculus (RCC), mereotopology (based on connection and parthood predicates).
- Extensions to objects with broad or imprecise boundaries.
- Spatial logics.

Main features:

- Formal logics (propositional, first order, modal...).
- Compromise between expressiveness, completeness with respect to a class of situations, and complexity.
- Reasoning: inference, satisfiability, composition tables, CSP...
Allen’s intervals: temporal reasoning

13 basic relations:

- p
- m
- o
- s
- e
- d
- f

Reasoning: based on geometrical or latticial representations.
Allen’s intervals: temporal reasoning
Geometrical / quantitative representation:
Allen’s intervals: temporal reasoning

Qualitative representation: lattice:

\[
\begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
\hline
0 & - & - & - & - \\
\hline
2 & - & - & - & - \\
\hline
3 & - & - & - & - \\
\hline
4 & - & - & - & - \\
\end{array}
\]

\( o = 02 \)

\[ \Rightarrow \text{conceptual neighborhoods (Freksa)} \]
Allen’s intervals: temporal reasoning

Extensions: rectangle, cube algebra

- Allen’s interval in each direction
- 2D (rectangles): $13^2 = 169$ relations
- 3D (cubes): $13^3 = 2197$ relations
- ⇒ high complexity, and fixed shaped objects
Cardinal directions (Frank, Egenhofer, Ligozat)
Qualitative directions: N, NE, E, SE, S, SW, W, NW

Cone–based  Projection–based

How to deal with complex shapes?
Only few compositions can be exactly determined.
RCC: Region Connection Calculus (Randell, Cui, Cohn - Vieu...)

- Spatial entities, defined in a qualitative way.
- No reference to points.
- Connection predicate $C$.
- Parthood predicate $P$:

$$P(x, y) : \forall z, C(z, x) \rightarrow C(z, y)$$
RCC: Region Connection Calculus (Randell, Cui, Cohn - Vieu...)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC(x, y)</td>
<td>x is disconnected from y</td>
<td>¬C(x, y)</td>
</tr>
<tr>
<td>P(x, y)</td>
<td>x is a part of y</td>
<td>∀z, C(z, x) → C(z, y)</td>
</tr>
<tr>
<td>PP(x, y)</td>
<td>x is a proper part of y</td>
<td>P(x, y) ∧ ¬P(y, x)</td>
</tr>
<tr>
<td>EQ(x, y)</td>
<td>x is identical with y</td>
<td>P(x, y) ∧ P(y, x)</td>
</tr>
<tr>
<td>O(x, y)</td>
<td>x overlaps y</td>
<td>∃z, P(z, x) ∧ P(z, y)</td>
</tr>
<tr>
<td>DR(x, y)</td>
<td>x is discrete from y</td>
<td>¬O(x, y)</td>
</tr>
<tr>
<td>PO(x, y)</td>
<td>x partially overlaps y</td>
<td>O(x, y) ∧ ¬P(x, y) ∧ ¬P(y, x)</td>
</tr>
<tr>
<td>EC(x, y)</td>
<td>x is externally connected</td>
<td>C(x, y) ∧ ¬O(x, y)</td>
</tr>
<tr>
<td>TPP(x, y)</td>
<td>x is a tangential proper</td>
<td>PP(x, y) ∧ ∃z[EC(z, x) ∧ EC(z, y)]</td>
</tr>
<tr>
<td>NTPP(x, y)</td>
<td>proper part of y</td>
<td>PP(x, y) ∧ ¬∃z[EC(z, x) ∧ EC(z, y)]</td>
</tr>
</tbody>
</table>
RCC: Region Connection Calculus (Randell, Cui, Cohn - Vieu...)

Diagram of RCC relationships:

- **T**
- **C**
- **DR**
- **O**
- **P**
- **PI**
- **PP**
- **PPI**
- **PO**
- **NTPP**
- **TPP**
- **EQ**
- **TPPI**
- **NTPPI**
- **EC**
- **DC**
Qualitative trajectory calculus (Cohn et al.)

- Extension of RCC to take time into account (dynamic scenes).
- RCC + Allen
- Example:
  - $X$, $Y$ objects
  - $I_i$ time intervals

\[
(P(X, Y), I_1) \land (PO(X, Y), I_2) \land (DR(X, Y), I_3) \\
\land meet(I_1, I_2) \land meet(I_2, I_3) \land before(I_1, I_3)
\]
Adding shape

- Varzi: predicates for part, hole, fill, convex hull.
- Dugat: mereogeometry based on spheres (congruence, distance, order...).
Modal logics of space

Topology:

- □A: A is locally true (A is true at point x iff A is true in a neighborhood of x).
- ◊A = ¬□¬A: A is true at x iff A is true at least one point of the neighborhood of x.

Reasoning axioms and inference rules of S4:

- A → (B → A)
- (A → (B → C)) → ((A → B) → (A → C))
- (¬A → ¬B) → (B → A)
- □(A → B) → (□A → □B)
- □A → A
- □A → □□A
Figure 1.1. Each modal formula identifies a region in a topological space. (a) A spoon, $p$. (b) The container part of the spoon, $\square p$. (c) The boundary of the spoon, $\Diamond p \land \Diamond \neg p$. (d) The container part of the spoon with its boundary, $\Diamond \square p$. (e) The handle of the spoon, $p \land \neg \Diamond \square p$. In this case the handle does not contain the junction handle-container point. (f) The junction handle-container point of the spoon, $\Diamond \square p \land \Diamond (p \land \neg \Diamond \square p)$: a singleton in the topological space.
Other spatial logics:

- Translation of RCC into modal logics.
- Logics of places ($\square = \text{everywhere}, \diamond = \text{somewhere}$).
- Modal logics of proximity ($\square A = \text{everywhere close to } A$).
- Modal logics of distance ($\square \leq^a = \text{everywhere in a neighborhood of radius } a$).
- Logics of inclusion and contact (inference in GIS).
- Modal logics of geometry, with modalities for points and for lines (affine, projective, parallelism...).
RCC-8 and modal logic

- $\Box X$: $X$ is valid at any point.
- $\Diamond X$: there exists a point where $X$ is valid.
- $C(X, Y) \equiv \Diamond (X \land Y)$
- $DC(X, Y) \equiv \Box (\neg X \lor \neg Y)$
- $P(X, Y) \equiv \Box (X \rightarrow Y)$
- $O(X, Y) \equiv \Diamond (i(X) \land i(Y))$ ($i = \text{interior}$)
- $TP(X, Y) \equiv \Box (X \rightarrow Y) \land \Diamond (X \land c(\neg Y))$ ($c = \text{closure}$)
- ...

A few important issues

- Context
- Representation issues
- Reasoning (inference, satisfiability, decidability, CSP...)
- Complexity
- Applications

State of the art:
- Very few applications
- Focus on topology
- Almost nothing on metric relations
- Almost nothing on uncertainty
### Composition tables

**Allen intervals:**

\[
\begin{array}{cccccccccccc}
\text{.} & \text{p} & \text{m} & \text{o} & \text{F} & \text{D} & \text{s} & \text{e} & \text{S} & \text{d} & \text{f} & \text{O} & \text{M} & \text{P} \\
\text{p} & (p) & (p) & (p) & (p) & (p) & (p) & (p) & (pmosd) & (pmosd) & (pmosd) & (pmosd) & \text{full} \\
\text{m} & (p) & (p) & (p) & (p) & (p) & (p) & (m) & (m) & (m) & (m) & (m) & (m) \\
\text{o} & (p) & (p) & (pmo) & (pmo) & (pmo) & (pmoFD) & (o) & (o) & (oFD) & (osd) & (osd) & (osd) & (DFsOMP) \\
\text{F} & (p) & (m) & (o) & (F) & (D) & (o) & (F) & (D) & (osd) & (DFsOMP) & (DFsOMP) & (DFsOMP) & (DFsOMP) \\
\text{D} & (pmoFD) & (oFD) & (oFD) & (D) & (D) & (oFD) & (D) & (D) & (D) & (D) & (D) & (D) & (D) \\
\text{s} & (p) & (p) & (pmo) & (pmo) & (pmo) & (pmoFD) & (s) & (s) & (seS) & (d) & (d) & (d) & (dfO) \\
\text{e} & (p) & (m) & (o) & (F) & (D) & (s) & (e) & (S) & (S) & (S) & (S) & (S) & (S) \\
\text{S} & (pmoFD) & (oFD) & (oFD) & (D) & (D) & (seS) & (S) & (S) & (S) & (S) & (S) & (S) & (S) \\
\text{d} & (p) & (p) & (pmosd) & (pmosd) & (pmosd) & (pmosd) & (full) & (d) & (d) & (dfOMP) & (dfOMP) & (dfOMP) & (dfOMP) & (dfOMP) \\
\text{f} & (p) & (m) & (osd) & (Fef) & (DFsOMP) & (d) & (f) & (OMP) & (d) & (f) & (OMP) & (OMP) & (OMP) \\
\text{O} & (pmoFD) & (oFD) & (oFD) & (D) & (D) & (seS) & (M) & (P) & (P) & (P) & (P) & (P) & (P) \\
\text{M} & (pmoFD) & (seS) & (dfO) & (dfO) & (M) & (P) & (dfO) & (M) & (P) & (dfO) & (M) & (P) & (P) \\
\text{P} & \text{full} & (dfOMP) & (dfOMP) & (P) & (P) & (P) & (P) & (P) & (P) & (P) & (P) & (P) & (P) \\
\end{array}
\]

\[\text{full}=(pmoFDseSdfOMP) \text{ and } \text{concur}=(oFDseSdfO)\]

From [http://www.ics.uci.edu/~alspaugh/cls/shr/allen.html](http://www.ics.uci.edu/~alspaugh/cls/shr/allen.html)
### RCC-8:

<table>
<thead>
<tr>
<th>α</th>
<th>DC</th>
<th>EC</th>
<th>PO</th>
<th>TPP</th>
<th>NTPP</th>
<th>TPPI</th>
<th>NTPPi</th>
<th>EQ</th>
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</thead>
<tbody>
<tr>
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<td>*</td>
<td>DC,EC,PO,TPP,NTPP</td>
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<td>DC,EC,PO,TPP,NTPP</td>
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<td>DC</td>
<td>DC</td>
</tr>
<tr>
<td>EC</td>
<td>DC,EC,PO,TPP,NTPPi</td>
<td>DC,EC,PO,TPP,NTPPi</td>
<td>DC,EC,PO,TPP,NTPPi</td>
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<td>EC</td>
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</tr>
<tr>
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<tr>
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<td>EQ</td>
</tr>
</tbody>
</table>

From wikipedia
Other approaches

- Ontologies and description logics.
- Graph-based reasoning.
- Grammars.
- Formal concept analysis.
- Decision trees.
- Constraint Satisfaction Problem.
- Relational algebras on temporal or spatial configurations.
- Graphical models.
- ...
Example using RCC: region identification (Le Ber et al.)
PLATEAU

VALLEY

forest

crops

cows pastures

hay meadows and pastures

crops

forest

PLATEAU
territory partially overlapping group

village territory

state forest

village

village forest

state forest

state forest externally connected group

F-ILOT - F-ILOTLIM - F-ILOTECDOM - F-POCOM - F-ECDOM - F-ENTREDOM
Semi-quantitative spatial reasoning: fuzzy approaches

- Limitations of purely qualitative reasoning
- Interest of adding semi-quantitative extension to qualitative value for deriving useful and practical conclusions
- Limitations of purely quantitative representations in the case of imprecise statements, knowledge expressed in linguistic terms, etc.
- Integration of both quantitative and qualitative knowledge using semi-quantitative (or semi-qualitative) interpretation of fuzzy sets
- Freeman (1975): fuzzy sets provide computational representation and interpretation of imprecise spatial constraints, expressed in a linguistic way, possibly including quantitative knowledge
- Granularity, involved in:
  - objects or spatial entities and their descriptions
  - types and expressions of spatial relations and queries
  - type of expected or potential result
Motivation: model-based recognition and spatial reasoning

- representation of imprecision
- spatial relations as structural information
  - topological relationships (set relations, adjacency)
  - distances
  - relative directional relationships
  - more complex relations (between, along...)
- two classes of relations
  - well defined in the crisp case (adjacency, distances...)
  - vague even in the crisp case (directional relationships...)
- fusion of several and heterogeneous pieces of knowledge and information

⇒ Fuzzy set theory, mathematical morphology
Imprecision and fuzziness

- objects (no clear boundaries, coarse segmentation...)
- relations (ex: left of, quite close)
- type of knowledge available (ex: the caudate nucleus is close to the lateral ventricle)
- question to be answered (ex: go towards this object while remaining at some security distance)
Types of representations: example of distances

- number in $\mathbb{R}^+$ (or in $[0,1]$)
- interval
- fuzzy number, fuzzy interval
- Rosenfeld:
  - distance density: degree to which the distance is equal to $n$
  - distance distribution: degree to which the distance is less than $n$
- linguistic value
- logical formula

$\Rightarrow$ unifying framework of fuzzy set theory

\[ d_{\text{min}} = 17, \quad d_{\text{Haus}} = 80 \]
Fuzzy sets in a nutshell (Zadeh, 1965)

- Space $S$ (image space, space of characteristics, etc.).
- Fuzzy set: $\mu : S \rightarrow [0, 1] - \mu(x) =$ membership degree of $x$ to $\mu$.
- Set theoretical operations: complementations, conjunctions (t-norms), disjunctions (t-conorms).
- Logic operators, aggregation and fusion operators...

Example: spatial fuzzy set

- $S$: $\mathbb{R}^3$ or $\mathbb{Z}^3$ in the digital case.
- $\mu : S \rightarrow [0, 1] - \mu(x) =$ degree to which $x$ belongs to the fuzzy object.
Linguistic variable

- size
- linguistic variable
- syntactic rules
- terms
- semantic rules
- membership functions

{very small, small, medium, large, very large}
Mathematical morphology

**Dilation**: operation in complete lattices that commutes with the supremum.

**Erosion**: operation in complete lattices that commutes with the infimum.

⇒ applications on sets, fuzzy sets, functions, logical formulas, graphs, etc.

**Using a structuring element:**

- dilation as a degree of conjunction: $\delta_B(X) = \{x \in S \mid B_x \cap X \neq \emptyset\}$,
- erosion as a degree of implication: $\varepsilon_B(X) = \{x \in S \mid B_x \subseteq X\}$.

A lot of other operations...
Fuzzy mathematical morphology

- Dilation as degree of intersection:
  \[ D_\nu(\mu)(x) = \sup \{ t[\nu(y - x), \mu(y)], \ y \in S \} \]

- Erosion as degree of inclusion:
  \[ E_\nu(\mu)(x) = \inf \{ l[\nu(y - x), \mu(y)], \ y \in S \} \]

  \[ l \text{ from a t-conorm } T \text{ or by residuation from the t-norm } t \]

- Opening and closing by composition

- Similar properties as in classical mathematical morphology
Fuzzy spatial relations

Fuzzy sets \(\rightarrow\) relations become a matter of degree

- Set theoretical relations
- Topology: connectivity, connected components, neighborhood, boundaries, adjacency
- Distances
- Relative direction
- More complex relations: between, along, parallel, around...

Most of them can be defined from mathematical morphology.
Distances between fuzzy sets: morphological approach

Expression of distances (minimum, Hausdorff...) in morphological (i.e. algebraic) terms ⇒ easy translation to the fuzzy case
Minimum (nearest point) distance distribution

\[ d_N(X, Y) = \inf\{n \in \mathbb{N}, X \cap D^n(Y) \neq \emptyset\} = \inf\{n \in \mathbb{N}, Y \cap D^n(X) \neq \emptyset\} \]

Degree to which the distance between \( \mu \) and \( \mu' \) is less than \( n \) (distance distribution):

\[ \Delta_N(\mu, \mu')(n) = f[\sup_{x \in S} t[\mu(x), D^n_\nu(\mu')(x)], \sup_{x \in S} t[\mu'(x), D^n_\nu(\mu)(x)]] \]

Hausdorff distance: similar equations
Minimum (nearest point) distance density

\[ d_N(X, Y) = n \iff D^n(X) \cap Y \neq \emptyset \text{ and } D^{n-1}(X) \cap Y = \emptyset \]

\[ d_N(X, Y) = 0 \iff X \cap Y \neq \emptyset \]

Degree to which the distance between \( \mu \) and \( \mu' \) is equal to \( n \) (distance density):

\[ \delta_N(\mu, \mu')(n) = t\left[ \sup_{x \in S} t[\mu'(x), D^n_\nu(\mu)(x)] \right] , c\left[ \sup_{x \in S} t[\mu'(x), D^{n-1}_\nu(\mu)(x)] \right] \]

\[ \delta_N(\mu, \mu')(0) = \sup_{x \in S} t[\mu(x), \mu'(x)] \]

Hausdorff distance: similar equations
Example: spatial representation of knowledge about distance
Directional relations

Reference object \((R)\)  

\(\nu_{\text{Right}}\)  

\[\mu_{\text{Right}}(R) = \delta_{\nu_{\text{Right}}}(R)\]
Complex relations

Example: the heart is between the lungs
Reasoning with mathematical morphology

- Chaining operations (image interpretation, recognition)
- Fusion of spatial relations (ex: structural recognition)
- Links with logics
  - propositional logics:
    - elegant tools for revision, fusion, abduction
    - links with mereotology, "egg-yolk" structures, logics of distances, nearness logics, linear logics, logics of convexity...
  - modal logics:
    - \((\Diamond, \Box) = (\text{dilation, erosion})\)
    - symbolic and qualitative representations of spatial relations
  - fuzzy logic
Example: dilation and erosion of a formula

Structuring element $B$: relation between worlds

Dilation:

$$\text{Mod}(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap \text{Mod}(\varphi) \neq \emptyset\}$$

Erosion:

$$\text{Mod}(E_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \models \varphi\}$$
Dilation and erosion as modal operators

Structuring element $B$: accessibility relation $R(\omega, \omega')$ iff $\omega' \in B(\omega)$

$$
\mathcal{M}, \omega \models \square \varphi \iff \forall \omega' \in \Omega, \ R(\omega, \omega') \Rightarrow \mathcal{M}, \omega' \models \varphi
$$

$$
\Rightarrow \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \models \varphi
$$

$$
\Rightarrow B(\omega) \models \varphi
$$

$$
\mathcal{M}, \omega \models \Diamond \varphi \iff \exists \omega' \in \Omega, \ R(\omega, \omega') \text{ et } \mathcal{M}, \omega' \models \varphi
$$

$$
\Rightarrow \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \cap \text{Mod}(\varphi) \neq \emptyset
$$

$$
\Rightarrow B(\omega) \cap \text{Mod}(\varphi) \neq \emptyset
$$

$$
\square \varphi \equiv E_B(\varphi) \quad \Diamond \varphi \equiv D_B(\varphi)
$$

Spatial interpretation: restriction or necessary region / extension or possible region
Example: logical expressions and links with mereotology

- Spatial entities represented as formulas.
- Structuring element: binary relationship between worlds, accessibility relation...
- **Adjacency:** \( \varphi \land \phi \rightarrow \bot \) and \( \delta \varphi \land \psi \not\rightarrow \bot \) and \( \varphi \land \delta \psi \not\rightarrow \bot \).
- **Tangential part:** \( \varphi \rightarrow \psi \) and \( \delta \varphi \land \neg \psi \not\rightarrow \bot \).
- **Proper tangential part in mereotopology:**
  \[
  \text{TPP}(\varphi, \psi) = P(\varphi, \psi) \land \neg P(\psi, \varphi) \land \neg P(\delta(\varphi), \psi).
  \]

RCC expression for \((\varphi = x, \psi = y)\):
\[
\text{TPP}(x, y) = (P(x, y) \land \neg P(y, x)) \land \exists z[(C(z, x) \land \neg(\exists z', P(z', z) \land P(z', x))) \land (C(z, y) \land \neg(\exists z', P(z', z) \land P(z', y)))]
\]
Model based image understanding

Models of various types:

- acquisition properties (geometry, noise statistics...)
- shape
- appearance
- spatial relations
- ...

Important

- to use available knowledge
- to guide the image exploration, for segmentation, recognition, scene understanding
- to solve ambiguities
- to deal with imprecision
- ...
Issues:

- semantic gap
- imprecisions and uncertainties
- pathological cases
- algorithms

Two main questions in structural recognition in images:

- given two objects (possibly fuzzy), assess the degree to which a relation is satisfied
- given one reference object, define the area of the space in which a relation to this reference is satisfied (to some degree)
Example in brain imaging

- **Concepts:**
  - *brain*: part of the central nervous system located in the head
  - *caudate nucleus*: a deep gray nucleus of the telencephalon involved with control of voluntary movement
  - *glioma*: tumor of the central nervous system that arises from glial cells
  - ...

- **Spatial organization:**
  - the *left caudate nucleus* is *inside* the *left hemisphere*
  - it is *close* to the *lateral ventricle*
  - it is *outside* (left of) the *left lateral ventricle*
  - it is *above* the *thalamus*, etc.
  - ...

- **Pathologies:** relations are quite stable, but more flexibility should be allowed in their semantics
Integration of ontologies, spatial relations and fuzzy models

Symbolic knowledge
- **Generic knowledge**
  - Brain anatomy ontology + brain structural description
- **Knowledge of specific cases**
  - Brain tumor ontology

Ontology-based segmented image database
- Healthy cases
- Pathological cases
  - Infiltrating tumors
  - Circumscribed tumors

Graph based representation of the generic model
- Fuzzy modeling of spatial relations
- Adjacent, Directional relation, Distance relation

Learning procedure
- **Step 1:**
  - learning spatial relations (adjacency, distance, orientation) of the generic model using healthy cases
- **Step 2:**
  - learning spatial relation for specific cases
  - deducing stable relations for each class of pathologies

Dealing with a specific case
- Generic model adaptation using knowledge of specific case and results of the learning procedure
  - Modified relations
  - Graph based propagation process to update the graph and to represent the tumor impact on the surrounding structures

Input
- Inside(PIR, GPR) is preserved

Enrichment of the database
Ontology of the anatomy (FMA) enriched with an ontology of spatial relations

Spatial relation ontology concepts

Spatial relations between anatomical concepts

Brain anatomy concepts
Learning spatial relations

caudate nucleus
putamen
thalamus
ventricles
tumor

\[ f(d) \]

1
0

\[ m \]
\[ m + 2\sigma \]
distances
Spatial reasoning for model-based recognition

Segmentation and recognition of some internal structures on a normal case (O. Colliot et al.):

- fusion of spatial relations (given by the model) to previously recognized objects
- deformable model constrained by spatial relations
Examples in pathological cases (H. Khotanlou, J. Atif, et al.)
Best segmentation path (G. Fouquier et al.)

- Generic Knowledge
- Image to segment
- Saliency Map
- Reference Structures
- Model Graph
- Specialized Graph
- Graph step $i$
- Graph step $i + 1$
- Saliency Map
- Image to segment
- Localizations
- Saliency Histogram
- Selection
- Evaluation
- Structure Segmentation

Symbols:
- : already segmented
- O : to segment

A priori knowledge
Visual information
Results
Best segmentation path (G. Fouquier et al.)

Evaluation and backtracking
Best segmentation path (G. Fouquier et al.)

Some results
Global approach based in CSP (O. Nempont et al.)

**Brain MRI**
- Inhomogeneities correction
- Brain surface extraction

**Structural model**

**Initial domains**

**Constraint propagation**
- Update domains
- until convergence
- Select a constraint and compute the associated operator

**Final segmentation**
- Metric definition
- Domains obtained by the propagation algorithm
- Minimal surface extraction
Global approach based in CSP (O. Nempont et al.)

Constraint Satisfaction Problem (CSP):

- Constraint network = \((\chi, \mathcal{D}, \mathcal{C})\)
- \(\chi\) = variables
- \(\mathcal{D}\) = set of associated domains
- \(\mathcal{C}\) = constraints involving variables of \(\chi\), relations on the variable domains
- Propagation of constraints:
  - Locally consistent constraint if all values of the domains can satisfy the constraint.
  - Suppression of inconsistent values: \((\chi, \mathcal{D}, \mathcal{C}) \rightarrow (\chi, \mathcal{D}', \mathcal{C})\)
  - Propagator = operator reducing the domains according to a constraint.
Global approach based in CSP (O. Nempont et al.)

- Variables = anatomical structures.
- Domain of a variable = interval of fuzzy sets $[A, \overline{A}]$.
- Example of constraint (1): inclusion

$$C_{A, B}^{in} : \mathcal{D}(A) \times \mathcal{D}(B) \rightarrow \{0, 1\}$$

$$(\mu_1, \mu_2) \mapsto \begin{cases} 1 & \text{if } \mu_1 \leq \mu_2, \\ 0 & \text{otherwise.} \end{cases}$$

- Associated propagator:

$$\langle A, B; (A, \overline{A}), (B, \overline{B}); C_{A, B}^{in} \rangle \quad \longrightarrow \quad \langle A, B; (A, \overline{A} \land \overline{B}), (B \lor A, \overline{B}); C_{A, B}^{in} \rangle$$
Global approach based in CSP (O. Nempont et al.)

- Example of constraint (2): directional relation

\[ C_{A,B}^{\text{dir}}: \mathcal{D}(A) \times \mathcal{D}(B) \rightarrow \{0, 1\} \]

\[ (\mu_1, \mu_2) \mapsto \begin{cases} 1 & \text{if } \mu_2 \leq \delta_\nu(\mu_1), \\ 0 & \text{otherwise}. \end{cases} \]

- Associated propagator:

\[ \langle A, B; (\overline{A}, \overline{A}), (B, \overline{B}); C_{A,B}^{\text{dir}} \rangle \]

- Other constraints: distance, partition, connectivity, adjacency, volume, contraste...

- Ordering of the propagators and iteration application.
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Propagation of constraint: example
Result: example
Examples in remote sensing (C. Vanegas)
Examples in remote sensing (C. Vanegas)

(a) Example image.

(b) Labeled image: The blue regions represent the sea, the red and orange represent ships or boats and the yellow regions represent the docks.

(c) Concept hierarchy $T_C$ in the context of symbolic AI.

(d) Conceptual graph representing the spatial organization of some elements of Figure 5.3.1.
Image understanding as an abduction problem
Formulation in DL:

- Knowledge base $\mathcal{K}$.
- TBox $T$, $m_i$ concepts defined in $T$.
- ABox $A$, such $\forall a \in A, \mathcal{K} \not\models \neg a$.
- ABox Abduction Problem $\langle \mathcal{K}, A \rangle$: finding a set of assertions $\gamma$ such that $\mathcal{K} \cup \gamma \models A$.

Link between FCA and DL: semantic context $\mathbb{K}_T = (G, M, I)$ defined as:

\[
\begin{align*}
G &= \{ (\mathcal{I}, d) \mid \mathcal{I} \text{ is a model of } T \text{ and } d \in \Delta^\mathcal{I} \} \\
M &= \{ m_1, \ldots, m_n \} \\
I &= \{ ((\mathcal{I}, d), m) \mid d \in m^\mathcal{I} \}
\end{align*}
\]
Tbox:

\[
\begin{align*}
\text{Man} & \sqsubseteq \text{Human} \\
\text{Woman} & \sqsubseteq \text{Human} \\
\text{Athlete} & \equiv \text{Human} \sqcap \\
& \quad \exists \text{hasProfession}.\text{Sport} \\
\text{Jumper} & \sqsubseteq \text{Athlete} \sqcap \\
& \quad \exists \text{use}.\text{SportEquipment} \\
\text{Foam_Mat} & \sqsubseteq \text{SportEquipment} \\
\text{Pole} & \sqsubseteq \text{SportEquipment} \\
\text{Javelin} & \sqsubseteq \text{SportEquipment} \\
\text{Horizontal_bar} & \sqsubseteq \text{SportEquipment} \\
\text{Jumping_Event} & \sqsubseteq \text{Event} \sqcap \\
& \quad \exists \text{hasPart}.\text{Jumper} \\
\text{Pole_Vault} & \sqsubseteq \text{Jumping_Event} \sqcap \\
& \quad \exists \text{hasPart}.\text{Pole} \\
& \quad \exists \text{hasPart}.\text{Horizontal_Bar} \\
& \quad \exists \text{hasPart}.\text{Foam_Mat} \\
\text{High_Jump} & \sqsubseteq \text{Jumping_Event} \sqcap \\
& \quad \exists \text{hasPart}.\text{Horizontal_Bar} \\
& \quad \exists \text{hasPart}.\text{Foam_Mat} \\
& \ldots
\end{align*}
\]
Semantic context:

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<th>Human</th>
<th>Man</th>
<th>Woman</th>
<th>Athlete</th>
<th>Jumper</th>
<th>SportEquipment</th>
<th>Foam_Mat</th>
<th>Pole</th>
<th>Javelin</th>
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<th>Jumping_Event</th>
<th>Pole_Vault</th>
<th>High_Jump</th>
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I. Bloch

Symbolic AI
Abox:
\{bar_1 : Horizontal\_Bar, fm_1 : Foam\_Mat, pole_1 : Pole, je_1 : Jumping\_Event\}.

An explanation $\gamma$ could be $\{pv_1 : Pole\_Vault\}$
Brain image interpretation

- Non-Enhanced Brain Tumor
- Lateral Ventricle
- Caudate Nuclei
- Putamen
Tbox:

Brain ⊑ HumanOrgan
CerebralHemisphere ⊑ BrainAnatomicalStructure
PeripheralCerebralHemisphere ⊑ CerebralHemisphereArea
SubCorticalCerebralHemisphere ⊑ CerebralHemisphereArea
GreyNuclei ⊑ BrainAnatomicalStructure
LateralVentricle ⊑ BrainAnatomicalStructure
BrainTumor ⊑ Disease ⊓∃ hasLocation. Brain
SmallDeformingTumor ≡ BrainTumor ⊓∃ hasBehavior. Infiltrating

SubCorticalSmallDeformingTumor ≡ SmallDeformingTumor ⊓∃ hasLocation. SubCorticalCerebralHemisphere

PeripheralSmallDeformingTumor ≡ BrainTumor ⊓∃ hasLocation. PeripheralCerebralHemisphere

LargeDeformingTumor ≡ BrainTumor ⊓∃ hasLocation. CerebralHemisphere

Edema ⊓∃ hasComponent. Edema
Necrosis ⊓∃ hasComponent. Necrosis
Enhanced ⊓∃ hasEnhancement. Enhanced
\[ \text{DiseasedBrain} \equiv \text{Brain} \ominus \exists \text{isAlteredBy}
\text{Disease} \]

\[ \text{TumoralBrain} \equiv \text{Brain} \ominus \exists \text{isAlteredBy}
\text{BrainTumor} \]

\[ \text{SmallDeformingTumoralBrain} \equiv \text{Brain} \ominus \exists \text{isAlteredBy}
\text{SmallDeformingTumor} \]

\[ \text{LargeDeformingTumoralBrain} \equiv \text{Brain} \ominus \exists \text{isAlteredBy}
\text{LargeDeformingTumor} \]

\[ \text{PeripheralSmallDeformingTumoralBrain} \equiv \text{Brain} \ominus \exists \text{isAlteredBy}
\text{PeripheralSmallDeformingTumor} \]

\[ \text{SubCorticalSmallDeformingTumoralBrain} \equiv \text{Brain} \ominus \exists \text{isAlteredBy}
\text{SubCorticalSmallDeformingTumor} \]

\ldots
Abox:

\[ t_1 : BrainTumor \]
\[ e_1 : NonEnhanced \]
\[ l_1 : LateralVentricle \]
\[ p_1 : PeripheralCerebralHemisphere \]

\[ (t_1, e_1) : hasEnhancement \]
\[ (t_1, l_1) : farFrom \]
\[ (t_1, p_1) : hasLocation \]

Most specific concept:

\[ C \equiv BrainTumor \sqcap \exists hasEnhancement. NonEnhanced \sqcap \exists farFrom. LateralVentricle \sqcap \exists hasLocation. PeripheralCerebralHemisphere \]
Concept abduction problem $\langle \mathcal{K}, C \rangle : \gamma \sqsubseteq_{\mathcal{K}} C$

Possible explanation set:
$\{ \text{DiseasedBrain}, \exists \text{isAlteredBy}. \top, \text{SmallDeformingTumoralBrain}, \text{PeripheralSmallDeformingTumoralBrain}... \}$.

A preferred solution with respect to some minimality criteria:
$\gamma \equiv \text{PeripheralSmallDeformingTumoralBrain}$
Some references