Propositional, first order and modal logics

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Role of logic in AI

- For 2000 years, people tried to codify "human reasoning" and came up with logic.
- AI until the 1980s: mostly designing machines that are able to represent knowledge and to reason using logic (e.g. rule-based systems).
- Current approach: mostly learning from data.
- But how communicate knowledge to a system? (was easier in earlier systems).
- Logic is still of prime importance!

Goals of logic:

1. Knowledge representation (KR).
2. Reasoning.

Natural language vs logic

Natural language: tricky, sentences are not necessarily true or false, wrong conclusions are easy...
Logic: restrictive and less flexible but removes ambiguity.

Challenges of KR and reasoning:

- representation of commonsense knowledge,
- ability of a knowledge-based system to trade-off computational efficiency for accuracy of inferences,
- criteria to decide whether a reasoning is correct or not,
- ability to represent and manipulate uncertain knowledge and information.

Main components in any logic

- Symbols, variables, formulas.
- Syntax.
- Semantics.
- Reasoning.

Syntax

- Propositional symbols or variables (atomic formulas): $p, q, r \ldots.$
- Connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\leftrightarrow$ (double implication).
- Formulas: propositional variables, combination of formulas using connectives (and no others).

Semantics Interpretation of a formula:

$$v : \mathcal{F} \rightarrow \{0, 1\}$$

$0 =$ false, $1 =$ true (truth value)
World $=$ assignment to all variables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

Notation: $A \equiv B$ iff A and B have the same truth tables.

Tautology $\top$: always true.
Antilogy or contradiction $\bot$: always false.

Determining the truth value of a formula: using decomposition trees.

Prove that $(A \rightarrow (B \vee C)) \vee (A \rightarrow B)$ is not a tautology.

Some useful equivalences:

$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$

$$A \rightarrow B \equiv \neg A \vee B$$

$$A \vee \neg A \equiv \top$$

$$A \wedge \neg A \equiv \bot$$

$$A \rightarrow A \equiv \top$$

$$A \wedge \top \equiv A$$

$$A \vee \bot \equiv A$$

$$\ldots$$

Find the right negation...

Tintin - On a marché sur la Lune - Hergé, Casterman, 1954.

1. *Le cirque Hipparque a besoin de deux clowns, vous feriez parfaitement l'affaire* ($a \wedge b$).
2. *Le cirque Hipparque n'a pas besoin de deux clowns, vous ne pouvez donc pas faire l'affaire.*

Other connectives

- nor $p \downarrow q = \neg(p \vee q)$
- nand $p \uparrow q = \neg(p \wedge q)$
- xor $p \oplus q$ iff one and only one of the two propositions is true.

Example: prove that $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q) \equiv \neg(p \leftrightarrow q)$

Finite languages

- Finite set of propositional variables $\{p_1 ... p_n\}$.
- Infinite set of formulas, but finite set of non-equivalent formulas.
- Complete formula: $q_1 \wedge ... \wedge q_n$ where $\forall n, q_i = p_i$ or $q_i = \neg p_i$.
- Disjunctive Normal Form (DNF): disjunction of complete formulas.
- By duality: Conjunctive Normal Form (CNF).
- Any formula of the language can be written as an equivalent formula in DNF (or CNF).

Example: Write in DNF form the formula $(p \vee q) \wedge r$.

Knowledge representation: example

w: the grass is wet.
r: it was raining.
s: sprinkle was on.

$$KB = \{r \rightarrow w, s \rightarrow w\}$$

Models: $\{w, r, s\}$ (stands for $v(w) = 1, v(r) = 1, v(s) = 1$), $\{w, \neg r, s\}$, $\{\neg w, \neg r, \neg s\}$...

Axioms and inference rules

For $\neg$ and $\rightarrow$:

$$\mathcal{A}_1 : A \rightarrow (B \rightarrow A)$$

$$\mathcal{A}_2 : (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

$$\mathcal{A}_3 : (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$$

Note that $A \vee B \equiv \neg A \rightarrow B$, $A \wedge B \equiv \neg(A \rightarrow \neg B)$.

Modus ponens:

$$\frac{A, A \rightarrow B}{B}$$

$\Rightarrow$ Deductive system S for proving theorems.

Consequence relation $\vdash$

$H \vdash C$ iff C can be proved from H using a deduction system S.

Theorem $\vdash T$ (without hypotheses)

$$A \vdash B \text{ iff } \vdash (A \rightarrow B)$$

Theorems of propositional logic are exactly the tautologies (completeness and non-contradiction).

Deduction rules using elimination and introduction

	Elimination	Introduction
Conjunction	$\frac{P \wedge Q}{P}$ and $\frac{P \wedge Q}{Q}$	$\frac{P, Q}{P \wedge Q}$
Disjunction	$\frac{P \vee Q, P \vdash M, Q \vdash M}{M}$	$\frac{P}{P \vee Q}$ and $\frac{Q}{P \vee Q}$
Implication	$\frac{P, P \rightarrow Q}{Q}$	$\frac{P \vdash Q}{P \rightarrow Q}$
Negation	$\frac{P, \neg P}{\perp}$	$\frac{P \vdash \perp}{\neg P}$

Example: prove that $\{p \rightarrow (q \wedge r), p\} \vdash r$

Satisfiability: A is true in the world m (m is a model for A, m satisfies A)

$$m \models A$$

For a knowledge base: KB is satisfiable iff $\exists m, \forall \varphi \in KB, m \models \varphi$ (i.e. $Mod(KB) \neq \emptyset$).

$$\begin{array}{lll}
m \models A \wedge B & \text{iff} & m \models A \text{ and } m \models B \\
m \models A \vee B & \text{iff} & m \models A \text{ or } m \models B \\
m \models \neg A & \text{iff} & m \not\models A \\
m \models A \rightarrow B & \text{iff} & m \models \neg A \text{ or } m \models B \\
A \text{ tautology} & \text{iff} & \forall m, m \models A \\
A \rightarrow B \text{ tautology} & \text{iff} & \forall m, m \models A \text{ implies } m \models B
\end{array}$$

$$A \vdash B \text{ iff } m \models A \text{ implies } m \models B$$

Knowledge representation: example (cont'd)

w: the grass is wet.

r: it was raining.

s: sprinkle was on.

$$KB = \{r \rightarrow w, s \rightarrow w, \neg w\}$$

Can we deduce $\neg r$ from KB?

Consistent formulas

$$A \text{ consistent with } B \text{ if } A \not\vdash \neg B$$

Equivalent expressions:

- B consistent with A.
- $\exists m, m \models A$ and $m \models B$.
- $A \wedge B$ satisfiable.

2. Predicate logic, first order logic

- Representation of entities (objects) and their properties, and relations among such entities.
- More expressive than propositional logic.
- Use of quantifiers ($\forall$, $\exists$).
- Predicates used to represent a property or a relation between entities.

Example of syllogism:

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

Syntax

Formulas are built from:

- Constants $a, b...$
- Variables $x, y, z...$
- Elementary terms are constants and variables.
- Functions: apply to terms to generate new terms.
- Predicates: apply to terms, as relational expressions (do not create new terms).
- Logical connectives: apply on formulas.
- Quantifiers: allow the representation of properties that hold for a collection of objects. For a variable x:
  - Universal: $\forall x P$ (for all x the property P holds).
  - Existential: $\exists x P$ (P holds for some x).
  - $\neg(\forall x P) \equiv \exists x(\neg P)$, $\neg(\exists x P) \equiv \forall x(\neg P)$.

Atomic formulas: All formulas that can be obtained by applying a predicate.

Formulas of the first order language: built from atomic formulas, connectives and quantifiers.

Free variable: has at least one non-quantified occurrence in a formula.

Bound variable: has at least one quantified occurrence.

Closed formula: does not contain any free variable.

Examples:

- $\exists x p(x, y, z) \vee (\forall z (q(z) \rightarrow r(x, z))$
  x and z are both free and bound, y is free and not bound.
- $\forall x \exists y ((p(x, y) \rightarrow \forall z r(x, y, z))$ is a closed formula.

Formula in prenex form: all quantifiers at the beginning.

Write in prenex form the following formula:

$$\forall x F \rightarrow \exists x G$$

Axioms and inference rules

Same as in propositional logic, plus:

$$\mathcal{A}_4 : (\forall x F(x)) \rightarrow F(t/x)$$

where t replaces x in $F(t/x)$ (substitution)

$$\mathcal{A}_5 : (\forall x(F \rightarrow G)) \rightarrow (F \rightarrow \forall x G) \text{ for } x \text{ non-free in } F$$

Generalization:

$$\frac{F}{\forall x F}$$

Proofs, consequences, theorems

Same definitions as in propositional logic.

Deduction theorem:

$$F \vdash G \text{ iff } \vdash (F \rightarrow G)$$

Socrates' syllogism:

- Predicate $H(x)$: x is a men.
- Functional symbol s: Socrates.
- Predicate $M(x)$: x is mortal.

From $\mathcal{A}_4$ and modus ponens:

$$\frac{\forall x(H(x) \rightarrow M(x)), H(s)}{M(s)}$$

Deduction rules using additional elimination and introduction for $\forall$ and $\exists$

	Elimination	Introduction
$\forall$	$\frac{\forall x F(x)}{F(t/x)}$	$\frac{F}{\forall x F(x)}$
$\exists$	$\frac{\exists x F, F \rightarrow G}{G}$ (if x non-free in G)	$\frac{F(t)}{\exists x F(x)}$

Prove that

$$\exists x(F(x) \vee G(x)) \vdash (\exists x F(x)) \vee (\exists x G(x))$$

Structures, interpretations and models

Establishing the validity of a formula requires an interpretation!

Structure: $\mathcal{M} = (D, I)$

- D: non-empty domain,
- I: interpretation in D of the symbols of the language
  - maps every functional symbol to a function in D with the same arity,
  - maps every relational symbol to a predicate in D with the same arity.

For a closed formula F:

$$\mathcal{M} \models F \text{ if the interpretation of } F \text{ is true in } \mathcal{M}$$

For a free formula $F(x)$, and $a \in D$:

$$\mathcal{M} \models F(a) \text{ if the interpretation of } F(a) \text{ is true in } \mathcal{M}$$

Example

- Constant a
- Unary functional symbol f
- Binay relational symbol P
- $\mathcal{T} = \{F_1, F_2, F_3\}$ with

$$F_1 = \forall x \forall y \forall z (P(x, y) \wedge P(y, z) \rightarrow P(x, z)) \tag{1}$$
$$F_2 = \forall x P(a, x) \tag{2}$$
$$F_3 = \forall x P(x, f(x)) \tag{3}$$

For $\mathcal{M} = \{\mathbb{N}, 0, x^2, \leq\}$, we have $\mathcal{M} \models \mathcal{T}$.

Properties for closed formulas F and G:

$$\begin{array}{lll}
\mathcal{M} \models \neg F & \text{iff} & \mathcal{M} \not\models F \\
\mathcal{M} \models (F \wedge G) & \text{iff} & \mathcal{M} \models F \text{ and } \mathcal{M} \models G \\
\mathcal{M} \models (F \vee G) & \text{iff} & \mathcal{M} \models F \text{ or } \mathcal{M} \models G \\
\mathcal{M} \models (F \rightarrow G) & \text{iff} & \mathcal{M} \not\models F \text{ or } \mathcal{M} \models G
\end{array}$$

Properties for $F(x)$ and $G(x)$ having x as free variable:

$$\begin{array}{lll}
\mathcal{M} \models \neg F(a) & \text{iff} & \mathcal{M} \not\models F(a) \\
\mathcal{M} \models (F \wedge G)(a) & \text{iff} & \mathcal{M} \models F(a) \text{ and } \mathcal{M} \models G(a) \\
\mathcal{M} \models (F \vee G)(a) & \text{iff} & \mathcal{M} \models F(a) \text{ or } \mathcal{M} \models G(a) \\
\mathcal{M} \models (F \rightarrow G)(a) & \text{iff} & \mathcal{M} \not\models F(a) \text{ or } \mathcal{M} \models G(a) \\
\mathcal{M} \models \forall x F(x) & \text{iff} & \forall a \in D, \mathcal{M} \models F(a) \\
\mathcal{M} \models \exists x F(x) & \text{iff} & \exists a \in D, \mathcal{M} \models F(a)
\end{array}$$

Logically (universally) valid formulas: whose interpretation is true in all structures.

F and G are equivalent iff they have the same models.

Completeness: $\vdash T$ iff $\mathcal{M} \models T$ for any structure $\mathcal{M}$.

Deduction theorem + completeness: $F \vdash G$ iff any model of F is a model of G.

Properties of the consequence relation:

1. Reflexivity: $F \vdash F$
2. Logical equivalence: if $F \equiv G$ and $F \vdash H$, then $G \vdash H$
3. Transitivity: if $F \vdash G$ and $G \vdash H$, then $F \vdash H$
4. Cut: if $F \wedge G \vdash H$ and $F \vdash G$, then $F \vdash H$
5. Disjunction of antecedents: if $F \vdash H$ and $G \vdash H$, then $F \vee G \vdash H$
6. Monotony: if $F \vdash H$, then $F \wedge G \vdash H$

Note: same as in propositional logic.

3. Modals Logics

- Back to Aristotle:

$$\textit{possible} = \begin{cases} \text{can be or not be} \\ \text{contingent} \end{cases}$$

  Three modalities: necessary, impossible, contingent (mutually incompatible).
- Carnap: semantics of possible worlds.
- Kripke: accessibility relation between possible worlds.
- Many different modal logics, e.g.:
  - deontic logic,
  - temporal logic,
  - epistemic logic,
  - dynamic logic,
  - logic of places,
  - ...

Here: bases of propositional modal logic

Modalities

- Modify the meaning of a proposition.
- Formalize modalities of the natural language.
- Universal modal operator $\Box$ = necessity.
- Existential modal operator: $\Diamond$ = possibility.

Examples:

$\Box A$ - Necessity	$\Diamond A$ - Possibility
It is necessary that A	It is possible that A
It will be always true that A	It will sometimes be true that A
It must be that A	It is allowed that A
It is known that A	The inverse of A is not known
...	...

Syntax

- All the syntax of propositional logic.
- If A is a formula, then $\Box A$ and $\Diamond A$ are formulas.

Duality constraint: $\Diamond A \equiv \neg \Box \neg A$.

Semantics

- P: atoms of a modal language.
- Structure $\mathcal{F} = (W, R)$
  - W = non-empty universe of possible worlds,
  - $R \subseteq W \times W$ = accessibility relation.
- Model $\mathcal{M} = (W, R, V)$ with

$$\begin{array}{rccc} V: & P & \rightarrow & 2^W \\ & p & \mapsto & V(p) \end{array}$$

  $V(p)$ = subset of W where p is true.
- Notation $\mathcal{M} \models_\omega A$: A is true at ω in the model $\mathcal{M}$.

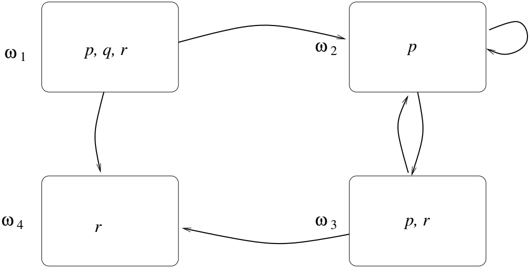
- $\mathcal{M} \models_\omega \top$
- $\mathcal{M} \not\models_\omega \bot$
- $\mathcal{M} \models_\omega p$ iff $\omega \in V(p)$
- $\mathcal{M} \models_\omega \neg A$ iff $\mathcal{M} \not\models_\omega A$
- $\mathcal{M} \models_\omega A_1 \wedge A_2$ iff $\mathcal{M} \models_\omega A_1$ and $\mathcal{M} \models_\omega A_2$
- $\mathcal{M} \models_\omega A_1 \vee A_2$ iff $\mathcal{M} \models_\omega A_1$ or $\mathcal{M} \models_\omega A_2$
- $\mathcal{M} \models_\omega A_1 \rightarrow A_2$ iff $\mathcal{M} \models_\omega A_1$ implies $\mathcal{M} \models_\omega A_2$
- $\mathcal{M} \models_\omega \Box A$ iff $\omega R t$ implies $\mathcal{M} \models_t A$ for all $t \in W$
- $\mathcal{M} \models_\omega \Diamond A$ iff $\mathcal{M} \models_t A$ for at least a $t \in W$ such that $\omega R t$

Valid formula

- A is valid in a model $\mathcal{M}$ if $\mathcal{M} \models_\omega A$ for all $w \in W$ (notation: $\mathcal{M} \models A$).
- A is valid in a structure $\mathcal{F}$ if it is valid in any model having this structure (notation: $\mathcal{F} \models A$).
- A is valid if it is valid in any structure (notation: $\models A$).

A simple example

$P = \{p, q, r\}$



$\mathcal{M}$: $W = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, V as in the figure,
$R = \{(\omega_1, \omega_2), (\omega_2, \omega_2), (\omega_2, \omega_3), (\omega_3, \omega_2), (\omega_3, \omega_4), (\omega_1, \omega_4)\}$.

Prove that

- $\mathcal{M} \models_{\omega_2} \Box p$
- $\mathcal{M} \models_{\omega_1} \Diamond(r \wedge \Box q)$

Schemas

K	$\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
P	$A \rightarrow \Box A$
L	$\Box(\Box A \rightarrow A) \rightarrow \Box A$
M	$\Box\Diamond A \rightarrow \Diamond\Box A$
T	$\Box A \rightarrow A$
B	$A \rightarrow \Box\Diamond A$
D	$\Box A \rightarrow \Diamond A$
4	$\Box A \rightarrow \Box\Box A$
5	$\Diamond A \rightarrow \Box\Diamond A$

...

Validity of schemas

Validity of	iff R is	
T	reflexive	$\forall s, sRs$
B	symmetric	$\forall s, t, sRT$ implies tRs
D	reproductive or serial	$\forall s, \exists t, sRt$
4	transitive	$\forall s, t, u, sRt$ and tRu implies sRu
5	Euclidean	$\forall s, t, u, sRt$ and sRu implies tRu
...		

Example: prove that $\Box A \rightarrow A$ is valid iff R is reflexive.

Typical examples

- Normal logics: contain K and the necessity inference rule $RN : \frac{A}{\Box A}$.
  - A is a theorem of logic K iff A is valid.
- KT logic
  - A is a theorem of logic KT iff A is valid in any structure where R is reflexive.
- $S4$ logic: contains $KT4$
  - A is a theorem of logic $S4$ iff A is valid in any structure where R is reflexive and transitive.
- $S5$ logic: contains $KT45$
  - A is a theorem of logic $S5$ iff A is valid in any structure where R is reflexive, transitive and Euclidean (R is an equivalence relation).

Theorems and inference rules

Depend on the schemas and axiomatic systems.

Example: Prove that

- $A \rightarrow \Diamond A$ is a theorem of $S5$,
- $A \rightarrow \Box\Diamond A$ is a theorem of $S5$,
- $RM : \frac{A \rightarrow B}{\Box A \rightarrow \Box B}$ is an inference rule of $S5$.

Algebraic approach for semantics

- Truth values can take other values than 0 and 1.
- $\Rightarrow$ multi-valued logics.
- Example: Lukasiewicz' 3-valued logic

Decidability

Is there an algorithm able to answer yes or no?

- Propositional logic: establishing that a formula is a tautology, that it is satisfiable, or that it is a consequence of a set of formulas are all decidable.
- First order logic: not decidable in general.
- Modal logic: decidable it if has the finite model property (i.e. every non-theorem is false in some finite model) and is axiomatizable by a finite number of schemas (ex: KT, KT4...).

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