Fuzzy Logic

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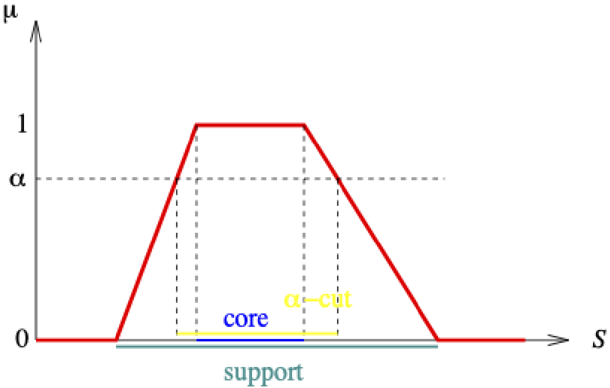
Imprecision and fuzziness

- usual way of speaking (ex: *a young person*)
- objects (no clear boundaries, coarse segmentation...)
- relations (ex: *left of, quite close*)
- observations (ex: *it is raining moderately*)
- type of knowledge available (ex: *the caudate nucleus is close to the lateral ventricle*)
- question to be answered (ex: *go towards this object while remaining at some security distance*)
- ...

Definitions: fuzzy sets (L. Zadeh, 1965)

- Space $\mathcal{S}$ (image space, space of characteristics, etc.)
- Fuzzy set: $\mu : \mathcal{S} \rightarrow [0,1]$ – $\mu(x) =$ membership degree of x to μ
- Support: $Supp(\mu) = \{x \in \mathcal{S}, \mu(x) > 0\}$
- Core / kernel: $\{x \in \mathcal{S}, \mu(x) = 1\}$
- α-cut: $\mu_\alpha = \{x \in \mathcal{S}, \mu(x) \geq \alpha\}$
- Cardinality: $|\mu| = \sum_{x \in \mathcal{S}} \mu(x)$ (for $\mathcal{S}$ finite)
- Convexity:
  $\forall (x,y) \in \mathcal{S}^2, \forall \lambda \in [0,1], \mu(\lambda x + (1-\lambda)y) \geq \min(\mu(x), \mu(y))$
- Fuzzy number: convex fuzzy set on $\mathbb{R}$, u.s.c., unimodal, with compact support. Example: LR-fuzzy sets.


μ
1
α
0
α−cut
core
support
S

First basic operations (L. Zadeh, 1965)

- Equality: $\mu = \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) = \nu(x)$
- Inclusion: $\mu \subseteq \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x)$
- Intersection: $\forall x \in \mathcal{S}, (\mu \cap \nu)(x) = \min(\mu(x), \nu(x))$
- Union: $\forall x \in \mathcal{S}, (\mu \cup \nu)(x) = \max(\mu(x), \nu(x))$
- Complementation: $\forall x \in \mathcal{S}, \mu^C(x) = 1 - \mu(x)$
- Properties:
  - consistency with binary set operations
  - $\mu = \nu \Leftrightarrow \mu \subseteq \nu$ and $\nu \subseteq \mu$
  - fuzzy complementation is involutive: $(\mu^C)^C = \mu$
  - intersection and union are commutative and associative
  - intersection and union are idempotent and mutually distributive
  - intersection and union are dual with respect to the complementation: $(\mu \cap \nu)^C = \mu^C \cup \nu^C$
  - $(\mu \cup \nu)_\alpha = \mu_\alpha \cup \nu_\alpha$, etc.

BUT: $\mu \cap \mu^C \neq \emptyset$, $\mu \cup \mu^C \neq \mathcal{S}$

Definitions: possibility theory (L. Zadeh, D. Dubois, H. Prade

Possibility measure: function Π from $2^{\mathcal{S}}$ into $[0, 1]$ such that:

1. $\Pi(\emptyset) = 0$
2. $\Pi(\mathcal{S}) = 1$
3. $\forall I \subseteq N, \forall A_i \subseteq \mathcal{S} (i \in I),\ \Pi(\cup_{i \in I} A_i) = \sup_{i \in I} \Pi(A_i)$

Necessity measure: $\forall A \subseteq \mathcal{S},\ N(A) = 1 - \Pi(A^C)$

1. $N(\emptyset) = 0$
2. $N(\mathcal{S}) = 1$
3. $\forall I \subseteq N, \forall A_i \subseteq \mathcal{S} (i \in I),\ N(\cap_{i \in I} A_i) = \inf_{i \in I} N(A_i)$

Useful properties:

- $\max(\Pi(A), \Pi(A^C)) = 1, \quad \min(N(A), N(A^C)) = 0$
- $\Pi(A) \geq N(A)$
- $N(A) > 0 \Rightarrow \Pi(A) = 1, \quad \Pi(A) < 1 \Rightarrow N(A) = 0$
- $N(A) + N(A^C) \leq 1, \quad \Pi(A) + \Pi(A^C) \geq 1$

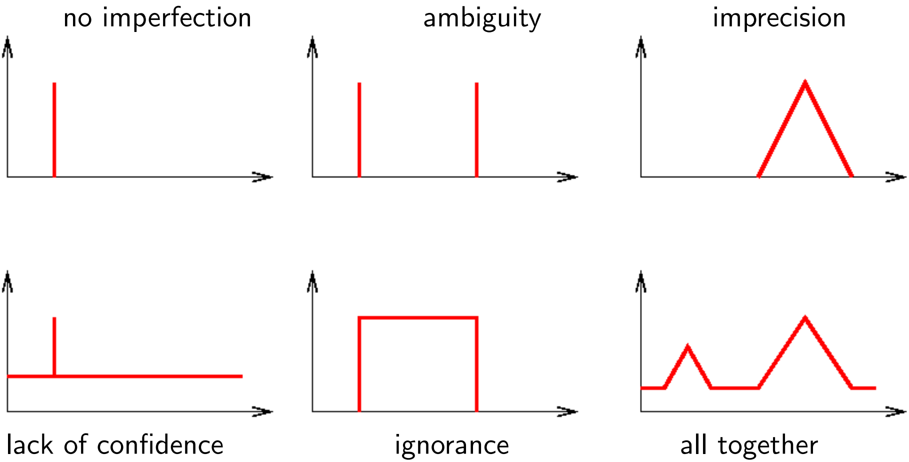
Possibility distribution: function π from $\mathcal{S}$ into $[0, 1]$ with the normalization condition $\sup_{x \in \mathcal{S}} \pi(x) = 1$

- Interpretation of π as the membership function defining the fuzzy set of possible values.
- In the finite case: $\Pi(A) = \sup\{\pi(x), x \in A\}$ (for $A \subseteq \mathcal{S}$)
  Conversely: $\forall x \in \mathcal{S}, \; \pi(x) = \Pi(\{x\})$
- $N(A) = 1 - \sup\{\pi(x), x \notin A\} = \inf\{1 - \pi(x), x \in A^C\}$

Semantics

- Degree of similarity (notion of distance).
- Degree of plausibility (that an object from which only an imprecise description is known is actually the one one wants to identify).
- Degree of preference (fuzzy class = set of “good” choices), close to the notion of utility function.

Representing different types of imperfection



Set theoretical operations

Fuzzy complementation / negation

function c from $[0, 1]$ into $[0, 1]$ such that:

1. $c(0) = 1$
2. $c(1) = 0$
3. c is involutive, i.e. $\forall x \in [0, 1], c(c(x)) = x$
4. c is strictly decreasing

General form of continuous complementations: $c(x) = \varphi^{-1}(1 - \varphi(x))$ with $\varphi : [0, 1] \rightarrow [0, 1]$, $\varphi(0) = 0$, $\varphi(1) = 1$, φ strictly increasing.

Example: $\varphi(x) = x^n \Rightarrow c(x) = (1 - x^n)^{1/n}$

Triangular norms: fuzzy intersection / conjunction

t-norm $t : [0,1] \times [0,1] \to [0,1]$ such that:

1. commutativity, i.e. $\forall (x,y) \in [0,1]^2,\ t(x,y) = t(y,x)$;
2. associativity, i.e. $\forall (x,y,z) \in [0,1]^3,\ t(t(x,y),z) = t(x,t(y,z))$;
3. 1 is unit element, i.e. $\forall x \in [0,1],\ t(x,1) = t(1,x) = x$;
4. increasingness with respect to the two variables:

$$\forall (x,x',y,y') \in [0,1]^4,\ (x \leq x' \text{ and } y \leq y') \Rightarrow t(x,y) \leq t(x',y').$$

Moreover: $t(0,1) = t(0,0) = t(1,0) = 0$, $t(1,1) = 1$, and 0 is null element $(\forall x \in [0,1], t(x,0) = 0)$.

Examples of t-norms: $\min(x,y)$, xy, $\max(0, x+y-1)$.

Triangular conorms: fuzzy union / disjunction

t-conorm $T : [0,1] \times [0,1] \to [0,1]$ such that:

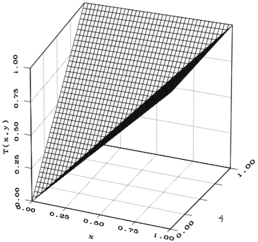
1. commutativity, i.e. $\forall (x,y) \in [0,1]^2,\ T(x,y) = T(y,x)$;
2. associativity, i.e. $\forall (x,y,z) \in [0,1]^3,\ T(T(x,y),z) = T(x,T(y,z))$;
3. 0 is unit element, i.e. $\forall x \in [0,1],\ T(x,0) = T(0,x) = x$;
4. increasingness with respect to the two variables

Moreover: $T(0,1) = T(1,1) = T(1,0) = 1,\ T(0,0) = 0$, and 1 is null element ($\forall x \in [0,1], T(x,1) = 1$).

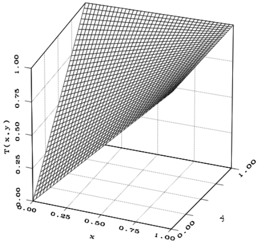
Examples of t-conorms: $\max(x,y)$, $x + y - xy$, $\min(1, x+y)$.

Duality: pair (t, T) such that $\forall (x,y) \in [0,1]^2,\ T(c(x),c(y)) = c(t(x,y))$

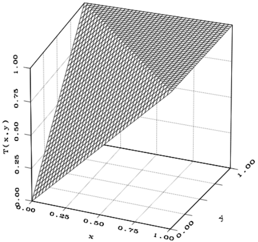
Other combination operators (mean, symmetrical sums, etc.) $\Rightarrow$ information fusion



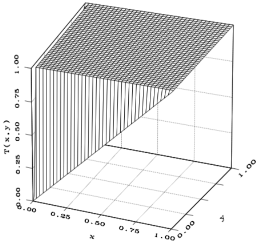
$\max(x, y)$



$x + y - xy$

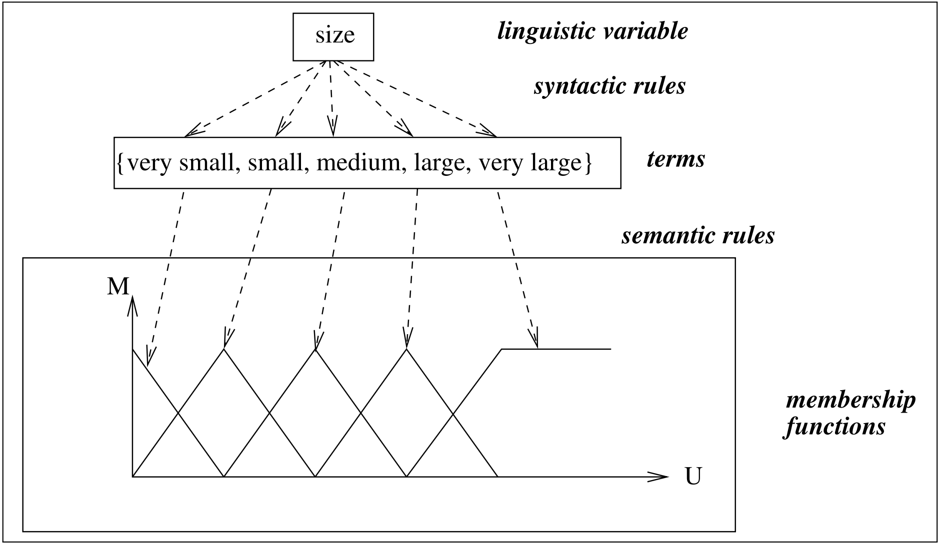


$\min(1, x + y)$



x if $y = 0$, y if $x = 0$, 1 otherwise

Linguistic variable (L. Zadeh)



Imprecise reasoning

- Difference between data and knowledge
- Classical logic:
  - language
  - semantics (interpretations, truth values)
  - syntax (axioms and inference rules)
- Human reasoning: flexible, allows for imprecise statements
- Gradual predicates:
  - continuous referential
  - typicality

Examples: *this person is young, this person is rather young...* –
Propositions that can be neither completely true nor completely false.

Uncertainty

= unable to say whether a proposition is true or not

- because information is incomplete, vague, imprecise
  $\Rightarrow$ possibility
- because information is contradicting or fluctuating
  $\Rightarrow$ probability

certainty degree $\neq$ truth degree

"It is probable that he is far from his goal"	"He is very far from his goal"

- Fuzzy logic: propositions with truth degrees
- Possibilistic logic: propositions with (un)certainty degrees

Fuzzy logic

- Basic fuzzy propositions: X *is* P
  $X =$ variable taking values in $\mathcal{U}$
  $P =$ fuzzy subset of $\mathcal{U}$
  Truth degrees in $[0, 1]$ defined from μ_P
- Conjunction: X *is* A *and* Y *is* B

$$\mu_{A \wedge B}(x, y) = t(\mu_A(x), \mu_B(y))$$

- Disjunction: X *is* A *or* Y *is* B

$$\mu_{A \vee B}(x, y) = T(\mu_A(x), \mu_B(y))$$

- Negation

$$\mu_{\neg A}(x) = c(\mu_A(x))$$

- Variables in a product space: X with values in $\mathcal{U}$, Y with values $\mathcal{V} \Rightarrow$ conjunction = Cartesian product: X *is* A *and* Y *is* B

$$\mu_{A \times B}(x, y) = t(\mu_A(x), \mu_B(y))$$

Fuzzy implications

- Classical propositional logic: $(A \rightarrow B) \equiv (B \vee \neg A)$
- Fuzzy logic: implication from a negation and a t-conorm
  - A and B crisp:

$$I(A, B) = T(c(A), B)$$

  - A and B fuzzy:

$$I(A, B) = \inf_x T(c(\mu_A(x)), \mu_B(x))$$

  - Examples ($c(a) = 1 - a$):

$T(a, b) = \max(a, b)$	$I(a, b) = \max(1 - a, b)$	Kleene-Diene
$T(a, b) = \min(1, a + b)$	$I(a, b) = \min(1, 1 - a + b)$	Lukasiewicz
$T(a, b) = a + b - ab$	$I(a, b) = 1 - a + ab$	Reichenbach

- Fuzzy logic: residual implications from a t-norm

$$I(A, B) = \sup\{X \mid t(X, A) \leq B\}$$

Adjunction: $t(X, A) \leq B \Leftrightarrow X \leq I(A, B)$

Fuzzy reasoning

- Classical propositional logic
  - Modus ponens: $(A \wedge (A \rightarrow B)) \vdash B$
  - Modus tollens: $((A \rightarrow B) \wedge \neg B) \vdash \neg A$
  - Syllogism: $((A \rightarrow B) \wedge (B \rightarrow C)) \vdash (A \rightarrow C)$
  - Cuntraposition: $(A \rightarrow B) \vdash (\neg B \rightarrow \neg A)$
  - ...
- Fuzzy (generalized) modus ponens
  - Rule :

$$\textit{if} \quad X \textit{ is } A \quad \textit{then} \quad Y \textit{ is } B$$

  - Knowledge or observation:

$$X \textit{ is } A'$$

  - Conclusion:

$$Y \textit{ is } B'$$

$$\mu_{B'}(y) = \sup_x t(\mu_{A'}(x), I(\mu_A(x), \mu_B(y)))$$

Fuzzy rules

$$\text{IF } (x \textit{ is } A \text{ AND } y \textit{ is } B) \text{ THEN } z \textit{ is } C$$

$$\text{IF } (x \textit{ is } A \text{ OR } y \textit{ is } B) \text{ THEN } z \textit{ is } C$$

...

α: truth degree of x *is* A
β: truth degree of y *is* B
γ: truth degree of z *is* C

Satisfaction degree of the rule:

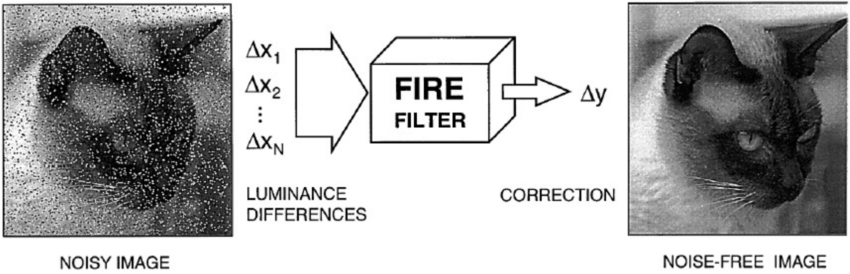
$$I(t(\alpha, \beta), \gamma) = T(c(t(\alpha, \beta)), \gamma))$$

$$I(T(\alpha, \beta), \gamma) = T(c(T(\alpha, \beta)), \gamma))$$

...

Example in image filtering

IF	a pixel is **darker** than its neighbors
THEN	**increase** its grey level
ELSE IF	a pixel is **brighter** than its neighbors
THEN	**decrease** its grey level
ELSE	**unchanged**



F. Russo et al.

Possibilistic logic

- Possibility measure on a Boolean algebra of logical formulas: $\Pi : B \to [0,1]$ such that:
  - $\Pi(\bot) = 0$
  - $\Pi(\top) = 1$
  - $\forall \varphi, \phi,\ \Pi(\varphi \vee \psi) = \max(\Pi(\varphi), \Pi(\psi))$
  - $\forall \varphi,\ \Pi(\exists x \varphi) = \sup\{\Pi(\varphi[a|x]), a \in D(x)\}$ (with $D(x) =$ domain of variable x, and $\varphi[a|x]$ obtained by replacing occurrences of x in φ by a)
- Normalized possibility distribution: $\pi : \Omega \to [0,1]$ such that $\exists \omega \in \Omega,\ \pi(\omega) = 1$ ($\Omega =$ set of interpretations)

$$\Pi(\varphi) = \sup\{\pi(\omega),\ \omega \models \varphi\}$$

- Necessity measure:

$$N(\varphi) = 1 - \Pi(\neg\varphi)$$

$\forall \varphi, \phi,\ N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$

- Example: default rule "if A then B"

$$\Pi(A \wedge B) \geq \Pi(A \wedge \neg B)$$

Possibilistic modus ponens

- Rule:

$$N(A \rightarrow B) = \alpha$$

- Knowledge or observation:

$$N(A) = \beta$$

- Conclusion:

$$\min(\alpha, \beta) \leq N(B) \leq \alpha$$

Stratified knowledge bases

$$KB = \{(\varphi_i, \alpha_i), i = 1...n\}$$

- $\alpha_i \in (0,1]$: certainty degree or priority of the (propositional) formula φ_i
- means $N(\varphi_i) \geq \alpha_i$
- knowledge: one is certain at level α_i that φ_i is true
- preference: goal φ_i with priority α_i

Representation as a possibility distribution on the set of interpretations Ω (induced by the underlying propositional logic):

- for one formula (φ, α):

$$\forall \omega \in \Omega,\ \pi_{(\varphi,\alpha)}(\omega) = \begin{cases} 1 & \textit{if } \omega \models \varphi \\ 1 - \alpha & \textit{otherwise} \end{cases}$$

- for the knowledge base:

$$\forall \omega \in \Omega,\ \pi_{KB}(\omega) = \min_{i=1...n} \{1-\alpha_i, \omega \models \neg\varphi_i\} = \min_{i=1...n} \max(1-\alpha_i, \varphi_i(\omega))\}$$

- Example of inference rule:

$$((\neg p \vee q, \alpha); (p \vee r, \beta)) \vdash (q \vee r, \min(\alpha, \beta))$$

- Inconsistency degree of KB : $1 - \max_{\omega \in \Omega} \pi_{KB}(\omega)$
- Complete base: $\forall \varphi$, either $KB \vdash \varphi$, or $KB \vdash \neg\varphi$
- Ignorance on φ: $KB \not\vdash \varphi$ and $KB \not\vdash \neg\varphi$
  $\Rightarrow$ simplest possibilistic model:

$$\Pi(\varphi) = \Pi(\neg\varphi) = 1$$

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