Formal Concept Analysis

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Objectives

- Symbolic learning
- Data mining.
- Knowledge discovery
- ...

Input

- data expressed as a table objects $\times$ attributes

Output

- concept lattice:
  - clusters = formal concepts (nodes of the Hasse diagram)
  - sub-concept / super-concept hierarchy (partial order)
- attribute implications:
  - representative set of dependencies among data

A small example

I	y_1	y_2	y_3
x_1	X	X	X
x_2	X		X
x_3		X	X

- x_i = objects
- y_i = attributes
- example of concept: $(\{x_1, x_2\}, \{y_1, y_3\})$
- example of attribute implication: $\{y_1\} \Rightarrow \{y_3\}$

Historical notion of concept

Port-Royal logic (traditional logic): formal notion of concept
A. Arnauld, P. Nicole : La logique ou l'art de penser, 1662.

concept = extent (objects) + intent (attributes)

Later:

- G. Birkhoff in the 1940's: Lattice theory
- M. Barbut, B. Monjardet in the 1970's: partial order, classification
- R. Wille in the 1980's: hierarchy of concepts
- B. Ganter and R. Wille in the 1990's : formal concept analysis

Formal Concept Analysis (FCA) (Ganter et al. 1997)

- Set of objects G.
- Set of attributes M.
- Relation $I \subseteq G \times M$: $(g, m) \in I$ = object g has attribute m.
- Formal context: $\mathbb{K} = (G, M, I)$.
- Derivation operators $\alpha : \mathcal{P}(G) \to \mathcal{P}(M)$ and $\beta : \mathcal{P}(M) \to \mathcal{P}(G)$:

$$\forall X \subseteq G, \alpha(X) = \{m \in M \mid \forall g \in X, (g, m) \in I\}$$

$$\forall Y \subseteq M, \beta(Y) = \{g \in G \mid \forall m \in Y, (g, m) \in I\}$$

(also denoted by $'$ or $\uparrow, \downarrow$)

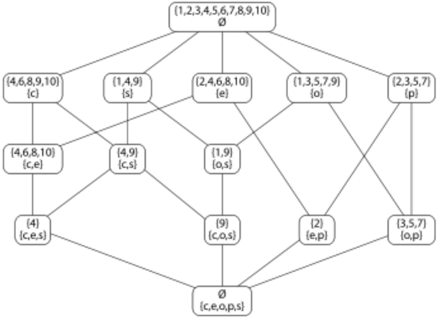
Example: $\alpha(\{x_1, x_2\}) = \{y_1, y_3\}, \beta(\{y_2\}) = \{x_1, x_3\}$

Formal concept and concept lattice

- $(X, Y), X \subseteq G, Y \subseteq M$ is a formal concept if
$$\alpha(X) = Y \text{ and } \beta(Y) = X$$
Formal concept $a = (e(a), i(a))$, extent $e(a) \subseteq G$, intent $i(a) \subseteq M$.
- (X, Y) formal concept iff it is maximal for the property $X \times Y \subseteq I$.
- Partial ordering:
$$(X_1, Y_1) \preceq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2 \;\; (\Leftrightarrow Y_2 \subseteq Y_1)$$
reflects the sub-concept / super-concept relation.
- $\mathbb{C}$: set of concepts of the context $\mathbb{K} = (G, M, I)$.
- $(\mathbb{C}, \preceq)$ is a complete lattice, called concept lattice.

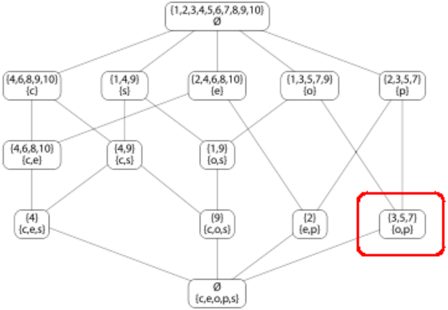
Example of a context and its concept lattice from Wikipedia

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			



- Objects are integers from 1 to 10.
- Attributes are composite (c) (i.e. non prime integer strictly greater than 1), even (e), odd (o), prime (p) and square (s).
- $\mathbb{K} = (G = \{1, 2...10\}, M = \{c, e, o, p, s\}, I)$.

$\mathbb{K}$	composite	even	odd	prime	square
1			×		×
2		×		×	
3			×	×	
4	×	×			×
5			×	×	
6	×	×			
7			×	×	
8	×	×			
9	×		×		×
10	×	×			



The pair $(\{3, 5, 7\}, \{o, p\})$ is a formal concept.

Galois connection

(α, β) is a Galois connection between the posets $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$:

$$\forall X \in \mathcal{P}(G), \forall Y \in \mathcal{P}(M), Y \subseteq \alpha(X) \Leftrightarrow X \subseteq \beta(Y)$$

Equivalently:

1. $\forall X_1, X_2 \subseteq G, X_1 \subseteq X_2 \Rightarrow \alpha(X_2) \subseteq \alpha(X_1)$
2. $\forall Y_1, Y_2 \subseteq M, Y_1 \subseteq Y_2 \Rightarrow \beta(Y_2) \subseteq \beta(Y_1)$
3. $\forall X \subseteq G, X \subseteq \beta(\alpha(X))$
4. $\forall Y \subseteq M, Y \subseteq \alpha(\beta(Y))$

Consequently:

- $\forall X \subseteq G, \alpha(X) = \alpha(\beta(\alpha(X)))$
- $\forall Y \subseteq M, \beta(Y) = \beta(\alpha(\beta(Y)))$
- $\alpha\beta$ and $\beta\alpha$ are closure operators, i.e. increasing, extensive and idempotent.
- $\alpha(\cup_i X_i) = \cap_i \alpha(X_i)$
- $\beta(\cup_j Y_j) = \cap_j \beta(Y_j)$

Conversely

- $A : \mathcal{P}(G) \to \mathcal{P}(M)$
- $B : \mathcal{P}(M) \to \mathcal{P}(G)$
- (A, B) Galois connection
- $\Rightarrow$ (A, B) is induced by a binary relation I
- $(g, m) \in I \Leftrightarrow m \in A(\{g\}) \Leftrightarrow g \in B(\{m\})$
- the derivation operators are then exactly $\alpha = A$ and $\beta = B$

- Infimum and supremum of a family $(X_t, Y_t)_{t \in T}$ of formal concepts:

$$\begin{aligned} \bigwedge_{t \in T} (X_t, Y_t) &= \left(\bigcap_{t \in T} X_t, \alpha(\beta(\bigcup_{t \in T} Y_t)) \right) \\ \bigvee_{t \in T} (X_t, Y_t) &= \left(\beta(\alpha(\bigcup_{t \in T} X_t)), \bigcap_{t \in T} Y_t \right) \end{aligned}$$

They are formal concepts.

Clarified context

= without redundant columns or rows

The concept lattices before and after clarification are isomorphic.

Example

	y_1	y_2	y_3	y_4
x_1	X	X	X	X
x_2	X		X	X
x_3		X	X	X
x_4		X	X	X
x_5	X			

– clarification $\rightarrow$

	y_1	y_2	y_3
x_1	X	X	X
x_2	X		X
x_3		X	X
x_5	X		

Another possible context reduction: based on reducible elements

Concept lattice construction

Remark:

- $\forall X \subseteq G$, $(\beta(\alpha(X)), \alpha(X))$ is a formal concept
- $\forall Y \subseteq M$, $(\beta(Y), \alpha(\beta(Y)))$ is a formal concept
- smallest concept: $(\beta(\alpha(\emptyset)), \alpha(\emptyset))$

$\Rightarrow$ starting from the smallest concept, iteratively add objects and compute closure.

Main issue: complexity $\Rightarrow$ several algorithms to reduce it.

Extension: add support information (frequent intents) $Supp(Y) = \frac{|\beta(Y)|}{|G|}$

Attribute implication

= description of some dependencies between data

Validity:

- $\mathbb{K} = (G, M, I)$, $A \subseteq M$, $B \subseteq M$
- Subset M' of attributes ($M' \subseteq M$)
- Attribute implication: $A \Rightarrow B$
- $A \Rightarrow B$ valid (true) in M' iff $A \subseteq M'$ implies $B \subseteq M'$.
- $A \Rightarrow B$ valid (true) in $\mathbb{K}$ iff $A \Rightarrow B$ valid in $M' = \{\alpha(\{g\}) \mid g \in G\}$.

Example

	y_1	y_2	y_3	y_4
x_1	X	X	X	X
x_2	X		X	X
x_3		X	X	X
x_4		X	X	X
x_5	X		X	

$\{y_1\} \Rightarrow \{y_3\}$ valid
$\{y_2, y_3\} \Rightarrow \{y_4\}$ valid
$\{y_1, y_3\} \Rightarrow \{y_4\}$ not valid

Reasoning with attribute implications

- Theories
- Models
- Inference
- Definition of non-redundant bases of implications
- Guigues-Duquenne basis:
  $T = \{A \Rightarrow \alpha(\beta(A)) \mid A \text{ pseudo-intent of } \mathbb{K}\}$
  Pseudo-intent: $A \subseteq M$ such that
  - $A \neq \alpha(\beta(A))$
  - $\alpha(\beta(B)) \subseteq A$ for each pseudo-intent $B \subset A$

Extensions

- Many-valued context: (G, M, W, I), $I \subseteq G \times M \times W$ (W = set of values for attributes).
- Fuzzy context: $I : G \times M \to [0, 1]$, $I(g, m)$ = degree to which object g satisfies property m.
- Links with possibility theory.
- ...

A few applications

- Classification and clustering.
- Recognition.
- Reinforcement learning (states / actions).
- Information retrieval, knowledge extraction.
- Social networks
- Spatial reasoning.
- Completing knowledge bases in description logics.
- Inference, abduction...
- ...

Several softwares available – See e.g.
`http://www.upriss.org.uk/fca/fcasoftware.html`

A few references

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