# Spatial Reasoning and model-based image understanding

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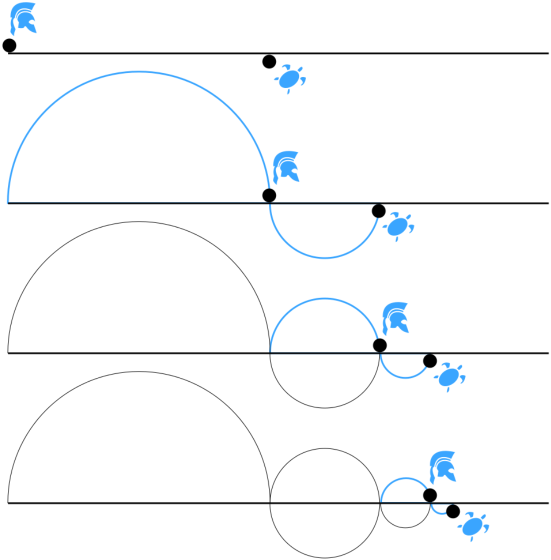
# Spatial Reasoning

Knowledge representation and reasoning on spatial entities and spatial relationships

- largely developed in the artificial intelligence community
  - mainly topological relations
  - formal logics (ex: mereotopology)
  - inference
- less developed in image interpretation
  - need for imprecise knowledge representation
  - (semi-)quantitative framework ($\Rightarrow$ numerical evaluation)
  - examples: structural recognition in images under imprecision
- main ingredients:
  - knowledge representation (including spatial relations)
  - imprecision representation and management
  - fusion of heterogeneous information
  - reasoning and decision making

# Philosophy

- From Pythagoras (c. 570-495 BC) to Zeno (c. 490-430 BC): concept of space linked to the first developments in arithmetics and Pythagorian geometry - Problem of infinitely subdivision possibility.
- Descartes (1596-1650): spatial extension is specific to material entities, governed by the only laws of mechanics.
- Newton (1643-1727): notion of absolute space.
- Hume (1711-1776): space reduced to a pure psychological function.
- Leibniz (1646-1716): space cannot be an absolute reality, motion and position are real and detectable only in relation to other objects, not in relation to space itself.
- Kant (1724-1804): objectivity of space.



- Poincaré (1854-1912): empiricist point of view where spatial knowledge is mainly derived from motor experience. Relativity of space.
- Bergson (1859-1941): a position in the space can be considered as an instantaneous cut of the movement, but the movement is more that a sum of positions in the space.
- Einstein (1879-1955): geometry is linked to the sensible and perceptible space. The geometrical configuration of the world itself becomes relative.
- Purely philosophical views of space developed by the phenomenologists and the existentialists.
- Reichenbach (1891-1953): geometry as a theory of relations.

# Linguistics

- Rich variety of lexical terms for describing spatial location of entities.
- Concern all lexical categories (nouns, verbs, adjectives, adverbs, prepositions).
  - French, and other Romance languages, shows a typological preference for the lexicalization of the path in the main verb.
  - In Germanic and Slavic languages, the path is rather encoded in satellites associated to the verb (particle or prefix).
- Source of inspiration of many works on qualitative spatial information representation and qualitative spatial reasoning.
- Asymmetry, importance of reference, of context, of functional properties of the considered physical entities
- Imprecision (too precise statements can even become inefficient if they make the message too complex).

# Human perception: example of distance

- Purely spatial measures, in a geometric sense, give rise to "metric distances", and are related to intrinsic properties of the objects.
- Temporal measures lead to distances expressed as travel time, and can be considered of extrinsic type, as opposed to the previous class.
- Economic measures, in terms of costs to be invested, are also of extrinsic type.
- Perceptual measures lead to distances of deictic type; they are related to an external point of view, which can be concrete or just a mental representation, which can be influenced by environmental features, by subjective considerations, leading to distances that are not necessarily symmetrical.
- Influence of other objects.

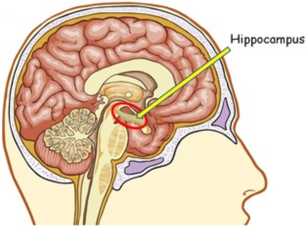
# Cognition

Cognitive understanding of a spatial environment is issued from two types of processes:

- route knowledge acquisition (first acquired during child development), which consists in learning from sensori-motor experience (i.e. actual navigation) and implies an order information between visited landmarks,
- survey knowledge acquisition, from symbolic sources such as maps, leading to a global view ("from above") including global features and relationships, which is independent of the order of landmarks.

Neuro-imaging:

- a right hippocampal activation can be observed for both mental navigation and mental map tasks,
- a parahippocampal gyrus activation is additionally observed only for mental navigation, when route information and object landmarks have to be incorporated.




Internal representation of space in the brain:

- egocentric representations,
- allocentric representations ("map in the head").

Intensively used in several works in the modeling and conception of geographic information systems, and in mobile robotics.

# Spatial reasoning formalisms

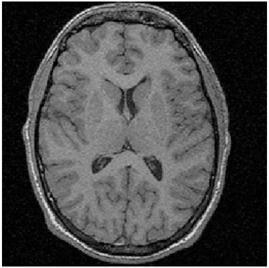
- Quantitative
- Qualitative (QSR)
- Fuzzy representations and reasoning: semi-quantitative / semi-qualitative approches

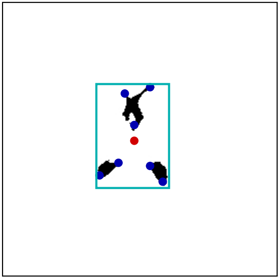
- Spatial entities
- Spatial relations
- Real world problems: dealing with imprecision and uncertainty.

Common to several representation and reasoning frameworks, used in the next parts of the course.

# Spatial entities

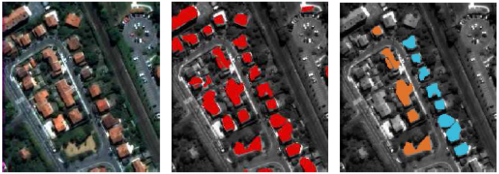
- Regions, fuzzy regions.
- Key points.
- Simplified regions (centroid, bounding box...).
- Abstract representations (e.g. in mereotopology, without referring to points, formulas in some logics...).





# Spatial relations

- Useful... (see e.g. Freeman 1975, Kuipers 1978...).
- Structural stability (more than shape, size, absolute position).
- Different types (binary / n-ary, simple / complex, well-defined / vague).

# Quantitative representations

- Precisely defined objects.
- Computation of well defined relations.
- Many limitations:
  - on the objects,
  - on the relations,
  - on the type of representations,
  - for reasoning.

But does not always match the usual way of reasoning (e.g. to the north, closer...).

# Qualitative / symbolic representations

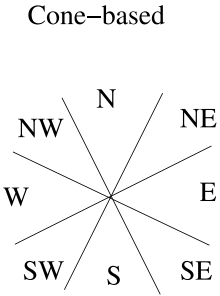
- Cardinal directions: 9 positions.
- Allen's intervals (temporal reasoning): 13 relations.
- Rectangle calculus (Allen on each axis): 169 relations.
- Cube calculus...
- Region Connection Calculus (RCC), mereotopology (based on connection and parthood predicates).
- Extensions to objects with broad or imprecise boundaries.
- Spatial logics.

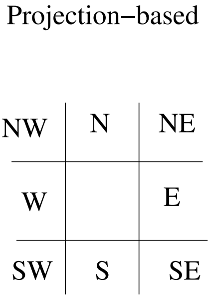
Main features:

- Formal logics (propositional, first order, modal...).
- Compromise between expressiveness, completeness with respect to a class of situations, and complexity.
- Reasoning: inference, satisfiability, composition tables, CSP...

Cardinal directions (Frank, Egenhofer, Ligozat)

Qualitative directions: N, NE, E, SE, S, SW, W, NW



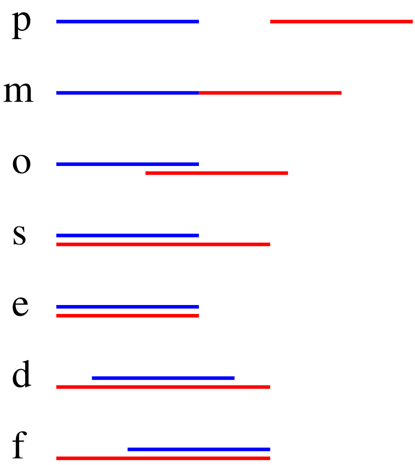




How to deal with complex shapes?

Only few compositions can be exactly determined.

## Allen's intervals

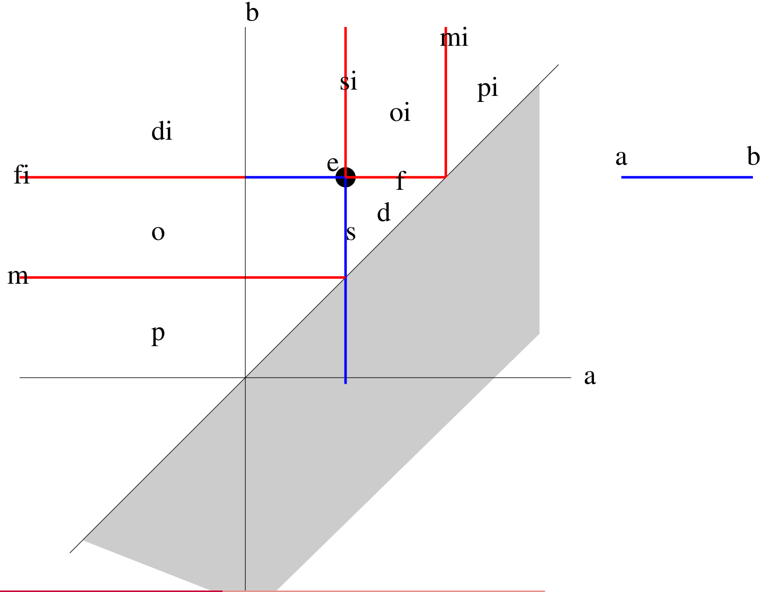
13 basic relations:




Reasoning: based on geometrical or latticial representations.

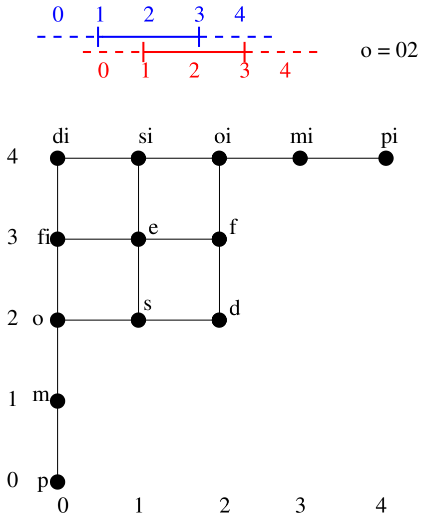
## Allen's intervals

Geometrical / quantitative representation:



## Allen's intervals

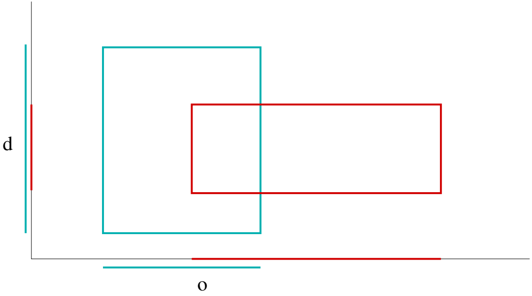
Qualitative representation: lattice:



## Allen's intervals

Extensions: rectangle, cube algebra

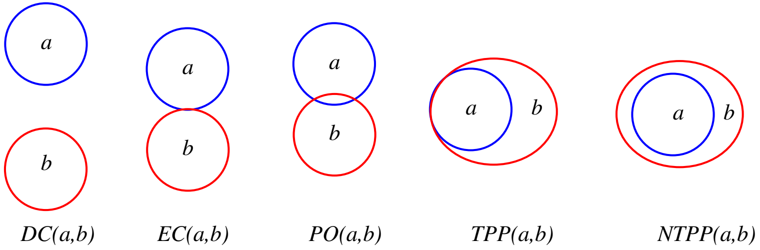
- Allen's interval in each direction
- 2D (rectangles): $13^2 = 169$ relations
- 3D (cubes): $13^3 = 2197$ relations
- $\Rightarrow$ high complexity, and fixed shaped objects



## RCC: Region Connection Calculus (Randell, Cui, Cohn - Vieu...)

- Spatial entities, defined in a qualitative way.
- No reference to points.
- Connection predicate $C$.
- Parthood predicate $P$:

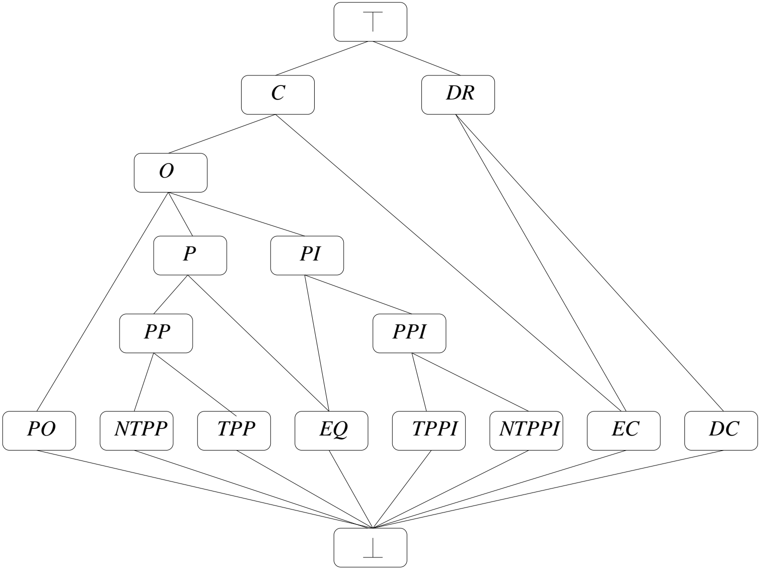
$$P(x, y) : \forall z, C(z, x) \rightarrow C(z, y)$$



## RCC: Region Connection Calculus (Randell, Cui, Cohn - Vieu...)

| | | |
|---|---|---|
| $DC(x,y)$ | $x$ is disconnected from $y$ | $\neg C(x,y)$ |
| $P(x,y)$ | $x$ is a part of $y$ | $\forall z, C(z,x) \rightarrow C(z,y)$ |
| $PP(x,y)$ | $x$ is a proper part of $y$ | $P(x,y) \wedge \neg P(y,x)$ |
| $EQ(x,y)$ | x is identical with y | $P(x,y) \wedge P(y,x)$ |
| $O(x,y)$ | $x$ overlaps $y$ | $\exists z, P(z,x) \wedge P(z,y)$ |
| $DR(x,y)$ | $x$ is discrete from $y$ | $\neg O(x,y)$ |
| $PO(x,y)$ | $x$ partially overlaps $y$ | $O(x,y) \wedge \neg P(x,y) \wedge \neg P(y,x)$ |
| $EC(x,y)$ | $x$ is externally connected to $y$ | $C(x,y) \wedge \neg O(x,y)$ |
| $TPP(x,y)$ | $x$ is a tangential proper part of $y$ | $PP(x,y) \wedge \exists z[EC(z,x) \wedge EC(z,y)$ |
| $NTPP(x,y)$ | $x$ is a non tangential proper part of $y$ | $PP(x,y) \wedge \neg\exists z[EC(z,x) \wedge EC(z,y)]$ |

## RCC: Region Connection Calculus (Randell, Cui, Cohn - Vieu...)

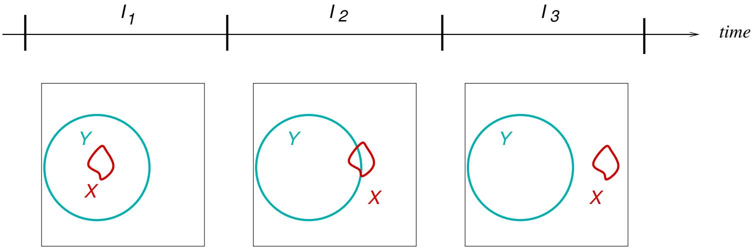


## Qualitative trajectory calculus (Cohn et al.)

- Extension of RCC to take time into account (dynamic scenes).
- RCC + Allen
- Example:
  - $X, Y$ objects
  - $I_i$ time intervals

$$(P(X,Y), I_1) \wedge (PO(X,Y), I_2) \wedge (DR(X,Y), I_3)$$

$$\wedge meet(I_1, I_2) \wedge meet(I_2, I_3) \wedge before(I_1, I_3)$$



# Modal logics of space

Topology:

- $\Box A$: $A$ is locally true ($A$ is true at point $x$ iff $A$ is true in a neighborhood of $x$).
- $\Diamond A = \neg\Box\neg A$: $A$ is true at $x$ iff $A$ is true at least one point of the neighborhood of $x$.
- Reasoning axioms and inference rules of S4:
  - $A \rightarrow (B \rightarrow A)$
  - $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
  - $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$
  - $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
  - $\Box A \rightarrow A$
  - $\Box A \rightarrow \Box\Box A$

Other examples:

- Translation of RCC into modal logics.
- Logics of places ($\Box$ = everywhere, $\Diamond$ = somewhere).
- Modal logics of proximity ($\Box A$ = everywhere close to $A$).
- Modal logics of distance ($\Box^{\leq a}$ = everywhere in a neighborhood of radius $a$).
- Logics of inclusion and contact (inference in GIS).
- Modal logics of geometry (affine, projective, parallelism...).

# A few important issues

- Context
- Representation issues
- Reasoning (inference, satisfiability, decidability, CSP...)
- Complexity
- Applications

State of the art:

- Very few applications
- Focus on topology
- Almost nothing on metric relations
- Almost nothing on uncertainty

Allen intervals:

| . | p | m | o | F | D | s | e | S | d | f | O | M | P |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **p** | (p) | (p) | (p) | (p) | (p) | (p) | (p) | (p) | (pmosd) | (pmosd) | (pmosd) | (pmosd) | *full* |
| **m** | (p) | (p) | (p) | (p) | (p) | (m) | (m) | (m) | (osd) | (osd) | (osd) | (Fef) | (DSOMP) |
| **o** | (p) | (p) | (pmo) | (pmo) | (pmoFD) | (o) | (o) | (oFD) | (osd) | (osd) | *concur* | (DSO) | (DSOMP) |
| **F** | (p) | (m) | (o) | (F) | (D) | (o) | (F) | (D) | (osd) | (Fef) | (DSO) | (DSO) | (DSOMP) |
| **D** | (pmoFD) | (oFD) | (oFD) | (D) | (D) | (oFD) | (D) | (D) | *concur* | (DSO) | (DSO) | (DSO) | (DSOMP) |
| **s** | (p) | (p) | (pmo) | (pmo) | (pmoFD) | (s) | (s) | (seS) | (d) | (d) | (dfO) | (M) | (P) |
| **e** | (p) | (m) | (o) | (F) | (D) | (s) | (e) | (S) | (d) | (f) | (O) | (M) | (P) |
| **S** | (pmoFD) | (oFD) | (oFD) | (D) | (D) | (seS) | (S) | (S) | (dfO) | (O) | (O) | (M) | (P) |
| **d** | (p) | (p) | (pmosd) | (pmosd) | *full* | (d) | (d) | (dfOMP) | (d) | (d) | (dfOMP) | (P) | (P) |
| **f** | (p) | (m) | (osd) | (Fef) | (DSOMP) | (d) | (f) | (OMP) | (d) | (f) | (OMP) | (P) | (P) |
| **O** | (pmoFD) | (oFD) | *concur* | (DSO) | (DSOMP) | (dfO) | (O) | (OMP) | (dfO) | (O) | (OMP) | (P) | (P) |
| **M** | (pmoFD) | (seS) | (dfO) | (M) | (P) | (dfO) | (M) | (P) | (dfO) | (M) | (P) | (P) | (P) |
| **P** | *full* | (dfOMP) | (dfOMP) | (P) | (P) | (dfOMP) | (P) | (P) | (dfOMP) | (P) | (P) | (P) | (P) |

full=(pmoFDseSdfOMP) and concur=(oFDseSdfO)

From http://www.ics.uci.edu/~alspaugh/cls/shr/allen.html

# Example: composition tables

RCC-8 :

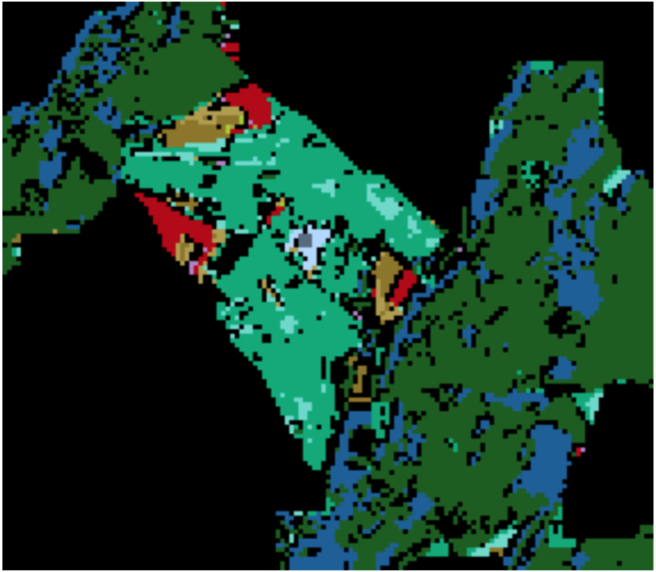
| o | DC | EC | PO | TPP | NTPP | TPPi | NTPPi | EQ |
|---|---|---|---|---|---|---|---|---|
| DC | * | DC,EC,PO,TPP,NTPP | DC,EC,PO,TPP,NTPP | DC,EC,PO,TPP,NTPP | DC,EC,PO,TPP,NTPP | DC | DC | DC |
| EC | DC,EC,PO,TPPi,NTPPi | DC,EC,PO,TPP,TPPi,EQ | DC,EC,PO,TPP,NTPP | EC,PO,TPP,NTPP | PO,TPP,NTPP | DC,EC | DC | EC |
| PO | DC,EC,PO,TPPi,NTPPi | DC,EC,PO,TPPi,NTPPi | * | PO,TPP,NTPP | PO,TPP,NTPP | DC,EC,PO,TPPi,NTPPi | DC,EC,PO,TPPi,NTPPi | PO |
| TPP | DC | DC,EC | DC,EC,PO,TPP,NTPP | TPP,NTPP | NTPP | DC,EC,PO,TPP,TPPi,EQ | DC,EC,PO,TPPi,NTPPi | TPP |
| NTPP | DC | DC | DC,EC,PO,TPP,NTPP | NTPP | NTPP | DC,EC,PO,TPP,NTPP | * | NTPP |
| TPPi | DC,EC,PO,TPPi,NTPPi | EC,PO,TPPi,NTPPi | PO,TPPi,NTPPi | PO,TPP,TPPi,EQ | PO,TPP,NTPP | TPPi,NTPPi | NTPPi | TPPi |
| NTPPi | DC,EC,PO,TPPi,NTPPi | PO,TPPi,NTPPi | PO,TPPi,NTPPi | PO,TPPi,NTPPi | PO,TPP,NTPP,TPPi,NTPPi,EQ | NTPPi | NTPPi | NTPPi |
| EQ | DC | EC | PO | TPP | NTPP | TPPi | NTPPi | EQ |

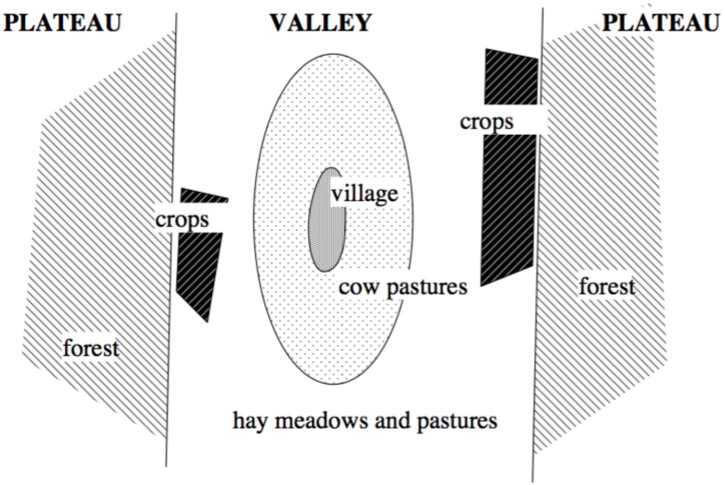
From wikipedia

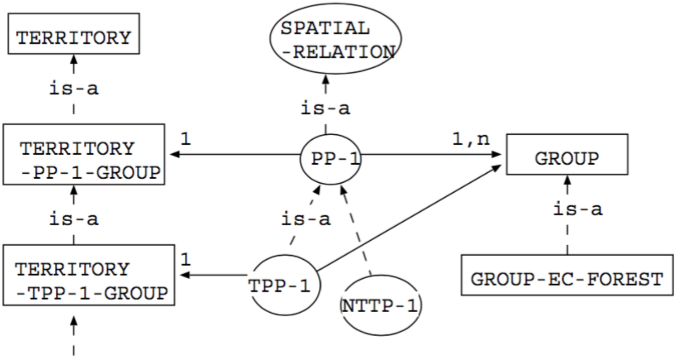
# Other approaches

- Ontologies and description logics.
- Graph-based reasoning.
- Grammars.
- Formal concept analysis.
- Decision trees.
- Constraint Satisfaction Problem.
- Relational algebras on temporal or spatial configurations.
- Graphical models.
- ...

## Example using RCC: region identification (Le Ber et al.)




PLATEAU
VALLEY
PLATEAU
crops
village
crops
cow pastures
forest
forest
hay meadows and pastures


TERRITORY
is-a
SPATIAL
-RELATION
is-a
TERRITORY
-PP-1-GROUP
1
PP-1
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GROUP
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is-a
is-a
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-TPP-1-GROUP
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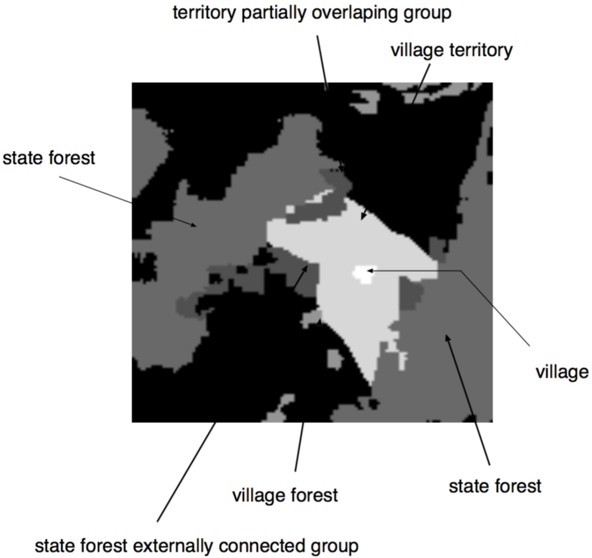
TPP-1
NTTP-1
GROUP-EC-FOREST




F-ILOT - F-ILOTLIM - F-ILOTECDOM - F-POCOM - F-ECDOM - F-ENTREDOM

# Semi-quantitative spatial reasoning: fuzzy approaches

- Limitations of purely qualitative reasoning
- Interest of adding semi-quantitative extension to qualitative value for deriving useful and practical conclusions
- Limitations of purely quantitative representations in the case of imprecise statements, knowledge expressed in linguistic terms, etc.
- Integration of both quantitative and qualitative knowledge using semi-quantitative (or semi-qualitative) interpretation of fuzzy sets
- Freeman (1975): fuzzy sets provide computational representation and interpretation of imprecise spatial constraints, expressed in a linguistic way, possibly including quantitative knowledge
- Granularity, involved in:
  - objects or spatial entities and their descriptions
  - types and expressions of spatial relations and queries
  - type of expected or potential result

# Motivation: model-based recognition and spatial reasoning

- representation of imprecision
- spatial relations as structural information
  - topological relationships (set relations, adjacency)
  - distances
  - relative directional relationships
  - more complex relations (between, along...)
- two classes of relations
  - well defined in the crisp case (adjacency, distances...)
  - vague even in the crisp case (directional relationships...)
- fusion of several and heterogeneous pieces of knowledge and information

$\Rightarrow$ Fuzzy set theory, mathematical morphology

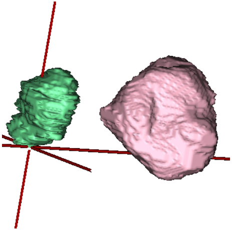
# Imprecision and fuzziness

- objects (no clear boundaries, coarse segmentation...)
- relations (ex: *left of, quite close*)
- type of knowledge available (ex: *the caudate nucleus is close to the lateral ventricle*)
- question to be answered (ex: *go towards this object while remaining at some security distance*)

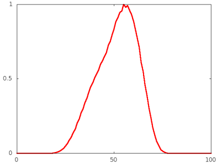
# Types of representations: example of distances

- number in $\mathbb{R}^+$ (or in $[0, 1]$)
- interval
- fuzzy number, fuzzy interval
- Rosenfeld:
  - distance density: degree to which the distance is equal to $n$
  - distance distribution: degree to which the distance is less than $n$
- linguistic value
- logical formula

$\Rightarrow$ unifying framework of fuzzy set theory

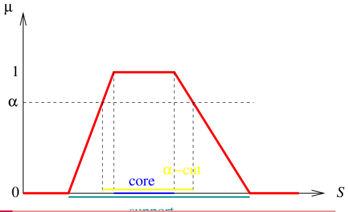


$d_{min} = 17, d_{Haus} = 80$



# Definitions: fuzzy sets

- Space $\mathcal{S}$ (image space, space of characteristics, etc.)
- Fuzzy set: $\mu : \mathcal{S} \rightarrow [0,1]$ – $\mu(x)$ = membership degree of $x$ to $\mu$
- Support: $Supp(\mu) = \{x \in \mathcal{S}, \mu(x) > 0\}$ – Core / kernel: $\{x \in \mathcal{S}, \mu(x) = 1\}$
- $\alpha$-cut: $\mu_\alpha = \{x \in \mathcal{S}, \mu(x) \geq \alpha\}$
- Cardinality: $|\mu| = \sum_{x \in \mathcal{S}} \mu(x)$ (for $\mathcal{S}$ finite)
- Convexity: $\forall (x,y) \in \mathcal{S}^2, \forall \lambda \in [0,1], \mu[\lambda x + (1-\lambda)y] \geq \min[\mu(x), \mu(y)]$
- Fuzzy number: convex fuzzy set on $\mathbb{R}$, u.s.c., unimodal, with compact support. Example: LR-fuzzy sets.



# Basic operations (Zadeh, 1965)

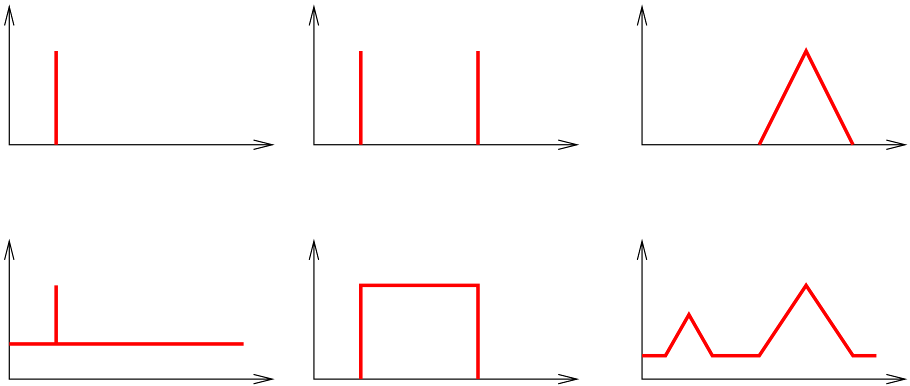
- Equality: $\mu = \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) = \nu(x)$
- Inclusion: $\mu \subseteq \nu \Leftrightarrow \forall x \in \mathcal{S}, \mu(x) \leq \nu(x)$
- Intersection: $\forall x \in \mathcal{S}, (\mu \cap \nu)(x) = \min[\mu(x), \nu(x)]$
- Union: $\forall x \in \mathcal{S}, (\mu \cup \nu)(x) = \max[\mu(x), \nu(x)]$
- Complementation: $\forall x \in \mathcal{S}, \mu^C(x) = 1 - \mu(x)$
- Properties:
  - consistency with binary set operations
  - $\mu = \nu \Leftrightarrow \mu \subseteq \nu$ and $\nu \subseteq \mu$
  - fuzzy complementation is involutive: $(\mu^C)^C = \mu$
  - intersection and union are commutative and associative
  - intersection and union are idempotent and mutually distributive
  - intersection and union are dual with respect to the complementation: $(\mu \cap \nu)^C = \mu^C \cup \nu^C$
  - $(\mu \cup \nu)_\alpha = \mu_\alpha \cup \nu_\alpha$, etc.

BUT: $\mu \cap \mu^C \neq \emptyset$, $\mu \cup \mu^C \neq \mathcal{S}$

# Semantics

- degree of similarity (notion of distance)
- degree of plausibility (that an object from which only an imprecise description is known is actually the one one wants to identify)
- degree of preference (fuzzy class = set of "good" choices), close to the notion of utility function

# Representing different types of imperfection



Fuzzy complementation

function $c$ from $[0,1]$ into $[0,1]$ such that:

1. $c(0) = 1$
2. $c(1) = 0$
3. $c$ is involutive, i.e. $\forall x \in [0,1], c(c(x)) = x$
4. $c$ is strictly decreasing

General form of continuous complementations: $c(x) = \varphi^{-1}[1 - \varphi(x)]$ with $\varphi : [0,1] \rightarrow [0,1]$, $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi$ strictly increasing.

Example: $\varphi(x) = x^n \Rightarrow c(x) = (1 - x^n)^{1/n}$

Triangular norms (fuzzy intersection)
t-norm $t : [0,1] \times [0,1] \to [0,1]$ such that:

1. commutativity, i.e. $\forall (x,y) \in [0,1]^2,\ t(x,y) = t(y,x)$;
2. associativity, i.e. $\forall (x,y,z) \in [0,1]^3,\ t[t(x,y),z] = t[x,t(y,z)]$;
3. 1 is unit element, i.e. $\forall x \in [0,1],\ t(x,1) = t(1,x) = x$;
4. increasingness with respect to the two variables:

$$\forall (x,x',y,y') \in [0,1]^4,\ (x \leq x' \text{ and } y \leq y') \Rightarrow t(x,y) \leq t(x',y').$$

Moreover: $t(0,1) = t(0,0) = t(1,0) = 0$, $t(1,1) = 1$, and 0 is null element $(\forall x \in [0,1], t(x,0) = 0)$.

Examples of t-norms: $\min(x,y)$, $xy$, $\max(0, x+y-1)$.

Triangular conorms (fuzzy union) t-conorm $T : [0,1] \times [0,1] \rightarrow [0,1]$ such that:

1. commutativity, i.e. $\forall (x,y) \in [0,1]^2,\ T(x,y) = T(y,x)$;
2. associativity, i.e. $\forall (x,y,z) \in [0,1]^3,\ T[T(x,y),z] = T[x, T(y,z)]$;
3. 0 is unit element, i.e. $\forall x \in [0,1],\ T(x,0) = T(0,x) = x$;
4. increasingness with respect to the two variables

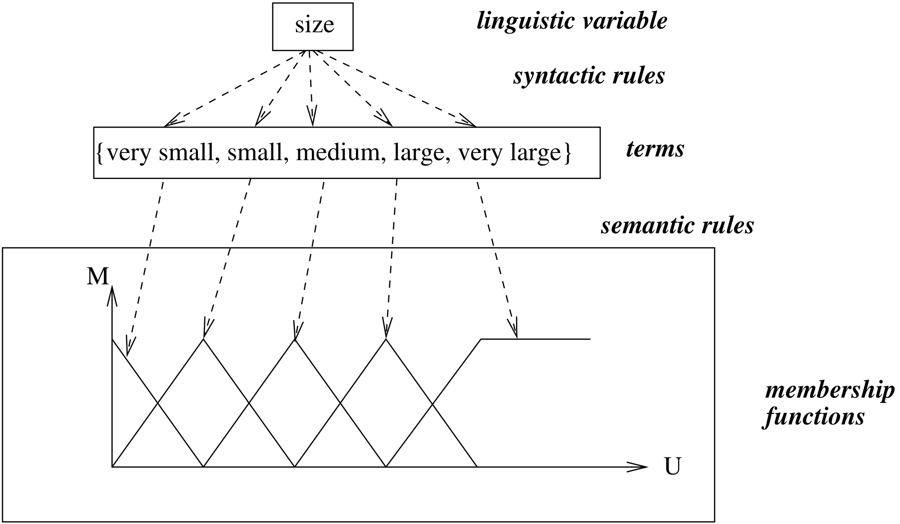
Moreover: $T(0,1) = T(1,1) = T(1,0) = 1,\ T(0,0) = 0$, and 1 is null element ($\forall x \in [0,1], T(x,1) = 1$).

Examples of t-conorms: $\max(x,y)$, $x + y - xy$, $\min(1, x+y)$.

Duality: $\forall (x,y) \in [0,1]^2,\ T[c(x), c(y)] = c[t(x,y)]$

Other combination operators (mean, symmetrical sums, etc.) $\Rightarrow$ information fusion

# Linguistic variable



# Imprecise reasoning

- Difference between data and knowledge
- Classical logic:
  - language
  - semantics (interpretations, truth values)
  - syntax (axioms and inference rules)
- Human reasoning: flexible, allows for imprecise statements
- Gradual predicates:
  - continuous referential
  - typicality

# Uncertainty

= unable to say whether a proposition is true or not

- because information is incomplete, vague, imprecise
  $\Rightarrow$ possibility
- because information is contradicting or fluctuating
  $\Rightarrow$ probability

certainty degree $\neq$ truth degree

| | |
|---|---|
| "It is probable that he is far from his goal" | "He is very far from his goal" |

- Fuzzy logic: propositions with truth degrees
- Possibilistic logic: propositions with (un)certainty degrees

# Fuzzy logic

- Basic fuzzy propositions: $X$ *is* $P$
  $X =$ variable taking values in $\mathcal{U}$
  $P =$ fuzzy subset of $\mathcal{U}$
  Truth degrees in $[0,1]$ defined from $\mu_P$
- Conjunction: $X$ *is* $A$ *and* $Y$ *is* $B$ $\quad \mu_{A \wedge B}(x,y) = t[\mu_A(x), \mu_B(y)]$
- Disjunction: $X$ *is* $A$ *or* $Y$ *is* $B$ $\quad \mu_{A \vee B}(x,y) = T[\mu_A(x), \mu_B(y)]$
- Negation: $\mu_{\neg A}(x) = c[\mu_A(x)]$
- Variables taking values in a product space: $X$ with values in $\mathcal{U}$, $Y$ with values $\mathcal{V} \Rightarrow$ conjunction $=$ cartesian product

$$X \textit{ is } A \textit{ and } Y \textit{ is } B \qquad \mu_{A \times B}(x,y) = t[\mu_A(x), \mu_B(y)]$$

# Fuzzy implications

- Classical logic: $(A \Rightarrow B) \Leftrightarrow \quad (B \text{ or } notA)$
- Fuzzy logic:
  - $A$ and $B$ crisp:

$$Imp(A, B) = T[c(A), B]$$

  - $A$ and $B$ fuzzy:

$$Imp(A, B) = \inf_x T[c(\mu_A(x)), \mu_B(x)]$$

  - Examples $(c(a) = 1 - a)$ :

| | | |
|---|---|---|
| $T(a, b) = \max(a, b)$ | $I(a, b) = \max(1 - a, b)$ | Kleene-Diene |
| $T(a, b) = \min(1, a + b)$ | $I(a, b) = \min(1, 1 - a + b)$ | Lukasiewicz |
| $T(a, b) = a + b - ab$ | $I(a, b) = 1 - a + ab$ | Reichenbach |

- Residual implications from a t-norm:

$$I(A, B) = \sup\{X \mid t(X, A) \leq B\}$$

Adjunction: $t(X, A) \leq B \Leftrightarrow X \leq I(A, B)$

# Fuzzy reasoning

- Classical logic
  - Modus ponens: $(A \wedge (A \Rightarrow B)) \Rightarrow B$
  - Modus tollens: $((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$
  - Syllogism: $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
  - Cuntraposition: $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
- Fuzzy modus ponens
  - Rule :

$$\textit{if} \quad X \textit{ is } A \quad \textit{then} \quad Y \textit{ is } B$$

  - Knowledge or observation:

$$X \textit{ is } A'$$

  - Conclusion:

$$Y \textit{ is } B'$$

$$\mu_{B'}(y) = \sup_x t[\mu_{A \Rightarrow B}(x,y), \mu_{A'}(x)]$$

- Other reasoning modes: similar extensions.

# Fuzzy rules

$$\text{IF } (x \textit{ is } A \text{ AND } y \textit{ is } B) \text{ THEN } z \textit{ is } C$$

$$\text{IF } (x \textit{ is } A \text{ OR } y \textit{ is } B) \text{ THEN } z \textit{ is } C$$

...

$\alpha$: truth degree of $x$ *is* $A$
$\beta$: truth degree of $y$ *is* $B$
$\gamma$: truth degree of $z$ *is* $C$

Satisfaction degree of the rule:

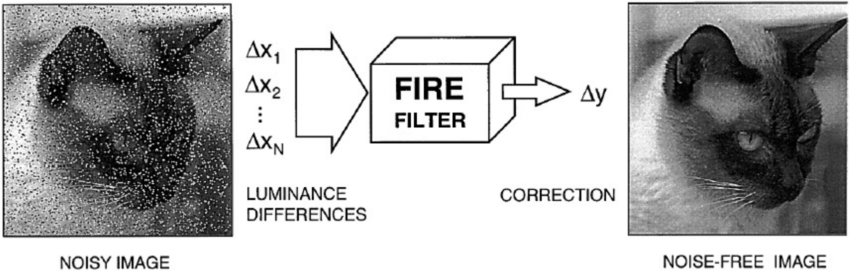
$$Imp(t(\alpha, \beta), \gamma) = T[c(t(\alpha, \beta)), \gamma)]$$

$$Imp(T(\alpha, \beta), \gamma) = T[c(T(\alpha, \beta)), \gamma)]$$

...

# Example in image filtering

| | |
|---|---|
| IF | a pixel is **darker** than its neighbors |
| THEN | **increase** its grey level |
| ELSE IF | a pixel is **brighter** than its neighbors |
| THEN | **decrease** its grey level |
| ELSE | **unchanged** |



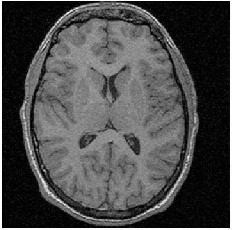

F. Russo et al.

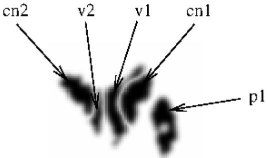
# Spatial fuzzy objects

$\mathcal{S}$: $\mathbb{R}^3$ or $\mathbb{Z}^3$ in the digital case

$$\mu : \mathcal{S} \to [0,1]$$

$\mu(x)$ = degree to which $x$ belongs to the fuzzy object





# Definition of membership functions

- often based on heuristics and ad hoc procedures
- from intensity function $I$ or gradient

$$\mu(x) = F_1[(I(x)]$$

$$\mu(x) = F_2[(\nabla I(x)]$$

- from the output values of some detector
- by introducing imprecision at the boundary of a crisp detection

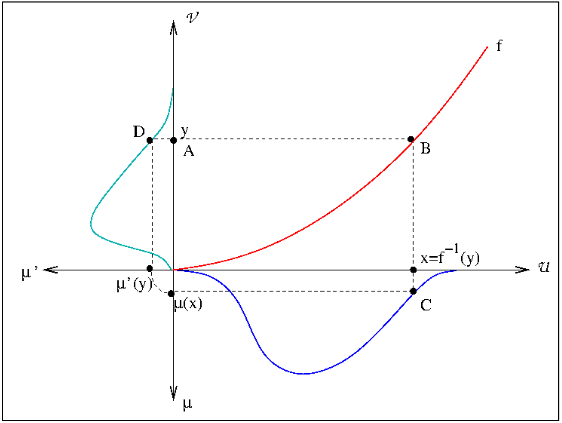
$$\mu(x) = \begin{cases} 1 & \text{if } x \in E^n(O) \\ 0 & \text{if } x \in \mathcal{S} - D^m(O) \\ F_3[d(x, E^n(O))] & \text{otherwise} \end{cases}$$

- from classification algorithms

# How can an operation be extended to the fuzzy case?

Extension principle: $f$ from $\mathcal{U}$ into $\mathcal{V}$

$$\forall y \in \mathcal{V},\ \mu'(y) = \begin{cases} 0 & \text{if } f^{-1}(y) = \emptyset, \\ \sup_{x \in \mathcal{U} | y = f(x)} \mu(x) & \text{otherwise} \end{cases}$$



# How can an operation be extended to the fuzzy case?

Using $\alpha$-cuts :

$$R(\mu) = \int_0^1 R_B(\mu_\alpha) d\alpha$$

$$R(\mu) = \sup_{\alpha \in [0,1]} \min(\alpha, R_B(\mu_\alpha))$$

$$R(\mu) = \sup_{\alpha \in [0,1]} (\alpha R_B(\mu_\alpha))$$

$$\dots$$

Extension principle based on $\alpha$-cuts:

$$\forall n, R(\mu, \nu)(n) = \sup_{R_B(\mu_\alpha, \nu_\alpha) = n} \alpha$$

# How can an operation be extended to the fuzzy case?

Formal translation:

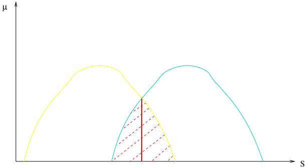
| set $X$ | fuzzy set $\mu$ |
|---|---|
| complementation $X^C$ | fuzzy complementation $c(\mu)$ |
| intersection $\cap$ | t-norm $t$ |
| union $\cup$ | t-conorm $T$ |
| $\exists$ | sup |
| $\forall$ | inf |

$\Rightarrow$ easy translation of algebraic and logical expressions

# Set relationships

Fuzzy sets $\Rightarrow$ relations become a matter of degree

- Degree of intersection: $\mu_{int}(\mu,\nu) = \sup_{x \in \mathcal{S}} t[\mu(x), \nu(x)]$




or: $\mu_{int}(\mu,\nu) = \frac{V_n[t(\mu,\nu)]}{\min[V_n(\mu), V_n(\nu)]}$

- Degree of inclusion:

$$\inf_{x \in \mathcal{S}} T[c(\nu(x)), \mu(x)]$$

# Mathematical morphology

Dilation: operation in complete lattices that commutes with the supremum.
Erosion: operation in complete lattices that commutes with the infimum.

$\Rightarrow$ applications on sets, fuzzy sets, functions, logical formulas, graphs, etc.

Using a structuring element:

- dilation as a degree of conjunction: $\delta_B(X) = \{x \in \mathcal{S} \mid B_x \cap X \neq \emptyset\}$,
- erosion as a degree of implication: $\varepsilon_B(X) = \{x \in \mathcal{S} \mid B_x \subseteq X\}$.

A lot of other operations...

# Fuzzy mathematical morphology

- Dilation as degree of intersection:

$$D_\nu(\mu)(x) = \sup\{t[\nu(y-x), \mu(y)],\ y \in \mathcal{S}\}$$

- Erosion as degree of inclusion:

$$E_\nu(\mu)(x) = \inf\{I[\nu(y-x), \mu(y)],\ y \in \mathcal{S}\}$$

  $I$ from a t-conorm $T$ or by residuation from the t-norm $t$
- Opening and closing by composition
- Similar properties as in classical mathematical morphology

# Fuzzy spatial relations

Fuzzy sets $\rightarrow$ relations become a matter of degree

- Set theoretical relations
- Topology: connectivity, connected components, neighborhood, boundaries, adjacency
- Distances
- Relative direction
- More complex relations: between, along, parallel, around...

Most of them can be defined from mathematical morphology.

# Distances between fuzzy sets

## Comparison between membership functions

- functional approach: distance from a $L_p$ norm

$$d_p(\mu, \nu) = [\sum_{x \in \mathcal{S}} |\mu(x) - \nu(x)|^p]^{1/p}$$

$$d_\infty(\mu, \nu) = \max_{x \in \mathcal{S}} |\mu(x) - \nu(x)|$$

- set theoretical approach

$$d(\mu, \nu) = 1 - \frac{\sum_{x \in \mathcal{S}} \min[\mu(x), \nu(x)]}{\sum_{x \in \mathcal{S}} \max[\mu(x), \nu(x)]}$$

- ...
- adapted to cases where the fuzzy sets to be compared represent the same structure or a structure and a model of it
  - model-based object recognition
  - case-based reasoning

# Distances between fuzzy sets

## Taking the spatial distance $d_E$ into account

- geometrical approach
  - space of dimension $n+1$
  - fuzzification: $d(\mu,\nu) = \int_0^1 D(\mu_\alpha, \nu_\alpha) d\alpha$
  - weighting

$$d(\mu,\nu) = \frac{\sum_{x\in\mathcal{S}} \sum_{y\in\mathcal{S}} d_E(x,y) \min[\mu(x),\nu(y)]}{\sum_{x\in\mathcal{S}} \sum_{y\in\mathcal{S}} \min[\mu(x),\nu(y)]}$$

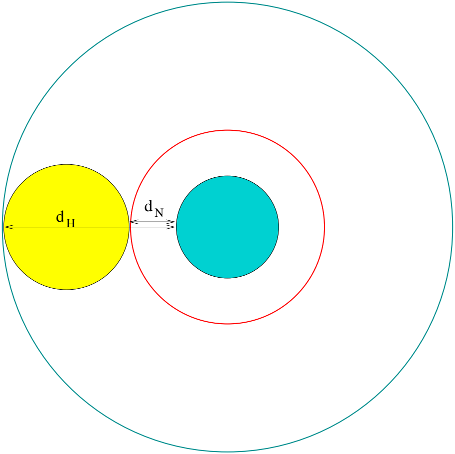
  - fuzzy number

$$d(\mu,\nu)(r) = \sup_{x,y,d_E(x,y)\leq r} \min[\mu(x),\nu(y)]$$

- morphological approach

# Distances between fuzzy sets: morphological approach

Expression of distances (minimum, Hausdorff...) in morphological (i.e. algebraic) terms ⇒ easy translation to the fuzzy case



# Minimum (nearest point) distance distribution

$$d_N(X, Y) = \inf\{n \in \mathbb{N}, X \cap D^n(Y) \neq \emptyset\} = \inf\{n \in \mathbb{N}, Y \cap D^n(X) \neq \emptyset\}$$

Degree to which the distance between $\mu$ and $\mu'$ is less than $n$ (distance distribution):

$$\Delta_N(\mu, \mu')(n) = f[\sup_{x \in \mathcal{S}} t[\mu(x), D^n_\nu(\mu')(x)], \sup_{x \in \mathcal{S}} t[\mu'(x), D^n_\nu(\mu)(x)]]$$

Hausdorff distance: similar equations

# Minimum (nearest point) distance density

$$d_N(X,Y) = n \Leftrightarrow D^n(X) \cap Y \neq \emptyset \text{ and } D^{n-1}(X) \cap Y = \emptyset$$

$$d_N(X,Y) = 0 \Leftrightarrow X \cap Y \neq \emptyset$$

Degree to which the distance between $\mu$ and $\mu'$ is equal to $n$ (distance density):

$$\delta_N(\mu,\mu')(n) = t[\sup_{x \in \mathcal{S}} t[\mu'(x), D^n_\nu(\mu)(x)], c[\sup_{x \in \mathcal{S}} t[\mu'(x), D^{n-1}_\nu(\mu)(x)]]]$$

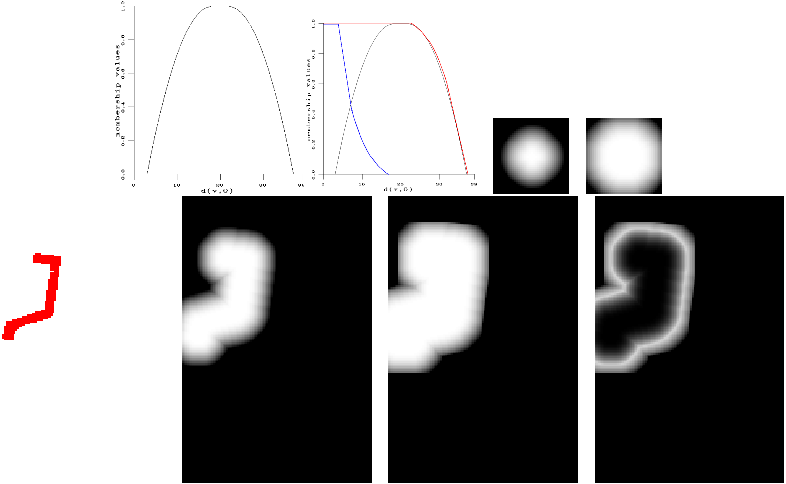
$$\delta_N(\mu,\mu')(0) = \sup_{x \in \mathcal{S}} t[\mu(x), \mu'(x)]$$

Hausdorff distance: similar equations

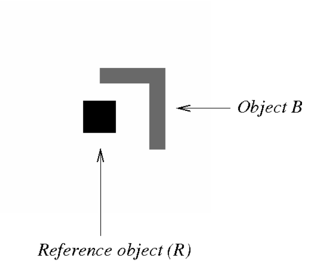
# Properties of fuzzy morphological distances

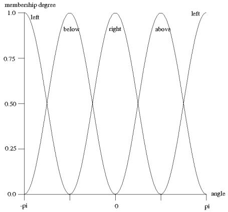
- fuzzy numbers
- positive: support included in $\mathbb{R}^+$
- symmetrical with respect to $\mu$ and $\mu'$
- if $\mu$ is normalized $\delta_N(\mu, \mu)(0) = 1$ and $\delta_N(\mu, \mu)(n) = 0$ for $n > 1$
- $\delta_H(\mu, \mu')(0) = 1$ implies $\mu = \mu'$ for $T$ being the bounded sum ($T(a, b) = \min(1, a + b)$), while it implies $\mu$ and $\mu'$ crisp and equal for $T = \max$
- triangular inequality not satisfied in general

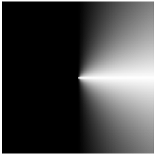
# Example: spatial representation of knowledge about distance



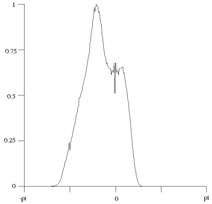
# Directional relations

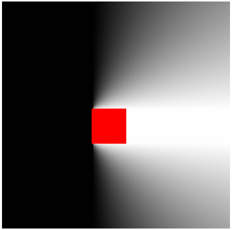


$\nu_{Right}$



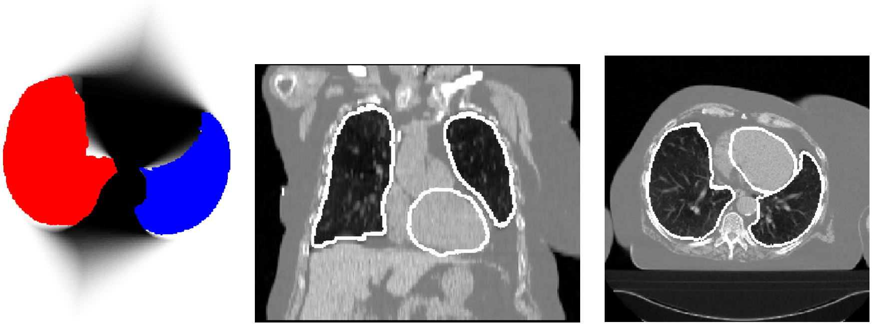



$$\mu_{Right}(R) = \delta_{\nu_{Right}}(R)$$

# Directional relative position: properties

- evaluation in the spatial domain, and with richer information (compared to other fuzzy methods)
- the possibility has a symmetry property
- invariance with respect to translation, rotation and scaling, for 2D and 3D objects (crisp and fuzzy)
- when the distance between the objects increases, the objects are seen as points
- nice behavior in case of concavities

Example: the heart is between the lungs



# Reasoning with mathematical morphology

- Chaining operations (image interpretation, recognition)
- Fusion of spatial relations (ex: structural recognition)
- Links with logics
  - propositional logics:
    - elegant tools for revision, fusion, abduction
    - links with mereotology, "egg-yolk" structures, logics of distances, nearness logics, linear logics, logics of convexity...
  - modal logics:
    - $(\Diamond, \Box) =$ (dilation, erosion)
    - symbolic and qualitative representations of spatial relations
  - fuzzy logic

# Example: dilation and erosion of a formula

Structuring element $B$: relation between worlds

Dilation:

$$Mod(D_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \cap Mod(\varphi) \neq \emptyset\}$$

Erosion:

$$Mod(E_B(\varphi)) = \{\omega \in \Omega \mid B(\omega) \models \varphi\}$$

# Dilation and erosion as modal operators

Structuring element $B$: accessibility relation $R(\omega, \omega')$ iff $\omega' \in B(\omega)$

$$
\begin{aligned}
\mathcal{M}, \omega \models \Box\varphi \quad & \Leftrightarrow \quad \forall \omega' \in \Omega,\ R(\omega, \omega') \Rightarrow \mathcal{M}, \omega' \models \varphi \\
& \Leftrightarrow \quad \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \models \varphi \\
& \Leftrightarrow \quad B(\omega) \models \varphi
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}, \omega \models \Diamond\varphi \quad & \Leftrightarrow \quad \exists \omega' \in \Omega,\ R(\omega, \omega') \text{ et } \mathcal{M}, \omega' \models \varphi \\
& \Leftrightarrow \quad \{\omega' \in \Omega \mid \omega' \in B(\omega)\} \cap Mod(\varphi) \neq \emptyset \\
& \Leftrightarrow \quad B(\omega) \cap Mod(\varphi) \neq \emptyset
\end{aligned}
$$

$$\boxed{\Box\varphi \equiv E_B(\varphi) \qquad \Diamond\varphi \equiv D_B(\varphi)}$$

Spatial interpretation: restriction or necessary region / extension or possible region

## Example: logical expressions and links with mereotology

- Spatial entities represented as formulas.
- Structuring element: binary relationship between worlds, accessibility relation...
- Adjacency: $\varphi \wedge \phi \rightarrow \bot$ and $\delta\varphi \wedge \psi \not\rightarrow \bot$ and $\varphi \wedge \delta\psi \not\rightarrow \bot$.
- Tangential part: $\varphi \rightarrow \psi$ and $\delta\varphi \wedge \neg\psi \not\rightarrow \bot$.
- Proper tangential part in mereotopology:
  $TPP(\varphi, \psi) = P(\varphi, \psi) \wedge \neg P(\psi, \varphi) \wedge \neg P(\delta(\varphi), \psi)$.


RCC expression for $(\varphi = x, \psi = y)$:
$TPP(x, y) = (P(x, y) \wedge \neg P(y, x)) \wedge \exists z[(C(z, x) \wedge \neg(\exists z', P(z', z) \wedge P(z', x)))) \wedge (C(z, y) \wedge \neg(\exists z', P(z', z) \wedge P(z', y)))]$

# Model based image understanding

Models of various types:

- acquisition properties (geometry, noise statistics...)
- shape
- appearance
- spatial relations
- ...

Important

- to use available knowledge
- to guide the image exploration, for segmentation, recognition, scene understanding
- to solve ambiguities
- to deal with imprecision
- ...

Issues:

- semantic gap
- imprecisions and uncertainties
- pathological cases
- algorithms

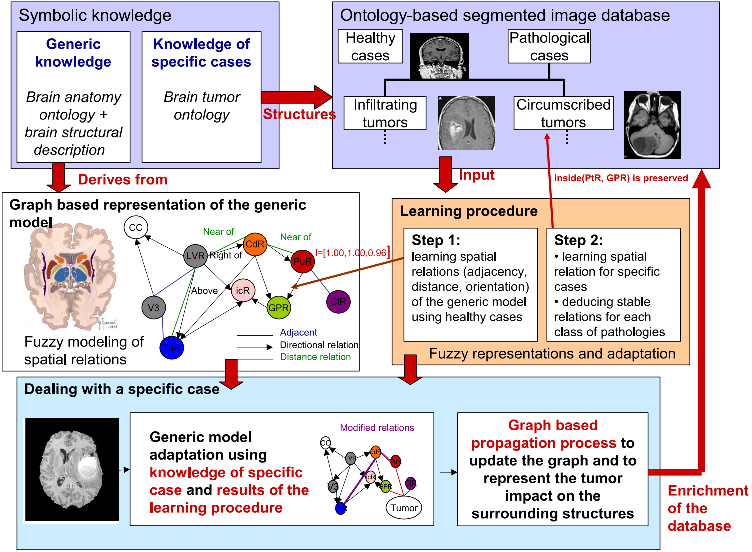
Two main questions in structural recognition in images:

- given two objects (possibly fuzzy), assess the degree to which a relation is satisfied
- given one reference object, define the area of the space in which a relation to this reference is satisfied (to some degree)

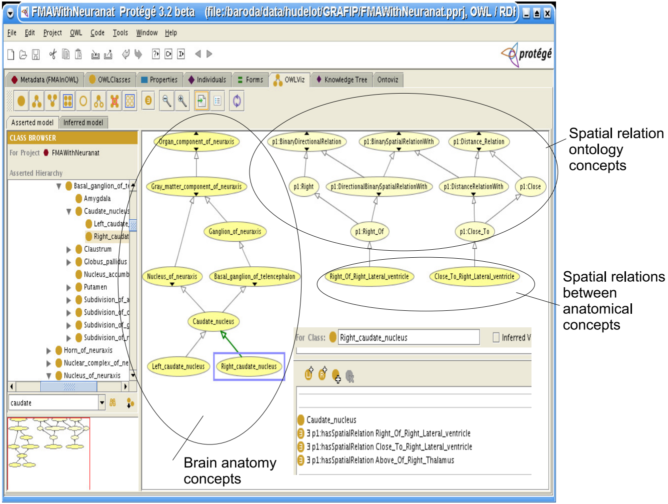
# Example in brain imaging

- Concepts:
  - brain: part of the central nervous system located in the head
  - caudate nucleus: a deep gray nucleus of the telencephalon involved with control of voluntary movement
  - glioma: tumor of the central nervous system that arises from glial cells
  - ...
- Spatial organization:
  - the left caudate nucleus is inside the left hemisphere
  - it is close to the lateral ventricle
  - it is outside (left of) the left lateral ventricle
  - it is above the thalamus, etc.
  - ...
- Pathologies: relations are quite stable, but more flexibility should be allowed in their semantics

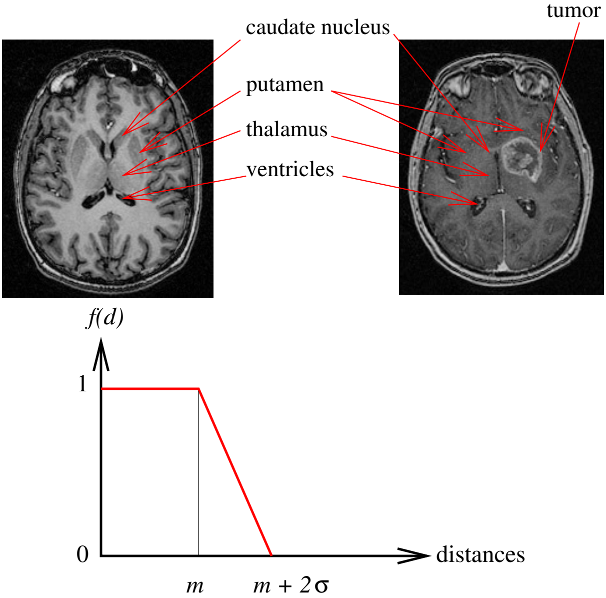
# Integration of ontologies, spatial relations and fuzzy models



# Ontology of the anatomy (FMA) enriched with an ontology of spatial relations



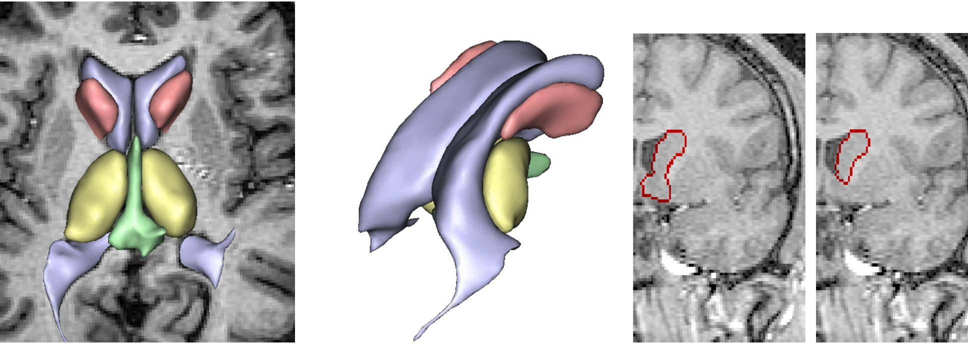
## Learning spatial relations



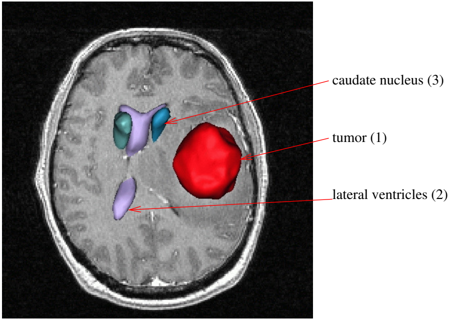
# Spatial reasoning for model-based recognition

Segmentation and recognition of some internal structures on a normal case (O. Colliot et al.):

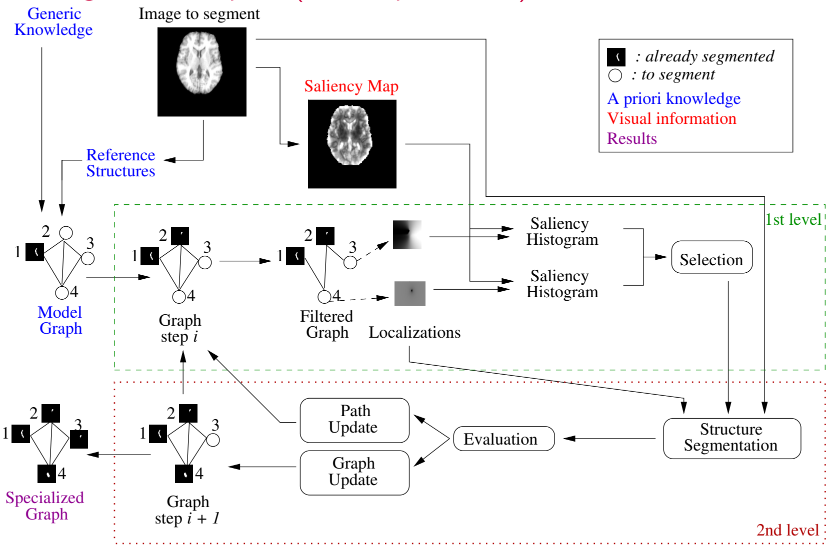
- fusion of spatial relations (given by the model) to previously recognized objects
- deformable model constrained by spatial relations



## Examples in pathological cases (H. Khotanlou, J. Atif, et al.)

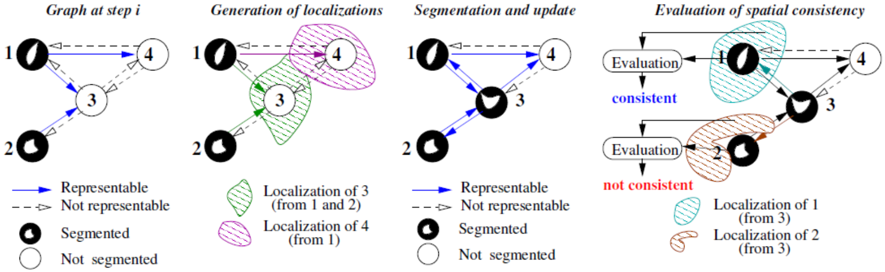



# Best segmentation path (G. Fouquier et al.)



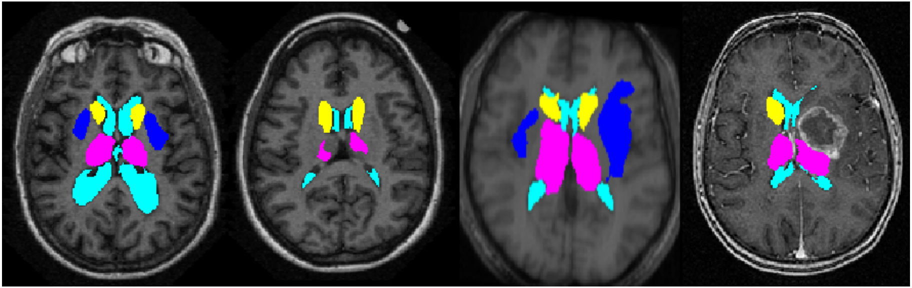
## Best segmentation path (G. Fouquier et al.)

### Evaluation and backtracking

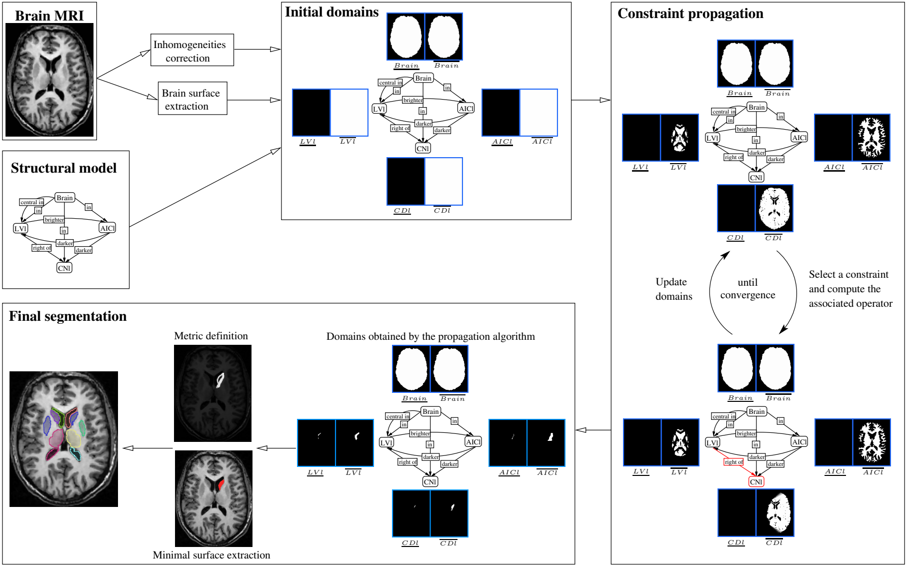


## Best segmentation path (G. Fouquier et al.)

Some results



# Global approach based in CSP (O. Nempont et al.)



Global approach based in CSP (O. Nempont et al.)

Constraint Satisfaction Problem (CSP):

- Constraint network $= (\chi, \mathcal{D}, \mathcal{C})$
- $\chi =$ variables
- $\mathcal{D} =$ set of associated domains
- $\mathcal{C} =$ constraints involving variables of $\chi$, relations on the variable domains
- Propagation of constraints:
  - Locally consistent constraint if all values of the domains can satisfy the constraint.
  - Suppression of inconsistent values: $(\chi, \mathcal{D}, \mathcal{C}) \rightarrow (\chi, \mathcal{D}', \mathcal{C})$
  - Propagator $=$ operator reducing the domains according to a constraint.

## Global approach based in CSP (O. Nempont et al.)

- Variables = anatomical structures.
- Domain of a variable = interval of fuzzy sets $[\underline{A}, \overline{A}]$.
- Example of constraint (1): inclusion

$$
\begin{aligned}
C^{in}_{A,B}: \quad \mathcal{D}(A) \times \mathcal{D}(B) &\rightarrow \{0,1\} \\
(\mu_1, \mu_2) &\mapsto \begin{cases} 1 & \text{if } \mu_1 \leq \mu_2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

- Associated propagator:

$$
\frac{\langle A, B; (\underline{A}, \overline{A}), (\underline{B}, \overline{B}); C^{in}_{A,B} \rangle}{\langle A, B; (\underline{A}, \overline{A} \wedge \overline{B}), (\underline{B} \vee \underline{A}, \overline{B}); C^{in}_{A,B} \rangle}
$$

## Global approach based in CSP (O. Nempont et al.)

- Example of constraint (2): directional relation

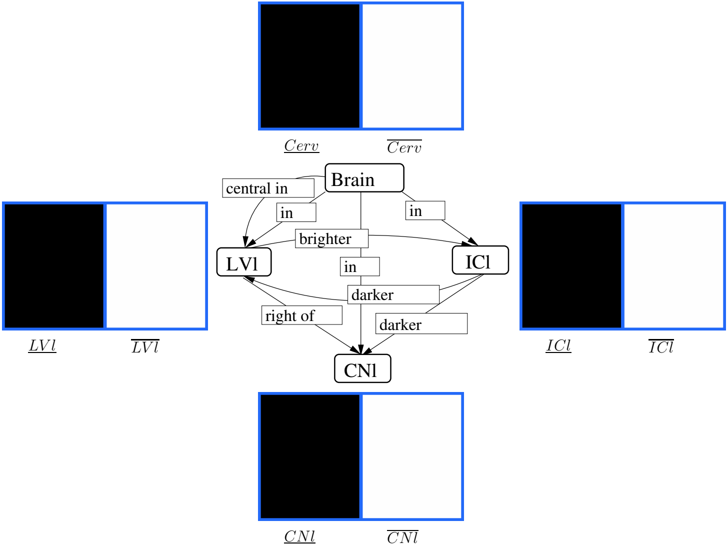
$$C_{A,B}^{dir\,\nu} : \quad \mathcal{D}(A) \times \mathcal{D}(B) \quad \to \{0,1\}$$
$$(\mu_1, \mu_2) \quad \mapsto \begin{cases} 1 & \text{if } \mu_2 \leq \delta_\nu(\mu_1), \\ 0 & \text{otherwise.} \end{cases}$$

- Associated propagator:

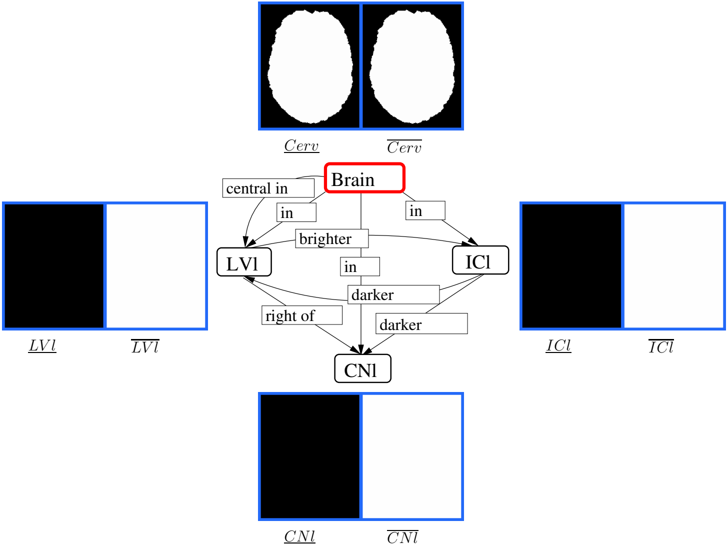
$$\frac{\langle A, B; (\underline{A}, \overline{A}), (\underline{B}, \overline{B}); C_{A,B}^{dir\,\nu} \rangle}{\langle A, B; (\underline{A}, \overline{A}), (\underline{B}, \overline{B} \wedge \delta_\nu(\overline{A})); C_{A,B}^{dir\,\nu} \rangle}$$

- Other constraints: distance, partition, connectivity, adjacency, volume, contraste...
- Ordering of the propagators and iteration application.

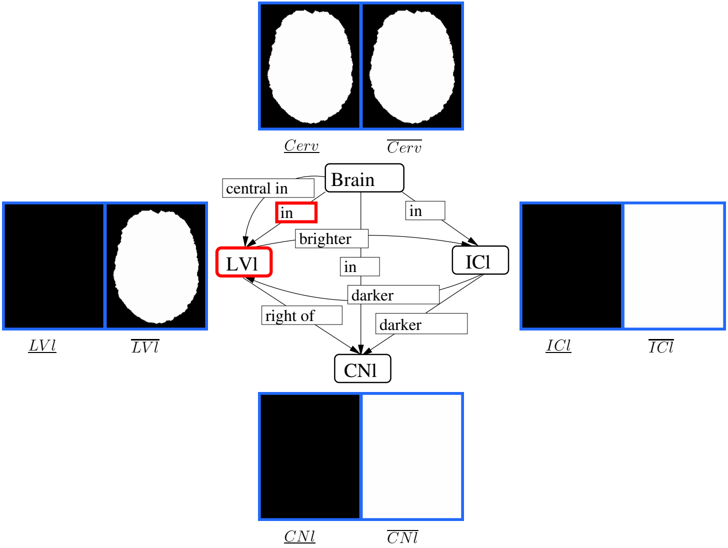
# Propagation of constraint: example



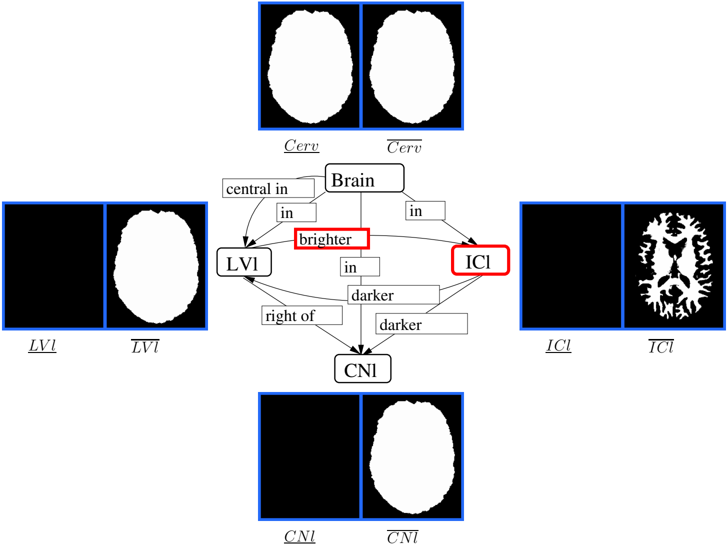
# Propagation of constraint: example



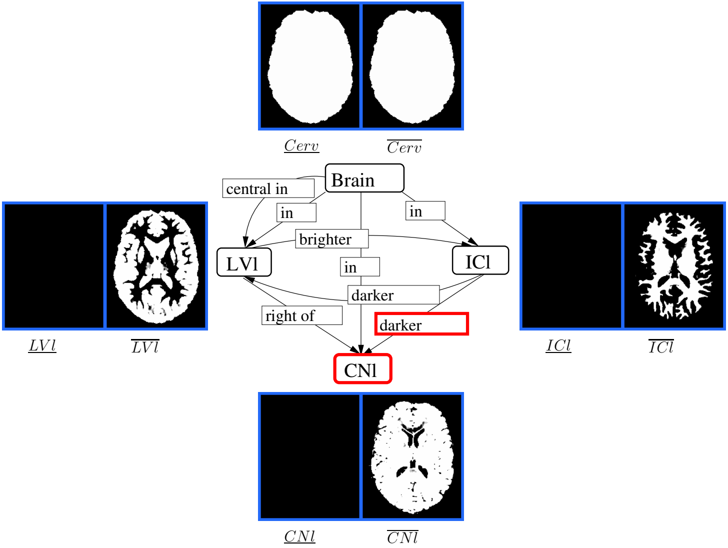
# Propagation of constraint: example



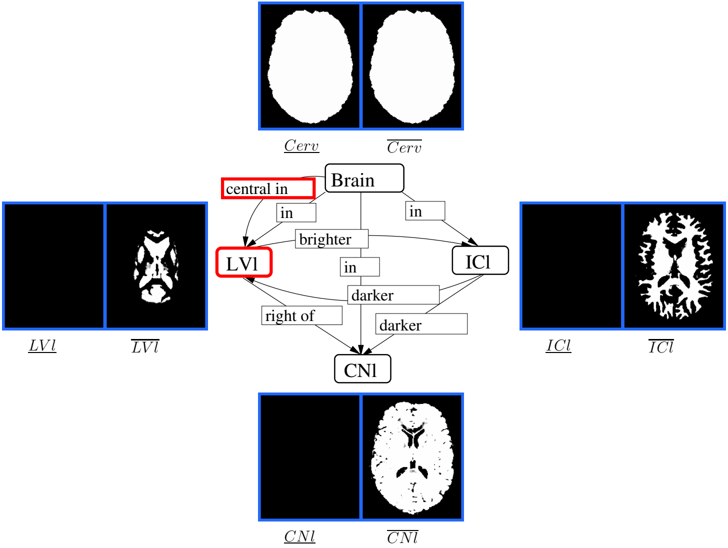
## Propagation of constraint: example



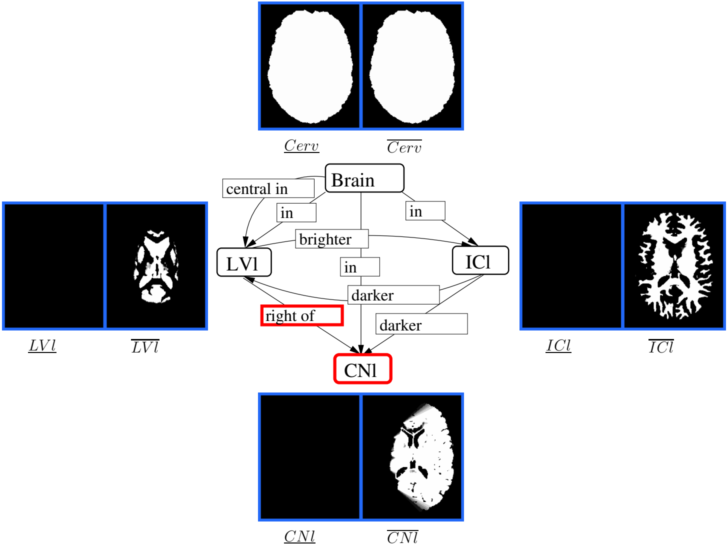
## Propagation of constraint: example

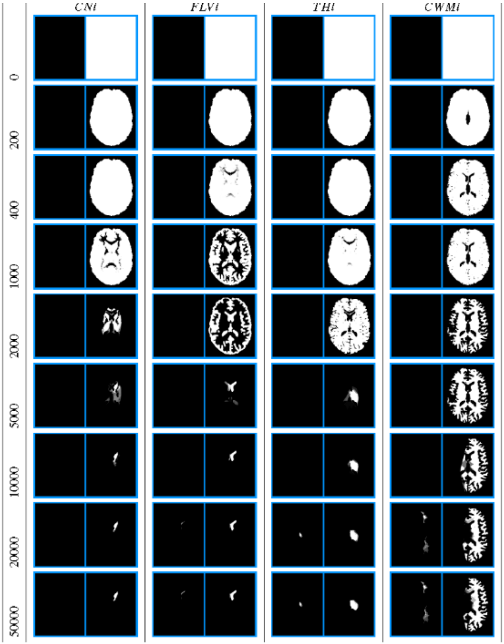


# Propagation of constraint: example

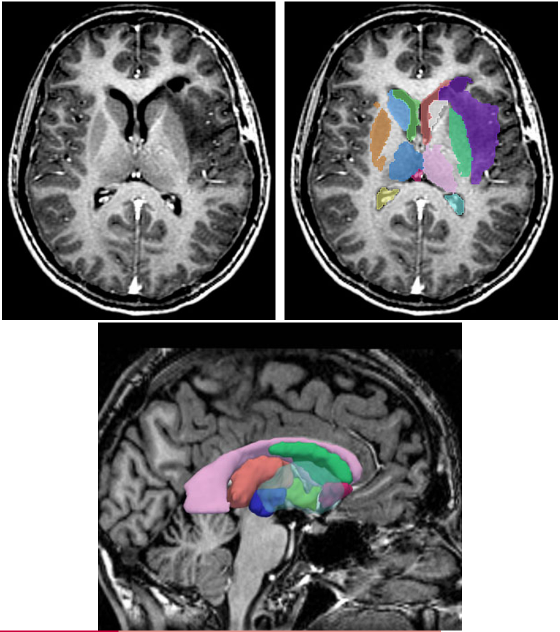


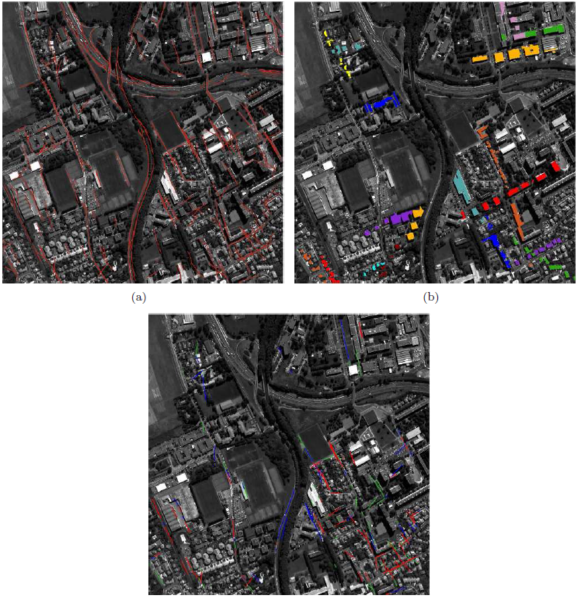
# Propagation of constraint: example




CNI
FLVI
THI
CWMI
0
200
400
1000
2000
5000
10000
20000
50000

# Result: example

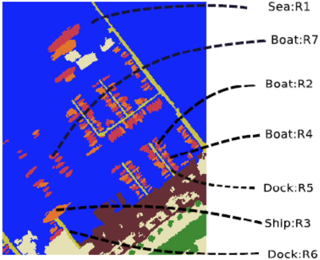




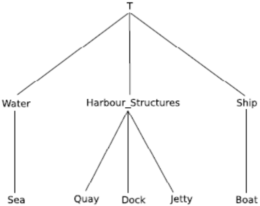
(a) (b) (c)

# Examples in remote sensing (C. Vanegas)

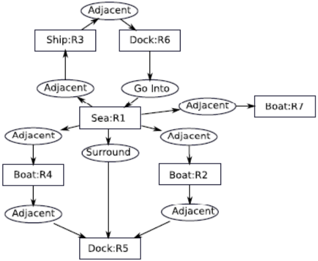
(a) Example image.




(b) Labeled image: The blue regions represent the sea, the red and orange represent ships or boats and the yellow regions represent the docks.




(c) Concept hierarchy $T_C$ in the context of




(d) Conceptual graph representing the spatial orga-