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A Local-Global Principle for the Real Continuum

Introduction. The logical foundations of mathematical analysis were developed in the 19th century by mathematicians such as Bolzano, Cauchy, Weierstrass, Dedekind, Cantor, Heine, Borel, and Cousin. They established rigorous proofs based on "completeness" axioms that characterize the real number continuum. As noticed in [16], rigor was not the most pressing question, one of the major motivations being *teaching*. Today, the foundation is still recognized as satisfactory, and all classical textbooks define $\mathbb{R}$ as any ordered field satisfying a "completeness" axiom. Here is a list of equivalent such axioms[1]:

[**sup**] (L.u.b. Property) Any set of reals has a supremum[2] (and an infimum[2]);
[**cut**] (Dedekind's Completeness) Any cut defines a (unique) real number;
[**nest+arch**] (Cantor's Property) Any sequence of nested closed intervals has a common point + Archimedean property;
[**cauchy+arch**] (Cauchy's Completeness) Any Cauchy sequence converges + Archimedean property;
[**mono**] (Monotone Convergence) Any monotonic sequence has a limit[2];
[**bw**] (Bolzano-Weierstrass) Any infinite set of reals (or any sequence) has a limit point[2];
[**bl**] (Borel-Lebesgue) Any cover of a closed interval by open intervals has a finite subcover[3];
[**cousin**] (Cousin's partition, see e.g., [6]) Any gauge defined on a closed interval admits a fine tagged partition of this interval;
[**ind**] (Continuous Induction, see e.g., [11, 9]).

It is somewhat striking that all these equivalent properties look so diverse. This calls for a need of a unifying principle from which all could be easily derived. In this work, we introduce and discuss yet another equivalent axiom in two equivalent versions (definitions to be given below):

[**lg**] (Local-Global) Any *local* and *additive* property is *global*;
[**gl**] (Global-Local) Any *global* and *subtractive* property has a *limit point*.

The earliest reference we could find that explicitly describes this principle is Guyou's little-known French textbook [8]. It was re-discovered independently many times in many various disguises in some American circles [5, 12, 17, 19, 15].

Present Situation. We have studied in detail the logical flow of proofs in graduate textbooks that are currently most influential in the U.S.A. [18, 1, 2], France [3, 14] and Brazil [7, 13]—not only proofs of the essential properties of the reals, but also of the basic theorems for continuity (intermediate value

[1]Some require the Archimedean property: Any real is upper bounded by a natural number.

[2]Possibly infinite. (For example, $\sup\mathbb{R}=+\infty$ and $\sup\varnothing=-\infty$.)

[3]This is Borel's statement, also (somewhat wrongly) attributed to Heine, and later generalized by Lebesgue and others.

theorem [**IVT**], extreme value theorem [**EVT**], Heine's theorem [**HEINE**]) and differentiation (essentially the mean value theorem [**MVT**]).

It appears that [**SUP**] is by far the preferred axiom, [**NEST+ARCH**] being the only considered alternative in [3, 2][4]. Other axioms ([**CAUCHY, ARCH**], [**MONO**], [**BW**], often [**BL**], and sometimes [**CUT**]) are derived as theorems. In contrast, [**COUSIN**] and [**IND**] are never used[5]. [**BW**] is often central to prove the basic theorems of real analysis (particularly [**IVT**], [**EVT**], [**HEINE**]) with sometimes [**BL**] as "topological" alternative. In our opinion, several classical proofs are difficult and subtle for the beginner (e.g., proofs of [**EVT**] or [**HEINE**] using [**BW**]). There have been recent attempts to improve this situation by advocating the use of [**COUSIN**] [6] or [**IND**] [11, 9], although using these methods can also be cumbersome at times.

Our Proposal. Let us explain the above [**LG**] and [**GL**] principles by defining the following intuitive notions. To simplify the assertions we consider any $[a,b]\subseteq[-\infty,+\infty]$ and assume that all closed intervals $[u,v]\subseteq[a,b]$ are nondegenerate ($u<v$). We shall always consider properties $\mathcal{P}$ of such intervals and write "$[u,v]\in\mathcal{P}$" if $[u,v]$ satisfies the property $\mathcal{P}$.

Definition 1. $\mathcal{P}$ is *additive* if $[u,v]\in\mathcal{P}\wedge[v,w]\in\mathcal{P}\implies[u,w]\in\mathcal{P}$.
$\mathcal{P}$ is *subtractive* if $\neg\mathcal{P}$ is additive, i.e., $[u,w]\in\mathcal{P}\implies[u,v]\in\mathcal{P}\vee[v,w]\in\mathcal{P}$.

A useful alternative definition can be given with overlapping intervals (this would not change the method).

Definition 2. $\mathcal{P}$ is *local at x* if there exists a neighborhood $V(x)$ in which all intervals $[u,v]$ containing x satisfy $\mathcal{P}$.
$\mathcal{P}$ has a *limit point* x if $\neg\mathcal{P}$ is not local at x, i.e., any neighborhood $V(x)$ contains an interval $[u,v]$ containing x and satisfying $\mathcal{P}$.
$\mathcal{P}$ is *local* if it is local at every point in $[a,b]$; $\mathcal{P}$ is *global* if $[a,b]\in\mathcal{P}$.

From these definitions it is immediate to see that [**LG**] $\iff$ [**GL**]. Interestingly, many usual properties/objects can be identified as local/limit points. For example, a function f is continuous iff for any $\epsilon>0$, "$|f(u)-f(v)|<\epsilon$" is local; a sequence x_k converges iff "$x_k\in[u,v]$ for sufficiently large k" has a limit point. We feel that local/global concepts are central in real analysis. Thus, taking [**LG**] or [**GL**] as the fundamental axiom for the real numbers it becomes easy and intuitive to prove all the other completeness properties, as well as all the above mentionned basic theorems of real analysis. Due to lack of space we provide only three exemplary proofs.

[4] Some textbooks also mention the possibility of "proving" the fundamental axiom by first *constructing* the reals from the rationals—themselves constructed from the natural numbers— the two most popular construction methods being Dedekind's cuts and Cantor's fundamental sequences. While this approach is satisfactory for logical consistency, the details are always tedious and not instructive for the student or for anyone using the real numbers, since the way they can be constructed never influences the way they are used.

[5] Cousin's [**COUSIN**], although proposed at the same time (1895) as Borel's [**BL**], has been largely overlooked since. It was only recently re-exhumed as a fundamental lemma for deriving the gauge (Kurzweil-Henstock) integral (see e.g., [6]). [**IND**] is much more recent, and in fact inspired from [**LG**] [4, 10].

PROOF OF **[BW]**. Let $A \subset [a,b]$ be infinite: the property that "$[u,v]$ contains infinitely many points of A" is global, and evidently subtractive. By **[GL]**, it has a limit point, i.e., A has a limit (accumulation) point. $\square$

PROOF OF **[BL]**. Let be given a covering of $[a,b]$ by open intervals: the property that "$[u,v]$ has a finite subcover" is local (with only one open interval), and evidently additive. By **[LG]**, it is global. $\square$

PROOF OF **[IVT]**. Let $f : [a,b] \to \mathbb{R}$ be continuous and let y be lying between $f(a)$ and $f(b)$: the property that y lies between $f(u)$ and $f(v)$ is global, and evidently subtractive. By **[GL]**, it has a limit point x, which by continuity of f satisfies $y = f(x)$. $\square$

The aim of this work is to draw attention to such local/global concepts in order to reform teaching of real analysis at undergraduate and graduate levels.

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