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On the Optimality of Mutual Information Analysis for Discrete Leakages

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Éloi de Chérisey*, Annelie Heuser**,
Sylvain Guilley** and Olivier Rioul**

* ENS Cachan, **Telecom ParisTech

Éloi de CHÉRISEY	**Sylvain GUILLEY**	**Annelie HEUSER**	**Olivier RIOUL**
is with	*is also with*	*is PhD fellow at*	*is also Prof at*

Research thread: "*Distinguishing distinguishers*"

- CHES 2014: known model + large Gaussian noise
  $\Longrightarrow$ **CPA is optimal**
- ASICRYPT 2014: idem with masking
  $\Longrightarrow$ **HO-CPA is optimal**
- CRYPTARCHI 2015: unknown model
  $\Longrightarrow$ **MIA is optimal**

Outline

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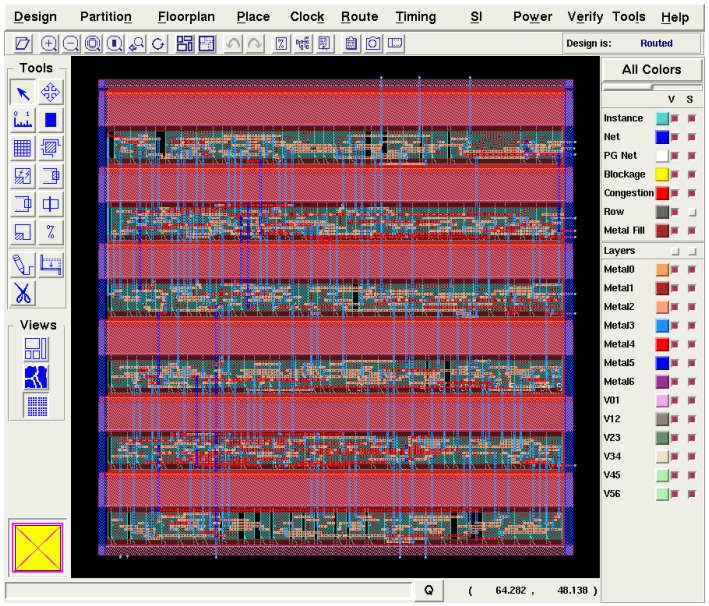
Motivations

1. How to attack when
   - no leakage model is available?
   - no profiling is possible?

   (e.g., balanced dual-rail circuits)



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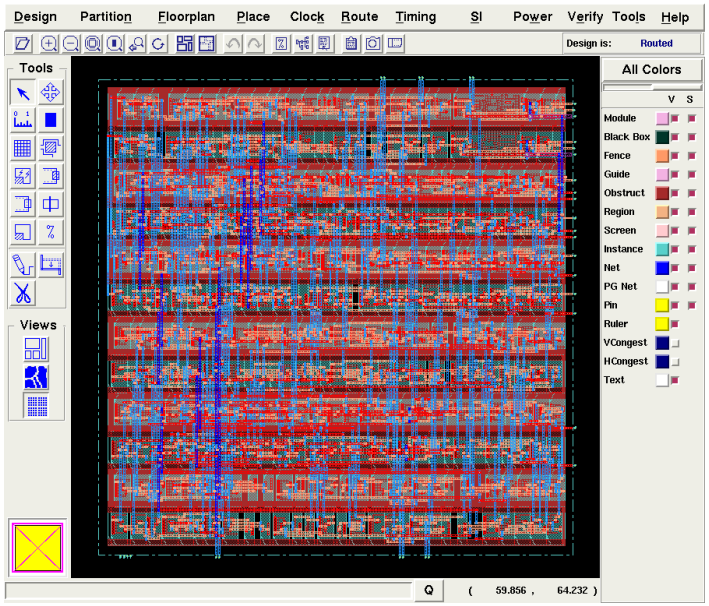

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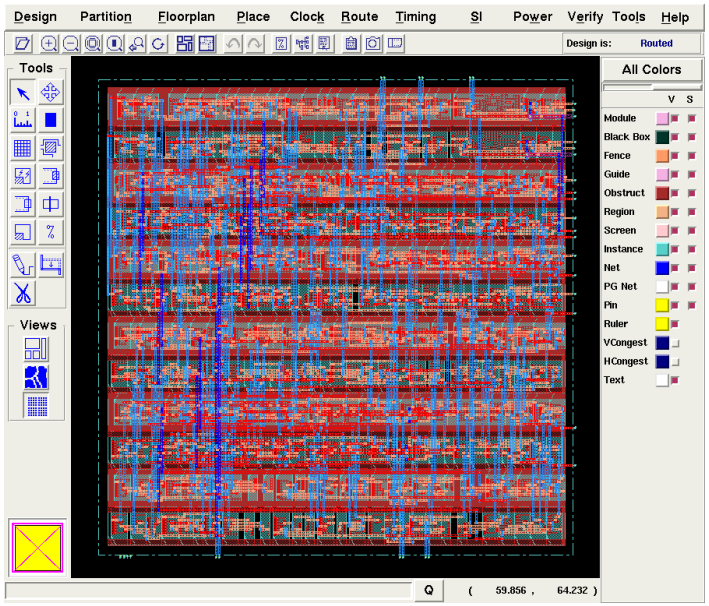

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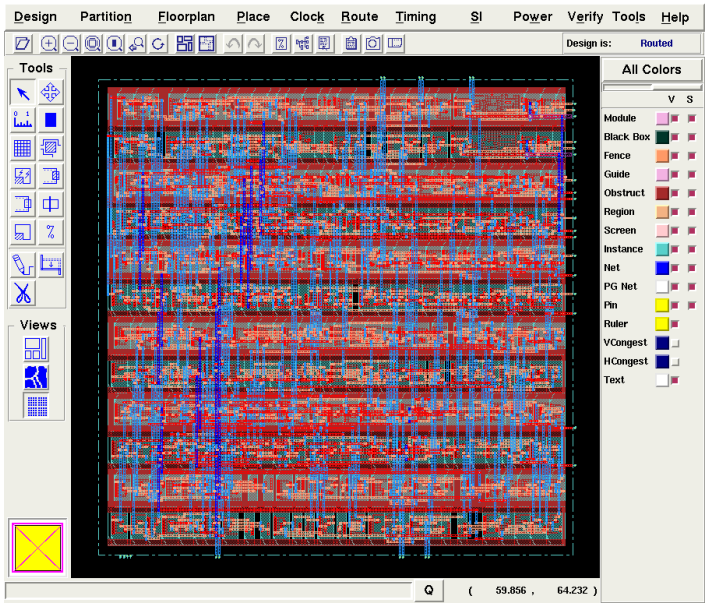

1. How to attack when
   - no leakage model is available?
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   (e.g., balanced dual-rail circuits)

   → Mutual Information Analysis (MIA) [Gierlichs et al., CHES 2008]
2. Is it possible for MIA to be:
   - Optimal?
   - Effective?
   - Efficient?

Results

We show that:

- MIA is“optimal” in that it is equivalent to a “universal” maximum likelihood (U-ML)
- MIA can even outperform CPA
- the computational complexity of MIA can be significantly reduced

Outline

Leakage model

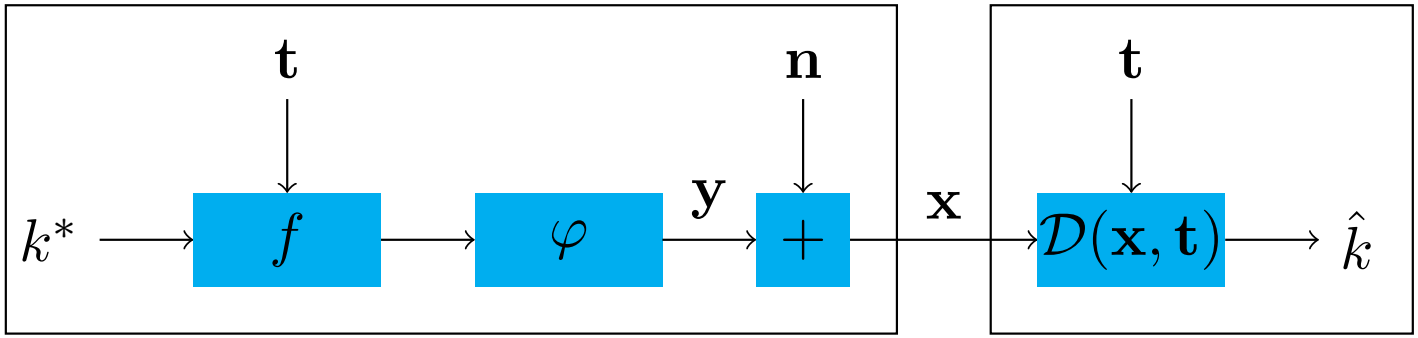
Assumptions:

- secret key k^*: deterministic but unknown
- uniformly distributed text bytes $\mathbf{t} = (t_1, \ldots, t_m)$: random but known
- m i.i.d. noisy measurements $\mathbf{x} = (x_1, \ldots, x_m)$

$$\mathbf{x} = \mathbf{y}(k^*) + \mathbf{n}$$

- leakage model:

$$\mathbf{y}(k) = \varphi(f(k, \mathbf{t}))$$



Markov Chain Property

Markov Condition

The leakage $\mathbf{x}$ depends on the secret key k^* only through the computed model $\mathbf{y}(k^*) = \varphi(f(k^*, \mathbf{t}))$

Thus the *conditional distribution* $\mathbb{P}(\mathbf{x}|\mathbf{y})$ *depends on the key only through the value of* $\mathbf{y}$.

Example

- $x_i = w_H(k^* \oplus t_i) + n_i$, where w_H is the Hamming weight
- $x_i = \varphi(S(k^* \oplus t_i)) + n_i$, where S is a substitution box

Empirical Distribution (Histogram)

Assumptions

Leakage is discrete (or discretized *via* binning).

Counting occurrences of each value of x and y gives the estimate:

$$\tilde{\mathbb{P}}(x|y) = \frac{\sum_{i=1}^{m} \mathbf{1}_{x_i=x, y_i=y}}{\sum_{i=1}^{m} \mathbf{1}_{y_i=y}}$$

Definition (Empirical Mutual Information)

$$\tilde{I}(X;Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \tilde{\mathbb{P}}(x,y) \log_2 \frac{\tilde{\mathbb{P}}(x,y)}{\tilde{\mathbb{P}}(x)\tilde{\mathbb{P}}(y)}$$

Best Distinguisher

The optimal distinguisher $\mathcal{D}(\mathbf{x}, \mathbf{y})$ and associated distinguishing rule $\hat{k} = \arg\max_{k \in \mathcal{K}} \mathcal{D}(\mathbf{x}, \mathbf{y})$ maximizes success:

Definition (Average Success Rate)

$$SR = \frac{1}{2^n} \sum_{k=0}^{2^n - 1} \mathbb{P}_k(\hat{k}(\mathbf{X}, \mathbf{T}) = k)$$

This is a theoretical definition since $\mathbb{P}_k(\mathbf{x}, \mathbf{t})$ is not known perfectly.

Maximum likelihood

Theorem (Optimal Distinguisher = Maximum Likelihood)

$$\mathcal{D}_{opt}(\mathbf{x}, \mathbf{t}) = \arg\max_{k \in \mathcal{K}} \mathbb{P}(\mathbf{x}|\mathbf{y})$$

Recall $\mathbf{y} = \varphi(f(k, \mathbf{t}))$ *depends on the key hypothesis* k*.*

Proved by [Heuser et al., in CHES 2014]

Optimality holds for perfectly known leakage model φ

Universal Maximum Likelihood

Universal = from the data without prior information.

Definition (Universal Maximum Likelihood)

$$\hat{k} = \arg\max_{k \in \mathcal{K}} \tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y})$$

where $\tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y}) = \prod_{i=1}^{m} \tilde{\mathbb{P}}(x_i|y_i)$

Loss in performance due to empirical probability estimation.

Universal Maximum Likelihood

Theorem (MIA = U-ML)

The universal ML key estimate is equivalent to MIA! [Gierlichs et al. CHES 2008]

$$\hat{k} = \arg\max_{k \in \mathcal{K}} \tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y}) = \arg\max_{k \in \mathcal{K}} \tilde{I}(X;Y)$$

This surprising theoretical result shows that MIA is "universally" optimal as it maximizes

$$\tilde{SR} = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \tilde{\mathbb{P}}_k(\hat{k}(\mathbf{X}, \mathbf{T}) = k)$$

Proof of the Theorem

Rearrange empirical probability:

$$\tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y}) = \prod_{i=1}^{m} \tilde{\mathbb{P}}(x_i|y_i) = \prod_{x\in\mathcal{X},y\in\mathcal{Y}} \tilde{\mathbb{P}}(x|y)^{n_{x,y}}$$

where

$$n_{x,y} = \sum_{i=1}^{m} \mathbf{1}_{x_i=x,y_i=y} = m\tilde{\mathbb{P}}(x,y)$$

Now take the logarithm:

$$\log \tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y}) = \sum_{x\in\mathcal{X},y\in\mathcal{Y}} m\tilde{\mathbb{P}}(x,y) \log(\tilde{\mathbb{P}}(x|y))$$

We recognize entropy!

$$\tilde{H}(X|Y) = -\sum_{x\in\mathcal{X},y\in\mathcal{Y}} \tilde{\mathbb{P}}(x,y) \log(\tilde{\mathbb{P}}(x|y)) \quad /\ldots$$

Proof of the Theorem (cont'd)

In other words:

$$\tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y}) = 2^{-m\tilde{H}(X|Y)}$$

Thus maximizing $\tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y})$ amounts to minimizing $\tilde{H}(X|Y)$, i.e., maximizing

$$\tilde{I}(X,Y) = \tilde{H}(X) - \tilde{H}(X|Y)$$

since $\tilde{H}(X)$ does not depend on k.

Conclusion:

$$\boxed{\arg\max_{k\in\mathcal{K}} \tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y}) = \arg\max_{k\in\mathcal{K}} \tilde{I}(X;Y)}$$

□

Outline

Case-Study

Consider the following leakage : $X = Y + N$ with $Y = \varphi(f(T \oplus K^*))$ where $Y = \pm 1$ and $N \sim \mathcal{U}(\pm\sigma)$ (uniformly distributed with values $\pm\sigma$).

Assumptions

- $T, K \in \mathbb{F}_2^n$
- $f : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is an Sbox function applied to a xor and φ is a one-bit selection function.
- $\sigma \in \mathbb{N}$ is an integer

Example

Dynamic Voltage Scaling (DVS) used as a countermeasure.

Case-Study: Distinguisher

Theorem (Optimal Distinguisher)

One easily finds

$$\mathcal{D}_{opt}(\mathbf{x},\mathbf{t}) = \arg\max_{k\in\mathcal{K}} \mathbb{P}(\mathbf{x}|\mathbf{y}) = \arg\max_{k\in\mathcal{K}} \frac{1}{2^m} \prod_{i=1}^{m} \begin{cases} 1 & \text{if } x-\varphi(f(t,k))=0 \\ 1 & \text{if } x-\varphi(f(t,k))=2\sigma \\ 0 & \text{otherwise} \end{cases}$$

To be compared to

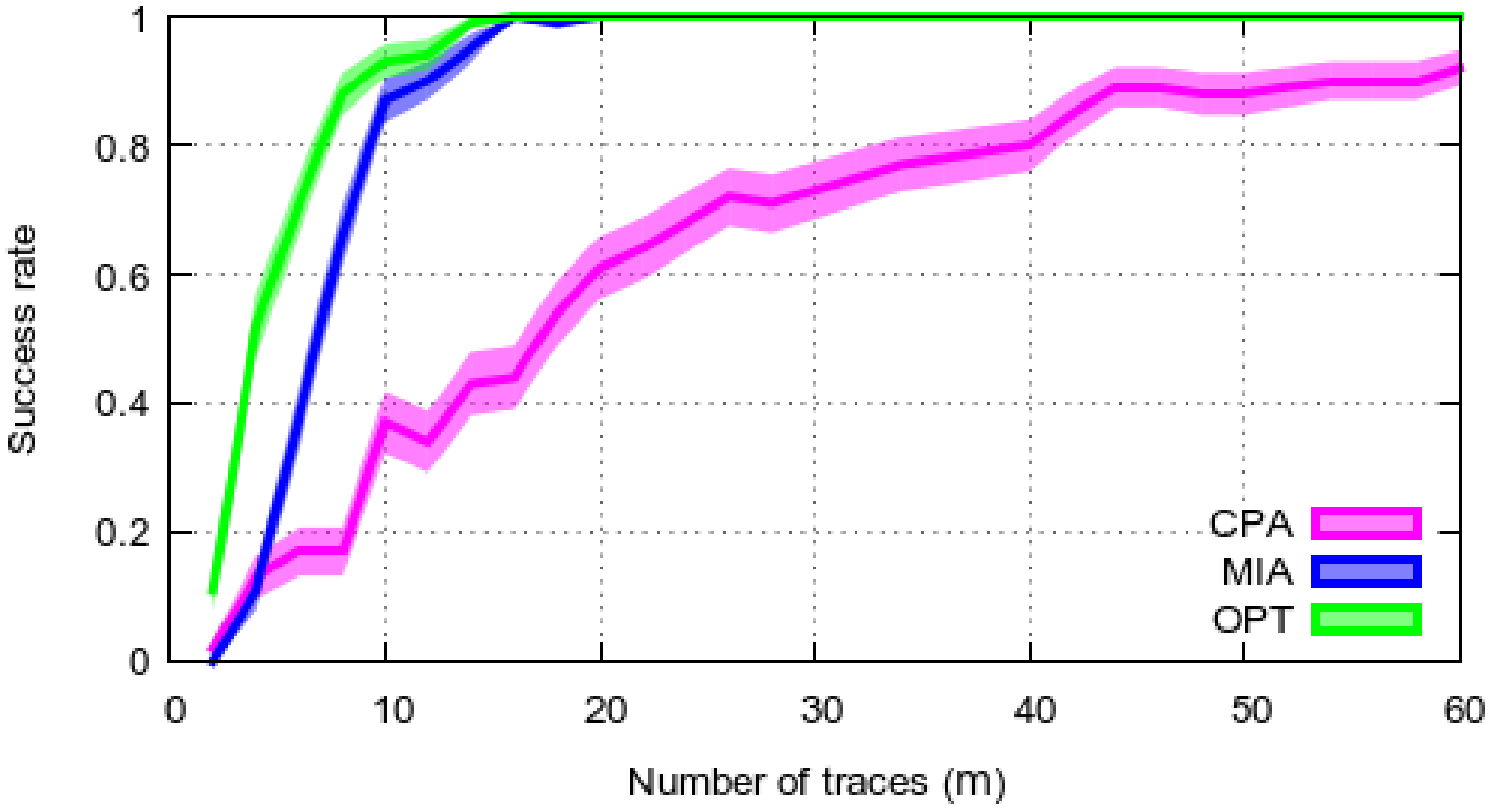
i CPA:

$$\mathcal{D}_{CPA}(\mathbf{x},\mathbf{t}) = \arg\max_{k\in\mathcal{K}} \frac{\widetilde{\mathsf{cov}}(X,Y)}{\tilde{\sigma}_X\tilde{\sigma}_Y}$$

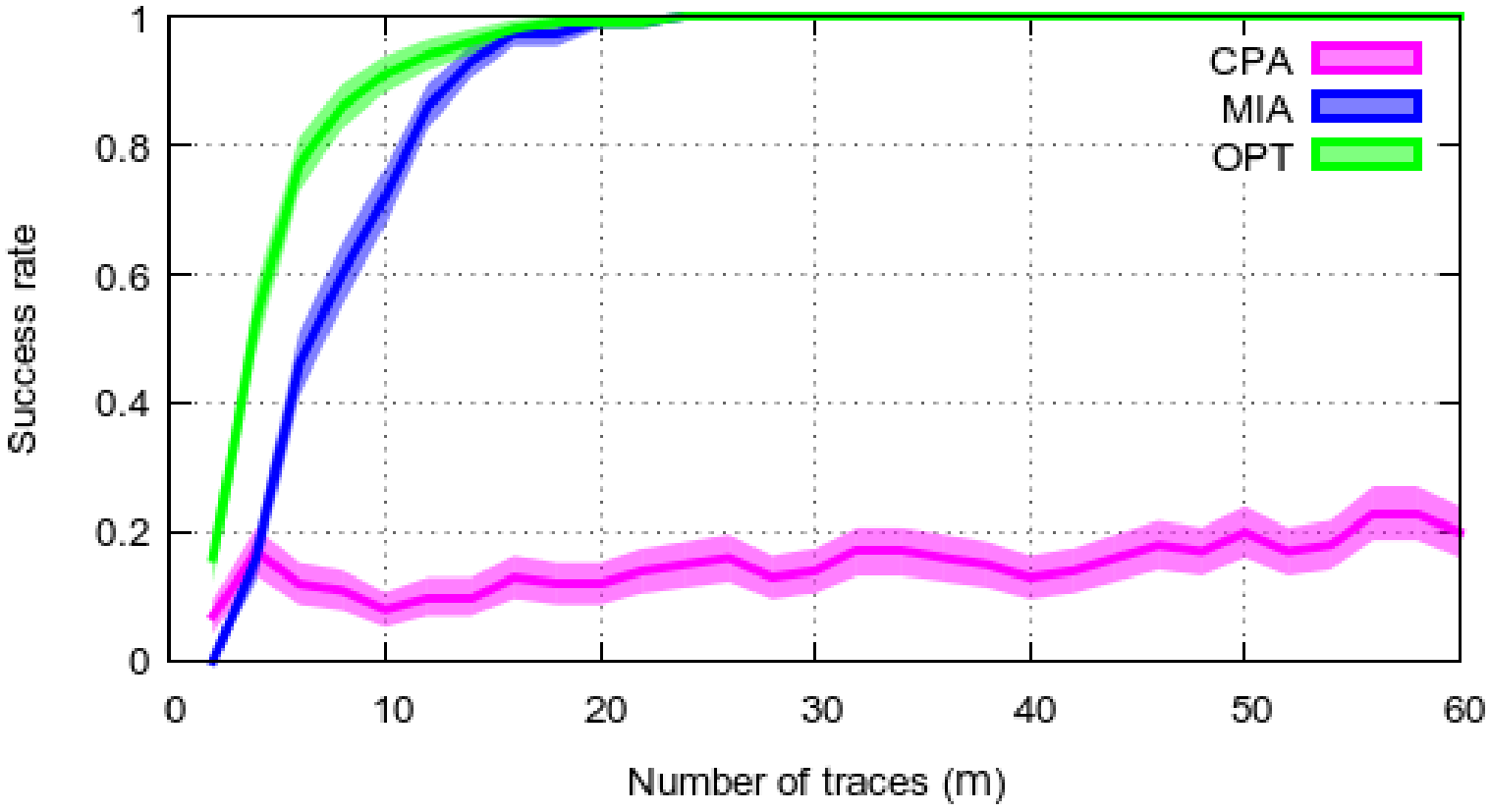
ii MIA :

$$\mathcal{D}_{MIA}(\mathbf{x},\mathbf{t}) = \arg\max_{k\in\mathcal{K}} \tilde{I}(X,Y)$$

Results for MIA and CPA ($\sigma = 1$)



Results for MIA and CPA ($\sigma = 4$)



Outline

Size of the Bins

Why Binning?

Due to noise, $\mathbf{x}$ assumes real values: $p(\mathbf{x}|\mathbf{y})$ as a pdf that must be estimated using bins as $\tilde{\mathbb{P}}(\mathbf{x}|\mathbf{y})$

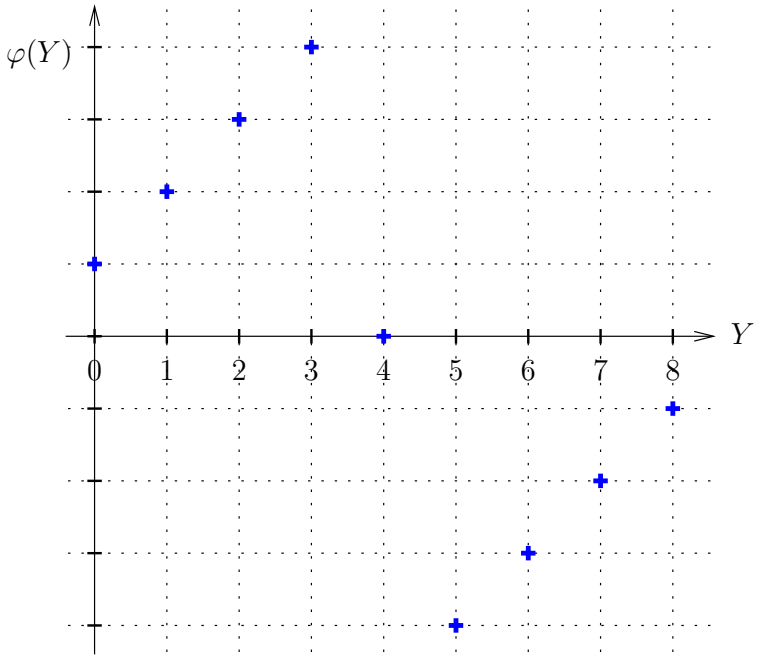
- what is the optimal size of bins?
- does it depend of the number of traces?

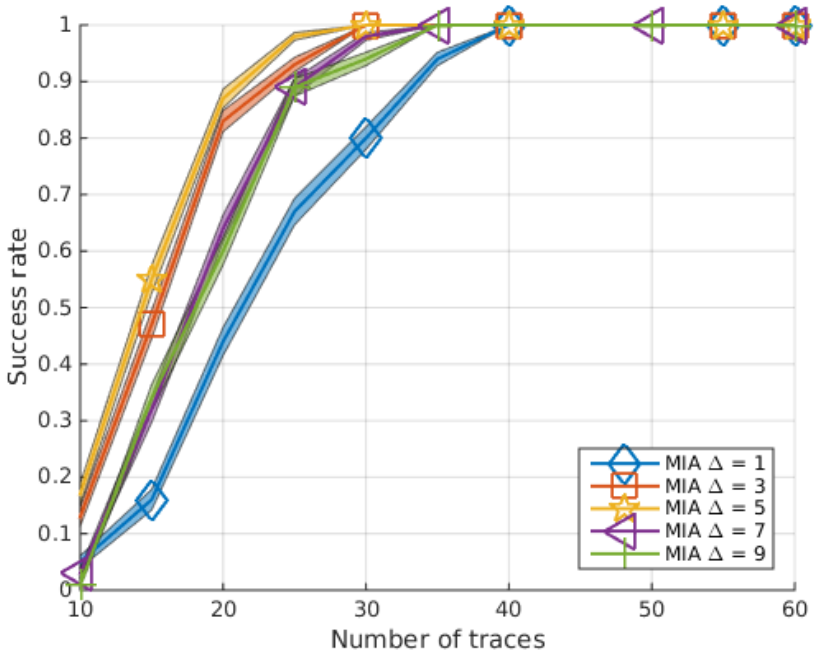
Leakage Example

Example

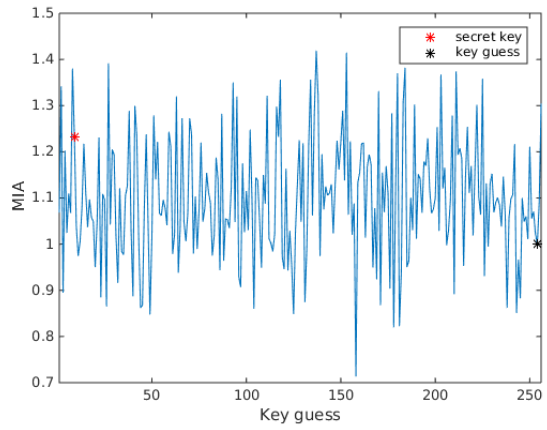
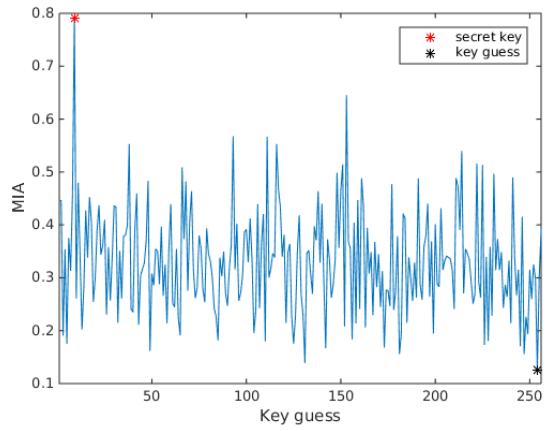
$\mathbf{y} = \varphi(f(\mathbf{y}(k^*, \mathbf{t})))$ with

- $f(k, \mathbf{t}) = \text{Sbox}(\mathbf{t} \oplus k^*)$
- $\varphi = \psi(w_H(\bullet))$ where ψ is a non-linear bijective function s.t. $\text{Cov}(Y, \varphi(Y)) = 0$

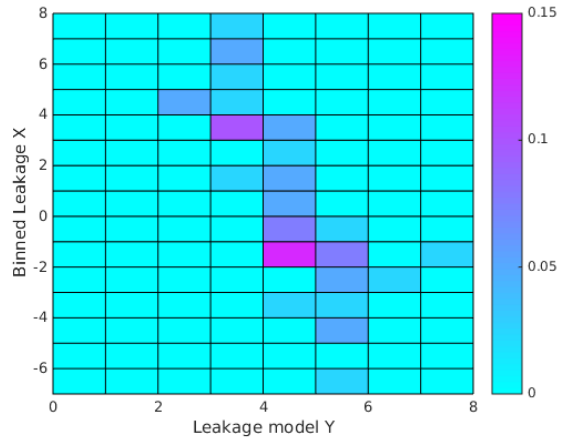
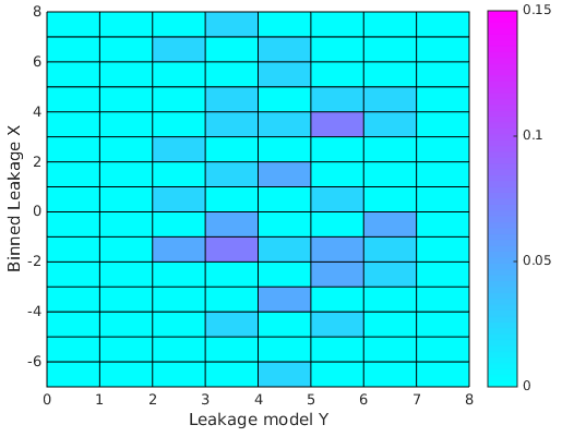
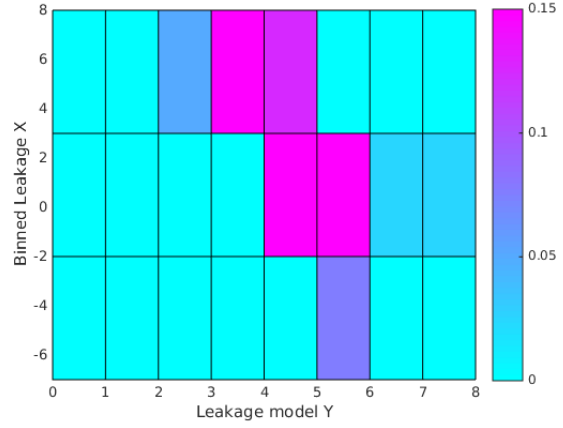
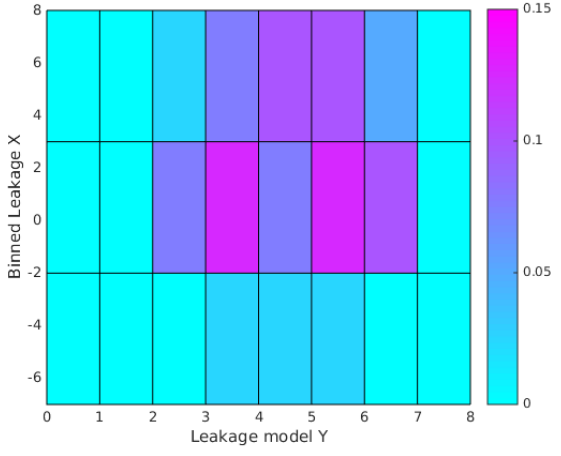




Leakage Example (cont'd)

	MIA using 30 traces
$\Delta = 1$	
$\Delta = 5$	

Leakage Example (cont'd)

	Correct key guess	False key guess
$\Delta = 1$		
$\Delta = 5$		

Fast MIA Algorithm Principle

See [Victor Lomné et al. in ASIACRYPT 2013] for CPA

Principle

The pmf is given by

$$\tilde{\mathbb{P}}(y, x) = \sum_{t|\varphi(f(t,k))=y} \tilde{\mathbb{P}}(t, x)$$

Implement this once and for all in place of empirical probability $\tilde{\mathbb{P}}(t, x)$.

Fast MIA Algorithm Flow

input : $\mathbf{x}$ a set of m traces which take discrete values, $\mathbf{t}$ a corresponding set of m plaintexts/ciphertexts

output: $(\tilde{I}(\mathbf{x}, \mathbf{y}(k)))_{k \in \mathcal{K}}$

`// From` $\mathbf{x}$ `and` $\mathbf{t}$`, build an hash table` $\mathsf{PMF}[t][x]$ `(i.e., an histogram);`

1 **for** $i \in \{1, \ldots, m\}$ **do**
2 $\quad \mathsf{PMF}[t_i][x_i] \mathrel{+}= 1;$
3 **end**

4 **for** $x \in \mathcal{X}$ **do**
5 $\quad \mathsf{P}(x) = 0$; `//` $\mathsf{P}(x)$ `holds` $m\tilde{\mathbb{P}}(x)$
6 $\quad$ **for** $t \in \mathbb{F}_2^n$ **do**
7 $\quad\quad \mathsf{P}(x) \mathrel{+}= \mathsf{PMF}[t][x]$;
8 $\quad$ **end**
9 **end**

Fast MIA Algorithm Flow

1 **for** $k \in \mathcal{K}$ **do** // Key enumeration
2 $\quad \forall x \in \mathcal{X}, y \in \mathcal{Y}, \mathsf{P}(x,y) = 0$; // $\mathsf{P}(x,y)$ holds $m\tilde{\mathbb{P}}(x,y)$
3 $\quad$ **for** $t \in \mathbb{F}_2^n$ **do**
4 $\quad\quad$ **for** $x \in \mathcal{X}$ **do**
5 $\quad\quad\quad \mathsf{P}(x, \varphi(f(k,t))) \mathrel{+}= \mathsf{PMF}[t][x]$; // $y = \varphi(f(k,t))$
6 $\quad\quad$ **end**
7 $\quad$ **end**
8 $\quad \tilde{I}(\mathbf{x}, \mathbf{y}(k)) = 0;$
9 $\quad$ **for** $y \in \mathcal{Y}$ **do**
10 $\quad\quad \mathsf{P}(y) = 0$; // $\mathsf{P}(y)$ holds $m\tilde{\mathbb{P}}(y)$
11 $\quad\quad$ **for** $x \in \mathcal{X}$ **do**
12 $\quad\quad\quad \mathsf{P}(y) \mathrel{+}= \mathsf{P}(x,y);$
13 $\quad\quad$ **end**
14 $\quad\quad$ **for** $x \in \mathcal{X}$ **do**
15 $\quad\quad\quad$ **if** $\mathsf{P}(x,y) \neq 0$ **then** // $\mathsf{P}(x,y) \neq 0 \implies (\mathsf{P}(x) \neq 0 \wedge \mathsf{P}(y) \neq 0)$
16 $\quad\quad\quad\quad \tilde{I}(\mathbf{x}, \mathbf{y}(k)) \mathrel{+}= \frac{\mathsf{P}(x,y)}{m} \log_2 \frac{m\mathsf{P}(x,y)}{\mathsf{P}(x)\mathsf{P}(y)}$;
17 $\quad\quad\quad$ **end**
18 $\quad\quad$ **end**
19 $\quad$ **end**
20 **end**
21 **return** $(\tilde{I}(\mathbf{x}, \mathbf{y}(k)))_{k \in \mathcal{K}}$

Fast MIA Algorithm Flow

1 **for** $k \in \mathcal{K}$ **do** // Key enumeration
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15 $\quad\quad\quad$ **if** $\mathsf{P}(x,y) \neq 0$ **then** // $\mathsf{P}(x,y) \neq 0 \implies (\mathsf{P}(x) \neq 0 \wedge \mathsf{P}(y) \neq 0)$
16 $\quad\quad\quad\quad \tilde{I}(\mathbf{x}, \mathbf{y}(k)) \mathrel{+}= \mathsf{P}(x,y) \log_2 \frac{\mathsf{P}(x,y)}{\mathsf{P}(y)}$; // no longer a mutual information, but OK...
17 $\quad\quad\quad$ **end**
18 $\quad\quad$ **end**
19 $\quad$ **end**
20 **end**
21 **return** $(\tilde{I}(\mathbf{x}, \mathbf{y}(k)))_{k \in \mathcal{K}}$

Conclusions

- MIA is equivalent to Universal ML
- MIA can be close to optimal
- Binning size is crucial
- Fast MIA

Thank you!

Questions?

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