SIAM J. MATH. ANAL.
Vol. 23, No. 6, pp. 1544–1576, November 1992

SIMPLE REGULARITY CRITERIA FOR SUBDIVISION SCHEMES*

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Abstract. Convergent subdivision schemes arise in several fields of applied mathematics (computer-aided geometric design, fractals, compactly supported wavelets) and signal processing (multiresolution decomposition, filter banks). In this paper, a polynomial description is used to study the existence and Hölder regularity of limit functions of binary subdivision schemes. Sharp regularity estimates are derived; they are optimal in most cases. They can easily be implemented on a computer, and simulations show that the exact regularity order is accurately determined after a few iterations. Connection is made to regularity estimates of solutions to two-scale difference equations as derived by Daubechies and Lagarias, and other known Fourier-based estimates. The former are often optimal, while the latter are optimal only for a subclass of symmetric limit functions.

Key words. subdivision algorithms, Hölder regularity, Sobolev regularity, two-scale difference equations, wavelets

AMS(MOS) subject classifications. 26A15, 26A16, 39B05, 42C15, 46E35, 94A12

1. Introduction.

This paper focuses on the behavior of real-valued discrete sequences u_n $(n \in \mathbf{Z})$ of finite length under repeated action of an operator $\mathcal{G}$ defined as

$$u_n \xrightarrow{\mathcal{G}} v_n = \sum_{k\in\mathbf{Z}} u_k\, g_{n-2k}. \tag{1.1}$$

The fixed sequence g_n that parameterizes $\mathcal{G}$ is called the subdivision *mask* [14], [15]. It plays a central role in the following. Starting from the initial "impulse" sequence

$$\delta_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

a *binary subdivision scheme* [14], [15], [17] (in one dimension) is an infinite collection of sequences g_n^j $(j \in \mathbf{N})$, defined by iteration as shown.

$$\begin{aligned} g_n^1 &= \mathcal{G}\{\delta_n\} = g_n, \\ g_n^2 &= \mathcal{G}\{g_n^1\}, \\ &\vdots \\ g_n^{j+1} &= \mathcal{G}\{g_n^j\}. \qquad (1.2) \\ &\vdots \end{aligned}$$

The g_n^j's are fully determined given the mask g_n. Of course other initial sequences can be considered. In addition, this scheme is said to be *interpolatory* [10]–[15] if it satisfies the extra condition $g_{2n} = \delta_n$, which means that all points g_n^j at some level j are carried unchanged to the next level: $g_{2n}^{j+1} = g_n^j$. In this paper we regard interpolatory subvision schemes as a special case to which general results will apply. However, we restrict ourselves to *binary* subdivision schemes, even though the results

* Received by the editors February 25, 1991; accepted for publication (in revised form) January 6, 1992.

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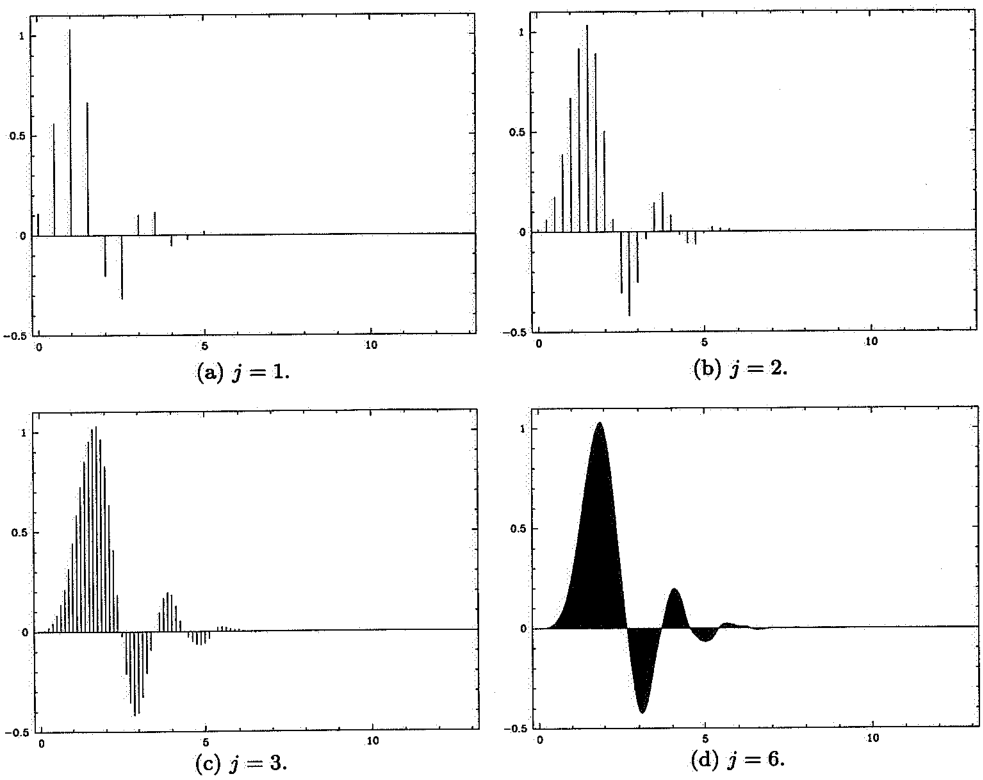


FIG. 1. *A binary subdivision scheme converging to a limit function (after* [6]). *The discrete sequences g_n^j are plotted as "pulses" against $n2^{-j}$ for $j = 1$, 2, 3, and 6. At each iteration step the up-scaling operator* (1.1) *is applied, which approximatively doubles the number of coefficients while preserving a global shape. When $j \to \infty$, these discrete curves converge to a "nice-looking," regular limit function, compactly supported on* $[0, 13]$.

of this paper easily extend to more general subdivision schemes, for which the number 2 in (1.1) is replaced by any integer $p \geq 2$ [8], [9].

Subdivision schemes arise in several fields of applied mathematics and signal processing. They have been used for curve fitting and to generate fractal or smooth curves and surfaces numerically [10]–[15], [17]. They also play an important role in wavelet theory [1], [3]–[7], [20]–[23], a newly born theory in functional analysis closely related to filter bank theory in signal processing [20], [18], [21], [22]. In all of these applications, the convergence of (1.2) to a function of a continuous variable $\varphi(x)$ as j indefinitely increases is important. It is also important to control several properties of the limit function $\varphi(x)$ from the choice of the mask g_n. For example, whether limit functions $\varphi(x)$ are regular (smooth) or not may be relevant for image coding applications using wavelets [1], [20], and this has motivated the work presented here.

The aim of this paper is to find necessary and sufficient conditions on the mask g_n for the existence and Hölder regularity of the limit function $\varphi(x)$. Figure 1 shows that $\varphi(x)$ can be thought of as a limit of discrete curves g_n^j plotted against $n2^{-j}$. (We then say that the sequences g_n^j "converge" to $\varphi(x)$ as $j \to \infty$.) In addition, we shall often be in the case of *uniform* convergence. Intuitively, this means that the discrete curves g_n^j converge "as a whole" to the limit curve $\varphi(x)$. Section 3 discusses several possible definitions for both types of convergence.

This paper is organized as follows. First, §2 describes binary subdivision schemes

(1.2) using the convenient polynomial notation. Then, various definitions of convergence are discussed (§3), and a basic necessary condition for the existence of a limit function is derived (§4). We show how the values of a limit function can be computed exactly on a computer (§5). The relation between the values of g_n^j and those of $\varphi(x)$ leads us to define "stable" subdivision schemes, to which the results of this paper fully apply (§6). Fortunately, almost all limit functions are stable.

To tackle the regularity problem, we characterize regularity of limit functions in terms of discrete sequences. Continuity is connected to uniform convergence and a necessary and sufficient condition for uniform convergence is derived in §7. Hölder regularity $\dot{C}^\alpha$ $(0 < \alpha \leq 1)$ is expressed by a similar property of the g_n^j's (§8). Finite differences of the g_n^j's play the role of derivatives and N-times continuously differentiable limit functions are, therefore, characterized by uniform convergence of finite differences (§9).

From these equivalences a full characterization of Hölder regularity $\dot{C}^r$ (for all $r > 0$) naturally emerges in terms of discrete sequences (§10). The main result of this paper is an easily implemented, optimal regularity estimate derived in §11. This estimate is then compared to other related work [3]–[12], [23]. A sharp upper bound for Hölder regularity is also derived in §13.

As a general rule, the first parts of the theorems derived in this paper show that a given property of the g_n^j's implies the corresponding regularity property of the limit function $\varphi(x)$. The second parts prove the converse implication, which is useful for proving optimality of regularity estimates and generally assumes the stability condition.

The purpose of this paper is close to the one of Daubechies and Lagarias in [8], [9]. They studied the existence, uniqueness, and regularity of solutions to "two-scale difference equations." We shall see in §5 that the limit function $\varphi(x)$ associated to mask g_n indeed satisfies the following two-scale difference equation.

$$\varphi(x) = \sum_k g_k \, \varphi(2x - k).$$

Although it can be shown [9] that a solution to this equation is not necessarily the limit function of the subdivision scheme with mask g_n, both approaches are closely related. In fact, the study of regularity of solutions to two-scale difference equations can be reduced, after suitable transformation [2], to that of limit functions $\varphi(x)$ of a binary subdivision scheme. However, the contents, formulation, and proofs of this paper differ notably from [8], [9]; Daubechies and Lagarias derive conditions for the existence of $\mathbf{L}^1$-solutions to two-scale difference equations and estimate global and local regularity of solutions that are, in fact, limit functions. This paper concentrates on the determination of *optimal* estimates for global regularity of limit functions, with interpretation in terms of discrete sequences and comparison with Fourier-based techniques. (Local regularity may also be investigated using the methods of this paper [19].)

It was pointed out to the author by one of the referees that the framework of this paper is very close to that of Dyn and Levin [14], [15]. I learned that several results were derived independently in [14], [15] for the study of C^N limit functions (see §§7 and 9).

2. Polynomial notation. Subdivision schemes have been mostly described using matrices or Fourier transforms [6], [8]–[11]. Throughout this paper we often use the polynomial description

$$U(X) = \sum_{n=0}^{L-1} u_n X^n$$

of any causal sequence u_n of length L ($u_n = 0$ for $n < 0$ and $n \geq L$). Since sequences of finite length can always be made causal by shifting, we assume all sequences causal in the following. This notation was adopted in [14], [15], which uses Laurent polynomials for noncausal sequences.

In polynomial notation, the up-scaling operator (1.1) reads

$$U(X) \xrightarrow{\mathcal{G}} V(X) = G(X)\, U(X^2), \tag{2.1}$$

which shows that it can be seen as resulting from two operations.

1. Change X to X^2 in $U(X)$, i.e., insert zeros between every two samples of u_n.
2. Multiply by $G(X)$, i.e., convolve the result with the mask g_n.

In other words, the operator (1.1), (2.1) "smooths" u_n at twice its rate, and (2.1) can be seen as a discrete version of a dilation by two: $f(x) \to f(x/2)$.

Iterating (2.1) gives the polynomial $G^j(X)$, associated to the sequence g_n^j (1.2).

$$G^j(X) = G(X)\, G(X^2)\, G(X^4) \cdots G(X^{2^{j-1}}). \tag{2.2}$$

This equation fully describes binary subdivision schemes in terms of polynomials (see §4 when the initial sequence is not δ_n). It can be rewritten in recursive form in two ways.

$$G^{j+1}(X) = G(X)\, G^j(X^2), \quad \text{i.e., } g_n^{j+1} = \sum_k g_k^j\, g_{n-2k}, \tag{2.3}$$

$$G^{j+1}(X) = G^j(X)\, G(X^{2^j}), \quad \text{i.e., } g_n^{j+1} = \sum_k g_k\, g_{n-2^j k}^j. \tag{2.4}$$

Both are useful in the sequel. Equation (2.3) is simply a rewriting of definition (1.2), while (2.4) links binary subdivision schemes to two-scale difference equations (see §5). We shall also consider (2.2) for polynomials other than $G(X)$. Given any polynomial $U(X)$, $U^j(X)$ (with a superscript index j) is

$$U^j(X) = U(X)\, U(X^2)\, U(X^4) \cdots U(X^{2^{j-1}}). \tag{2.5}$$

In this paper we use l^1 and l^∞-norms of discrete sequences in terms of polynomials,

$$\|U(X)\|_\infty = \max_k |u_k|,$$
$$\|U(X)\|_1 = \sum_k |u_k|,$$

and the following well-known inequality:

$$\|U(X)\, V(X)\|_\infty \leq \|V(X)\|_1 \|U(X)\|_\infty. \tag{2.6}$$

For polynomials with real coefficients, the following useful inequality holds whenever $V(X)$ has no roots *on* the unit circle.

$$\|U(X)\|_\infty \leq c_V\, \|U(X)\, V(X)\|_\infty, \tag{2.7}$$

where c_V is a constant depending only on $V(X)$.

Proof. This is trivially true for infinite sequences when the roots of $V(X)$ lie outside the unit circle; the constant c_V is then the converging l^1-norm of the Laurent series coefficients of $1/V(X)$, which is analytic in the complex-domain region $|X| \le 1$. Here, since v_n is a sequence of finite length L, the index reversal $n \leftrightarrow L-1-n$ in v_n transforms roots of $V(X)$ inside the unit circle into roots outside the unit circle. Hence (2.7) holds when $V(X)$ has no roots *on* the unit circle. □

3. Definition of convergent subdivision schemes. Various definitions of convergent binary subdivision schemes have been proposed in the literature [6], [10]–[15], [17]. In this paper we restrict ourselves to pointwise or uniform convergence. It is easy to define a limit function in the case of interpolatory subdivision schemes as defined in the introduction: Since for such schemes one has $g_n^{j+1} = g_n^j$, the function $\varphi(x)$ can always be defined on dyadic rationals by

$$\varphi(n2^{-j}) = g_n^j. \tag{3.1}$$

For example, determining a continuous limit function amounts to finding a continuous extension of (3.1) to the real axis [11], [12].

The situation is more complex for general subdivision schemes since the values of g_n^j are not necessarily preserved as j increases. In order to "converge" to a limit function, the sequence g_n^j must be somehow interpolated. The idea is that the resulting sequence of functions of the continuous variable x, indexed by j, converges (pointwise or uniformly) to a limit function $\varphi(x)$ under some conditions on the mask g_n.

In [6], Daubechies chose to interpolate the sequence g_n^j by stepwise constant functions: she defined $\varphi(x)$ as the limit of $\varphi^j(x) = g^j_{\lfloor 2^j x + 1/2 \rfloor}$ as $j \to \infty$. Other kinds of interpolation are possible and yield similar results. Among possible choices are $g^j_{\lfloor 2^j x \rfloor}$, $g^j_{\lceil 2^j x \rceil}$, and the continuous linear interpolation function $\varphi^j_{\mathcal{L}}(x)$ obtained by connecting the points g_n^j by segments as in Figs. 2 and 3. All such interpolation functions $\varphi^j(x)$ agree at the "knots" $n2^{-j}$, i.e., $\varphi^j(n2^{-j}) = g_n^j$. In this paper we use a stronger definition that gives some flexibility on the way the subdivision scheme is interpolated.

DEFINITION 3.1. A binary subdivision scheme g_n^j (1.2) converges (pointwise) to a limit function $\varphi(x)$ if, for any sequence of integers n_j satisfying

$$|n_j 2^{-j} - x| \le c\, 2^{-j} \tag{3.2}$$

(where c is a constant independent of j), we have

$$\varphi(x) = \lim_{j\to\infty} g^j_{n_j}. \tag{3.3}$$

The convergence is, moreover, *uniform* if

$$\sup_x |\varphi(x) - g^j_{n_j}| \to 0 \quad \text{as } j \to \infty. \tag{3.4}$$

Note that the sequence n_j depends on x, hence $g^j_{n_j}$ can be regarded as a function of x. The flexibility comes from the arbitrary choice of n_j satisfying (3.2)[1]. In particular,

[1] It seems natural to impose the more general condition $n_j 2^{-j} \to x$ as $j \to \infty$ in place of (3.2). But then $n_j - 2^j x$ is allowed to increase indefinitely as $j \to \infty$, and the resulting definition becomes too strong for deriving some of the results of this paper.

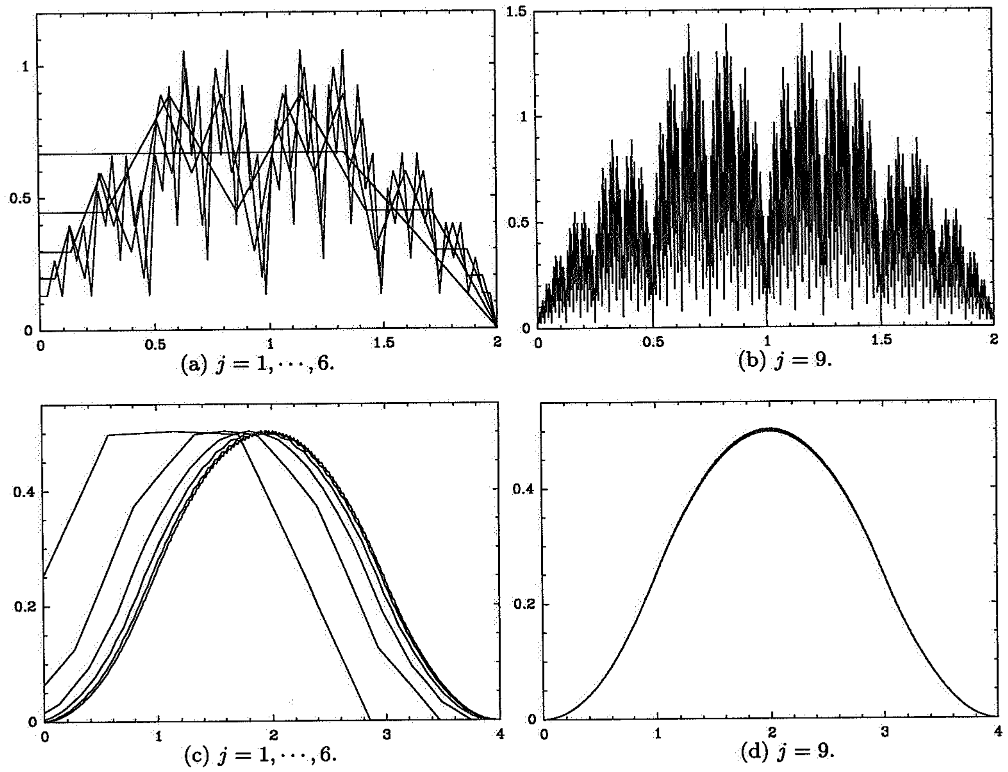


FIG. 2. *Two examples of diverging dyadic up-scaling schemes. Figures* (a), (c) *show six plots of the discrete sequences* g_n^j ($j = 1, \cdots, 6$), *represented with values joined by segments and plotted against* $n2^{-j}$. *Figures* (b), (d) *show the obtained curve after* 9 *iterations.* (a), (b) $g_0 = g_1 = g_2 = \frac{2}{3}$, $g_n = 0$ *elsewhere. Here* $G(-1) = \frac{2}{3} \neq 0$. *Note that up-scaling follows a fractal law.* (c), (d) $g_0 = g_4 = 0.5$, $g_1 = g_3 = 0.99$, $g_2 = 1$, $g_n = 0$ *elsewhere, renormalized such that* $G(1) = 2$. *Here* $G(-1) \approx 0.01$ *is so small that divergence is not obvious at the level of the figure. Divergence is here due to oscillations that occur in the graph of* g_n^j. *Although very small, these oscillations are so rapid that they preclude convergence.*

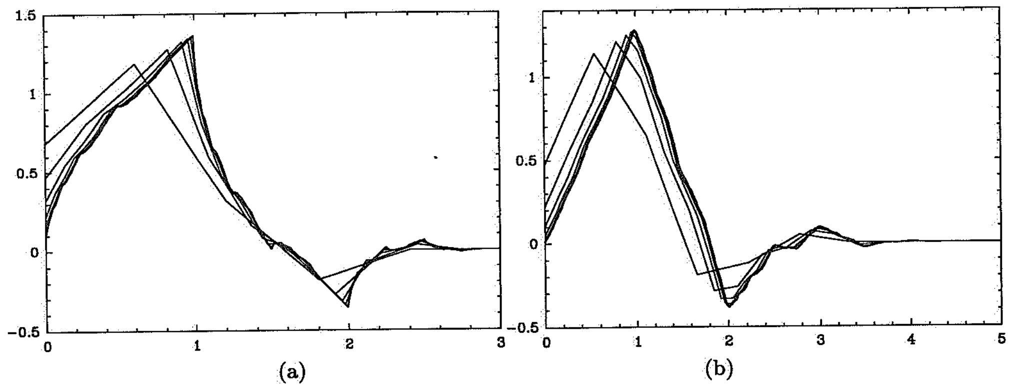


FIG. 3. *Two examples of converging dyadic up-scaling schemes (after* [6]). *The* g_n^j*'s are plotted against* $n2^{-j}$ *for* $j = 1, \cdots, 6$, *with coefficients joined by segments, so that the behavior of the "slopes" can be observed.* (a) *The limit function is* $\dot{C}^{0.5500\cdots}$ *and not* C^1; *therefore, slopes are allowed to increase indefinitely near the peaks of the limit function.* (b) *The limit function is* $\dot{C}^{1.0878\cdots}$; *therefore,* C^1. *Slopes are constrained to be bounded, especially near the apparent "peaks" of the limit function.*

the above "stepwise interpolation" examples are recovered by letting $n_j = \lfloor 2^j x + \frac{1}{2} \rfloor$, $\lfloor 2^j x \rfloor$, $\lceil 2^j x \rceil$, respectively.

Convergence of the linear interpolations $\varphi^j_{\mathcal{L}}(x)$ of the g^j_n is also implied by Definition 3.1. This comes from the inequality $|\varphi^j_{\mathcal{L}}(x) - g^j_{n_j}| \le |g^j_{n_j+1} - g^j_{n_j}|$, which holds for $n_j = \lfloor 2^j x \rfloor$ because $\varphi^j_{\mathcal{L}}$ is monotonous on each interval $[n2^{-j},\ (n+1)2^{-j}]$. From (3.3) this clearly implies $|\varphi^j_{\mathcal{L}}(x) - g^j_{n_j}| \to 0$; hence $|\varphi(x) - \varphi^j_{\mathcal{L}}(x)| \to 0$. Convergence of smoother interpolation functions such as splines are similarly implied by Definition 3.1.

Still another definition of convergence was proposed in [14], [15] by Dyn and Levin. For example, uniform convergence is expressed as the existence of a continuous function $\varphi(x)$ such that $\sup_n |g^j_n - \varphi(n2^{-j})| \to 0$ as $j \to \infty$. Note that this is implied by the uniform convergence of the linear interpolations $\varphi^j_{\mathcal{L}}(x)$; since the $\varphi^j_{\mathcal{L}}(x)$'s are continuous, their uniform limit is continuous and we have $\sup_n |g^j_n - \varphi(n2^{-j})| = \sup_n |\varphi^j(n2^{-j}) - \varphi(n2^{-j})|$, which $\to 0$ as $j \to \infty$.

Therefore, Definition 3.1 implies all the others. In fact, §7 shows that all definitions of uniform convergence presented in this section are equivalent. Since the results of this paper are mostly based on uniform convergence, they remain valid for various frameworks used in other works (in particular [6], [14]).

It is possible, however, to find examples for which *pointwise* convergence holds for one definition and not for another. Consider, for example, $G(X) = X$ for which $g^j_n = 1$ if $n = 2^j - 1$ and zero otherwise. Here pointwise convergence of stepwise—or linear—interpolations holds and we easily find that the limit exists and is $\varphi(x) \equiv 0$. (This is a typical example of a pointwise, nonuniform convergence to a continuous function.) But convergence does not hold for all x in the sense of Definition 3.1 because the scheme diverges for $x = 1$ (take $n_j = 2^j - 1$). Therefore, Definition 3.1 forbids "sharp discontinuities" about which $|g^j_{n_j+1} - g^j_{n_j}|$ does not tend to zero as $j \to \infty$.

The choice $G(X) = 1 + X$ behaves similarly. For $x \ne 0$ and $x \ne 1$, the scheme converges to a limit function equal to 1 for $0 < x < 1$, and zero for $x < 0$ or $x > 1$; however, depending on the choice of the definition of pointwise convergence, it either converges or diverges for $x = 0$ and $x = 1$.

4. Basic properties. Several basic properties and simplifications for the study of convergent binary subdivision schemes follow easily from the description (1.2), (2.2). First note that all functions considered in this paper are compactly supported because the mask g_n is of finite length L. In fact, we easily find that the length of g^j_n is $(2^j - 1)(L - 1) + 1$ by estimating the polynomial degree of (2.2). Therefore, $\varphi(x)$, if it exists, has compact support $[0,\ L - 1]$. This property makes many technical proofs easier.

Second, we can restrict the initial sequence in (1.2) to δ_n. For an arbitrary initial sequence of finite length h_n, (2.2) is simply multiplied by $H(X^{2^j})$:

$$H^j(X) = G^j(X)\, H(X^{2^j}). \tag{4.1}$$

The iterated sequence is, therefore, $h^j_n = \sum_k h_k\, g^j_{n-2^j k}$. From Definition 3.1, the limit function becomes $\psi(x) = \sum_k h_k \varphi(x - k)$ instead of $\varphi(x)$. Moreover, since both functions are compactly supported, $\varphi(x)$ itself can be written as $\varphi(x) = \sum_k (h^{-1})_k \psi(x - k)$, where $(h^{-1})_n$ is the convolutional inverse of h_n, i.e., $\sum_k (h^{-1})_k h_{n-k} = \delta_n$. The convergence and regularity properties of $\varphi(x)$ and $\psi(x)$ are, therefore, the same, and we can restrict ourselves to the study of the g_n^j's and $\varphi(x)$.[2]

In order that $\varphi(x)$ is well defined or does not vanish for all x, the iterated sequences g_n^j should neither diverge nor tend to zero as $j \to \infty$. The following proposition shows that this requires some basic conditions to be fulfilled by mask g_n.

PROPOSITION 4.1. If $\varphi(x) \neq 0$ exists for some $x \in \mathbf{R}$, then

$$\sum_k g_{2k} = \sum_k g_{2k+1} = 1, \quad \text{i.e., } G(1) = 2 \text{ and } G(-1) = 0. \tag{4.2}$$

Proof. The key point is to consider the even and odd-indexed sequences g_{2n}^j and g_{2n+1}^j separately. Let $y = x/2$ and $n_j = n_j(y)$ be a sequence of integers satisfying (3.2) for y. On one hand, from Definition 3.1, the common limit of $g_{2n_j}^j$ and $g_{2n_j+1}^j$ as $j \to \infty$ is $\varphi(2y) = \varphi(x)$. But from (2.3) we also have

$$g_{2n}^j = \sum_k g_{2k}\, g_{n-k}^{j-1},$$

$$g_{2n+1}^j = \sum_k g_{2k+1}\, g_{n-k}^{j-1}.$$

Letting $n = n_j$ and applying Definition 3.1 to the right-hand sides of these equations, we obtain that their respective limits as $j \to \infty$ are $(\sum_k g_{2k})\varphi(2y)$ and $(\sum_k g_{2k+1})\varphi(2y)$. By identification we therefore have

$$\varphi(2y) = \left(\sum_k g_{2k}\right)\varphi(2y) = \left(\sum_k g_{2k+1}\right)\varphi(2y).$$

Dividing the members of this equality by $\varphi(2y) \neq 0$ gives (4.2). □

Condition (4.2) may be interpreted as follows. On one hand, $G(1) = 2$ is just a normalization condition that ensures that the order of magnitude of g_n^j is preserved when $j \to \infty$. On the other hand, the fact that $G(X)$ must have at least one zero at $X = -1$ is a "local" requirement. For example, it ensures that the g_n^j's, for large j, do not rapidly oscillate in n between two different limits, $(\sum_k g_{2k})\varphi(2y)$ and $(\sum_k g_{2k+1})\varphi(2y)$. Figure 2 illustrates this phenomenon on a particular example (see also [20]).

Note that (4.2) is not sufficient to ensure convergence. As an example, consider $G(X) = 1 + X^3$. Here $G^j(X)$ is a polynomial in X^3; therefore, g_n^j vanishes for $n \neq 3k$ $(k \in \mathbf{Z})$, whereas $g_{3k}^j = 1$. It, therefore, cannot converge to a limit function. (Section 7 gives a necessary and sufficient condition for uniform convergence.)

5. Exact computation of limit functions. Assume that the limit function $\varphi(x)$ of a binary subdivision scheme g_n^j exists for all $x \in \mathbf{R}$. This section derives a simple, easily implementable method for computing the *exact* values of $\varphi(x)$ at dyadic rationals $x = n2^{-j}$, $n \in \mathbf{Z}$, with a finite number of operations. The starting point is the *two-scale difference equation* [8], [9] satisfied by $\varphi(x)$:

$$\varphi(x) = \sum_k g_k\, \varphi(2x - k). \tag{5.1}$$

[2] Note that this restriction works only for initial sequences of *finite* length. If, e.g., $h_n = 1$ for all $n \in \mathbf{Z}$, then using the definition $h_n^{j+1} = \mathcal{G}\{h_n^j\}$ and Proposition 4.1, it easily follows by induction on j that $h_n^j \equiv 1$. Hence $\psi(x) = \sum_k \varphi(x-k) \equiv 1$ is C^∞, whatever the regularity order of $\varphi(x)$.

This equation, which was mentioned in the introduction, is easily derived by using (2.4) for $n = n_j$ (3.2) and applying Definition 3.1.

Now, let

$$\Phi^j(X) = \sum_n \varphi(n2^{-j})X^n \tag{5.2}$$

be the polynomial associated to the sequence $\varphi(n2^{-j})$. Taking $x = n2^{-j-1}$ in the two-scale difference equation yields $\varphi(n2^{-j-1}) = \sum_k g_k\, \varphi(2^{-j}(n - 2^j k))$, i.e., $\Phi^{j+1}(X) = \Phi^j(X)G(X^{2^j})$. By iteration we have

$$\Phi^j(X) = \Phi(X)G^j(X), \tag{5.3}$$

where

$$\Phi(X) = \Phi^0(X) = \sum_n \varphi(n)X^n. \tag{5.4}$$

Equation (5.3) is very useful, since it links the values of the iterated sequences g_n^j to the ones of the limit function $\varphi(n2^{-j})$. The latter are simply obtained from the g_n^j's by convolving them with the sequence $\varphi(n)$, provided that the $\varphi(n)$'s can be predetermined.

There are several methods for precomputing $\varphi(n)$, which is, by definition, the limit of $g_{n2^j}^j$ as $j \to \infty$. First note that we have, from (2.4),

$$g_{n2^{j+1}}^{j+1} = \sum_k g_k\, g_{(2n-k)2^j}^j = \mathcal{G}^*\{g_{n2^j}^j\},$$

where $\mathcal{G}^*$ is the following transposed operator [6], [18] of $\mathcal{G}$ (1.1):

$$u_n \xrightarrow{\mathcal{G}^*} v_n = \sum_{k\in\mathbf{Z}} g_k\, u_{2n-k}. \tag{5.5}$$

Therefore, $\varphi(n)$ can be determined as the limit of $(\mathcal{G}^*)^j\{\delta_n\}$ as $j \to \infty$.

Another method stems from the resulting equality

$$\varphi(n) = \mathcal{G}^*\{\varphi(n)\}. \tag{5.6}$$

The sequence $\varphi(n)$, $n = 0, \cdots, L-1$ (where L is the length of the mask g_n), is here determined, up to normalization, as the eigenvector of the operator $\mathcal{G}^*$ associated to the eigenvalue 1. To obtain a normalization for $\varphi(n)$, rewrite (5.5) under polynomial form

$$V(X^2) = (U(X)G(X) + U(-X)G(-X))/2. \tag{5.7}$$

Since we have, by Proposition 4.1, $G(1) = 2$, and $G(-1) = 0$, it follows that $\mathcal{G}^*$ preserves the sum of sequences; hence $\sum_n \varphi(n) = \sum_n g_{n2^j}^j = \sum_n g_{2n} = 1$, i.e.,

$$\Phi(1) = 1. \tag{5.8}$$

6. Stability. There is an exceptional class of limit functions $\varphi(x)$ for which the regularity estimates derived in this paper will not always be optimal. Optimality, as well as some other results of this paper, will be proven only in the case of "stability," in the sense of the following definition.

DEFINITION 6.1. A binary subdivision scheme converging to a nonzero limit function (or its limit function $\varphi(x) \not\equiv 0$) is *stable* if no root of $\Phi(X)$ (5.4) lies on the unit circle, i.e.,

$$\sum_n \varphi(n)\, e^{in\omega} \neq 0 \quad \text{for all } \omega \in \mathbf{R}.$$

The terminology "stable" comes from (2.6), (2.7) written for $V(X) = \Phi(X)$,

$$c_1 \|U(X)\|_\infty \leq \|U(X)\Phi(X)\|_\infty \leq c_2 \|U(X)\|_\infty,$$

which means that the filter of impulse response $\varphi(n)$ and its inverse are numerically stable for finite length sequences. The stability condition slightly restricts the choice of the scaling sequence g_n. For example, if the mask length is $L = 4$, "unstable" $\varphi(x)$'s are such that $g_0 = g_3$ and $g_1 = g_2$. All (real-valued) limit functions are stable for lengths up to 3. Note that an interpolatory subdivision scheme is always stable since it has the property that $\varphi(n/2) = g_n$ (see (3.1) for $j = 1$); hence $\Phi(X) \equiv 1$.

In fact, in the rest of this paper, stability can be replaced by the even weaker condition that there exists $x \in \mathbf{R}$ such that

$$\sum_n \varphi(n + x)\, e^{in\omega} \neq 0 \quad \text{for all } \omega \in \mathbf{R} \tag{6.1}$$

(a similar, but stronger stability condition appears in [15]). Condition (6.1) comes from (5.3) where n is replaced by $n + x$, for any fixed number x. That is,

$$\Phi_x^j(X) = \Phi_x(X) G^j(X)$$

where

$$\Phi_x^j(X) = \sum_n \varphi((n + x)2^{-j}) X^n$$

and $\Phi_x(X) = \Phi_x^0(X)$.

Although almost all convergent subdivision schemes are stable, it is easy to construct unstable ones (even with definition (6.1)). As in the preceding section we have the following generalization of (5.6), (5.7).

$$\Phi_x(X^2) = (\Phi_{2x}(X)G(X) + \Phi_{2x}(-X)G(-X))/2.$$

Therefore, any polynomial mask $G(X)$ divisible by $(X^2 - e^{i\omega})$, $\omega \neq 0$, yields unstability since we have $\Phi_x(e^{i\omega}) = 0$.

I conjecture that the converse holds, i.e., stability (in the weak form (6.1)) is equivalent to the condition that $G(X)$ has no pair of opposite zeros $(e^{i\omega/2}, e^{i(\omega/2+\pi)})$ on the unit circle. If this conjecture is true, then the regularity estimates presented below will be optimal under the simple condition above on the mask coefficients g_n, which is easy to check. When this condition is not satisfied, it is possible to apply a trick as shown at the end of §9 which allows one to consider another, *stable* binary subdivision scheme which has the same regularity properties.

7. Continuous limit functions. The framework of uniform convergence (3.4) is shown to be very convenient in the sequel, and the following theorem shows that all stable continuous limit functions are obtained by uniform convergence. We shall then derive a necessary and sufficient condition for uniform convergence in all cases.

THEOREM 7.1. *Assume that a binary subdivision scheme converges pointwise to a limit function $\varphi(x)$ for all $x \in \mathbf{R}$. If the convergence is uniform, then $\varphi(x)$ is continuous. The converse is true if $\varphi(x)$ is stable.*

Proof. ($\Rightarrow$) In §3 we have seen that uniform convergence (3.4) implies uniform convergence of linear interpolations $\varphi_{\mathcal{L}}^j(x)$ of the g_n^j's to $\varphi(x)$. Since this is a uniformly convergent sequence of compactly supported continuous functions, $\varphi(x)$ is continuous.

($\Leftarrow$) We have

$$\sup_x |\varphi(x) - g_{n_j}^j| \leq \sup_x |\varphi(x) - \varphi(n_j 2^{-j})| + \sup_{n_j} |\varphi(n_j 2^{-j}) - g_{n_j}^j|$$

where n_j is a sequence of integers satisfying (3.2). Since $\varphi(x)$ is compactly supported and continuous, it is uniformly continuous. Therefore $\sup_x |\varphi(x) - \varphi(n_j 2^{-j})| \to 0$. The other term can be written $\sup_n |\varphi(n2^{-j}) - g_n^j| = \|\Phi^j(X) - G^j(X)\|_\infty$. From (5.3) we have $\Phi(X)(\Phi^j(X) - G^j(X)) = (\Phi(X) - 1)\Phi^j(X)$. Since (5.8) holds, $X - 1$ divides $\Phi(X) - 1$ and we can write, using (2.6), $\|\Phi(X)(\Phi^j(X) - G^j(X))\|_\infty \leq c\,\|(X-1)\,\Phi^j(X)\|_\infty$. The latter norm is $\sup_n |\varphi(n2^{-j}) - \varphi((n-1)2^{-j})|$, which tends to zero as $j \to \infty$ because $\varphi(x)$ is uniformly continuous. Now we can use (2.7) with $V(X) = \Phi(X)$ since $\Phi(X)$ is stable. This yields $\|\Phi^j(X) - G^j(X)\| \to 0$ as $j \to \infty$, which ends the proof. □

It is an open problem to find a limit function $\varphi(x)$ for which the convergence is not uniform (in the sense of Definition 3.1). That would imply that $\varphi(x)$ is unstable or discontinuous.

We now derive a necessary and sufficient condition for uniform convergence of a binary subdivision scheme g_n^j to a (continuous) limit function $\varphi(x)$. (By Theorem 7.1, this also gives a necessary and sufficient condition for the continuity of a stable limit function.) We need the following lemma, which will also be useful for deriving an optimal regularity estimate in §11.

LEMMA 7.2. *Assume $G(-1) = 0$ and let $F(X) = G(X)/(1 + X)$. The sequence of the first-order differences of g_n^j,*

$$d_n^j = g_n^j - g_{n-1}^j,$$

follows a binary subdivision scheme with polynomial mask $F(X)$ and initial sequence's polynomial $1 - X$.

In addition, for any fixed positive integer i, we have

$$\max_n |g_{n+1}^j - g_n^j| \leq c \left(\max_{0 \leq n \leq 2^i - 1} \sum_k |f_{n-2^i k}^i| \right)^{j/i}, \tag{7.1}$$

where f_n is the mask associated to the polynomial $F(X)$ and c is a constant independent of j.

Proof. Let $D^j(X) = (1 - X)G^j(X)$ be the polynomial associated to d_n^j. We have

$$\begin{aligned} D^j(X) &= (1 - X)(1 + X)(1 + X^2) \cdots (1 + X^{2^{j-1}})\, F^j(X) \\ &= (1 - X^{2^j}) F^j(X), \end{aligned}$$

which shows, by (4.1), that d_n^j follows the announced subdivision scheme.

Using (2.2) in the above equation, we can write $D^{i+\ell}(X) = F^i(X)\, D^\ell(X^{2^i})$, which also reads $d_n^{i+\ell} = \sum_k f^i_{n-2^ik} d^\ell_k$. Majorating yields

$$\|D^{i+\ell}(X)\|_\infty \le \left(\max_n \sum_k |f^i_{n-2^ik}|\right) \|D^\ell(X)\|_\infty,$$

which, by iteration for $j = \ell + ni$, $0 \le \ell \le i-1$, gives (7.1) where c depends only on the fixed integers i and ℓ. □

THEOREM 7.3. *A binary subdivision scheme g_n^j converges uniformly to a (continuous) limit function if and only if $G(1) = 2$, $G(-1) = 0$ and*

$$\max_n |g^j_{n+1} - g^j_n| \to 0 \quad \textit{as } j \to \infty. \tag{7.2}$$

Moreover, there exists $\alpha > 0$ such that

$$\max_n |g^j_{n+1} - g^j_n| \le c\, 2^{-j\alpha}. \tag{7.3}$$

Proof. ($\Rightarrow$) immediately results from Proposition 4.1 and the inequality

$$\max_n |g^j_{n+1} - g^j_n| \le \sup_x |\varphi(x) - g^j_{n_j+1}| + \sup_x |\varphi(x) - g^j_{n_j}|.$$

($\Leftarrow$) We first prove that (7.3) is implied by conditions $G(1) = 2$, $G(-1) = 0$ and (7.2). First note that from the first part of Lemma 7.2 we have $(1-X)G^j(X) = (1-X^{2^j})F^j(X)$, i.e., $g^j_n - g^j_{n-1} = f^j_n - f^j_{n-2^j}$. Write

$$\begin{aligned} f^j_n &= (f^j_n - f^j_{n-2^j}) + (f^j_{n-2^j} - f^j_{n-2\cdot 2^j}) + \cdots \\ &= (g^j_n - g^j_{n-1}) + (g^j_{n-2^j} - g^j_{n-2^j-1}) + \cdots. \end{aligned}$$

The number of terms in the sums is bounded by L because the length of f^j_n is bounded by 2^jL. From (7.2) each term tends to zero uniformly with respect to n; hence so does f^j_n. Therefore, there exists a (sufficiently large) index i such that $\max_n |f^i_n| \le \varepsilon_i < 1/L$. Now since the number of terms in the sum in (7.1) is bounded by L, the second part of Lemma 7.2 gives (7.3) with $\alpha = -\log_2(L\varepsilon_i)/i > 0$.

To prove the converse part of the theorem, consider $\sup_{n_j} |g^{j+1}_{n_{j+1}} - g^j_{n_j}|$, where n_j satisfies (3.2). This equals $\sup_{n_j} |g^{j+1}_{2n_j+m_j} - g^j_{n_j}|$, where $m_j = n_{j+1} - 2n_j$ is a bounded integer. Now, from (2.3) we can write $g^{j+1}_{2n+m} = \sum_k g_{2k+m}\, g^j_{n-k}$. Therefore, the sequence $g^{j+1}_{2n+m} - g^j_n$ is a convolved version of g^j_n; its associated polynomial can be written in the form $U_m(X)G^j(X)$. But from (4.2), we have $\sum_k g_{2k+m} = 1$ (for all m), and, therefore, $U_m(1) = 0$. Using (2.6), it follows that

$$\sup_{n_j} |g^{j+1}_{2n_j+m_j} - g^j_{n_j}| = \|U_{m_j}(X)G^j(X)\|_\infty \le c'\, \|(1-X)G^j(X)\|_\infty,$$

where c' is a constant (independent of j since m_j is bounded). From (7.3) the latter norm is bounded by $c\,c'\,2^{-j\alpha}$. We, therefore, end up with $\sup_{n_j} |g^{j+1}_{n_{j+1}} - g^j_{n_j}| \le cc'\,2^{-j\alpha}$. Iterating this inequality, we obtain, for any $\ell > 0$,

$$\sup_{n_j} |g^{j+\ell}_{n_{j+\ell}} - g^j_{n_j}| \le c\,c'\,(2^{-(j+\ell-1)\alpha} + \cdots + 2^{-(j+1)\alpha} + 2^{-j\alpha}) \le c''\,2^{-j\alpha}, \tag{7.4}$$

which shows that the sequence of functions $g^j_{n_j(x)}$ is a uniform Cauchy sequence, which converges uniformly to a continuous limit function $\varphi(x)$. □

This theorem has several interesting consequences. First, we shall see in §8 that (7.3), in fact, implies that $\varphi(x)$ is Lipschitz of order α, which is stronger than continuity.[3] Therefore, by Theorem 7.1, a continuous stable limit function is automatically Lipschitz of order α for some $\alpha > 0$.

Second, note that the necessary and sufficient condition is quite weak and intuitive: it is sufficient that the differences $g^j_{n+1} - g^j_n \to 0$ uniformly as $j \to \infty$ to obtain a continuous limit function.[4] In fact, we easily find that (7.2) holds for any definition of uniform convergence presented in §3.1. (For example, any uniformly convergent sequence of interpolating functions $\varphi^j(x)$ of the g^j_n's such that $g^j_n = \varphi^j(n2^{-j})$ clearly gives (7.2).) Since we have seen that Definition 3.1 for uniform convergence implies the others, it follows that *all* these definitions of uniform convergence are equivalent.

In particular, some results derived in this paper have been derived in the framework of Dyn and Levin [14], [15] as well. The necessary and sufficient condition (7.2) appears in [14] for interpolatory subdivision schemes and was first derived in [15, Thm. 3.2] for general binary subdivision schemes—using the (apparently) weaker definition mentioned in §3.1—in a slightly different but equivalent form, namely, $\max_n |f^j_n| \to 0$ as $j \to \infty$. Theorem 7.3 was included here in order that this paper be self-contained, since some material presented in this section is also useful in the sequel.

8. Lipschitz limit functions. In this section we want to characterize Lipschitz limit functions. Recall that $\varphi(x)$ is said to be *Lipschitz of order* α $(0 < \alpha \leq 1)$, $\varphi(x) \in \dot{C}^\alpha$, if we have for all x and $h \in \mathbf{R}$,

$$|\varphi(x+h) - \varphi(x)| \leq c\,|h|^\alpha, \tag{8.1}$$

where c is a constant. Here, $\varphi(x)$ is compactly supported, and (8.1) needs to be satisfied only for small h's. Since the spaces $\dot{C}^\alpha$, for $0 < \alpha \leq 1$, interpolate between C^0 and C^1, a $\dot{C}^\alpha$-function will be said to be *regular of order* α. There is a slight irritation in that C^1 and $\dot{C}^1$ do not coincide; for example, a linear spline function is $\dot{C}^1$ but not differentiable at its knots.

THEOREM 8.1. *If* $G(1) = 2$, $G(-1) = 0$, *and*

$$\max_n |g^j_{n+1} - g^j_n| \leq c\,2^{-j\alpha} \tag{8.2}$$

for some $0 < \alpha \leq 1$, *then the binary subdivision scheme converges uniformly to a* $\dot{C}^\alpha$ *limit function. The converse is true if* $\varphi(x)$ *is stable.*

In addition, the more regular the limit, the faster the convergence to this limit:

$$\sup_x |\varphi(x) - g^j_{n_j}| \leq c\,2^{-j\alpha} \tag{8.3}$$

for any sequence n_j *of integers satisfying* (3.2).

Proof. (⇒) Let us first prove (8.3). Since (8.2) holds, we are in the framework of Theorem 7.3, and (7.4) holds. Letting $\ell \to \infty$ in (7.4) gives (8.3).

[3] Using the same proof as the one of Theorem 7.3, we can show that when (3.2) is replaced by $n_j 2^{-j} \to x$ as $j \to \infty$, uniform convergence requires (7.3) for $\alpha = 1$, which corresponds to almost continuously differentiable functions.

[4] In contrast, the *slopes* $(g^j_{n+1} - g^j_n)/(2^{-j})$ may indefinitely increase (see next section).

We now prove that $\varphi(x)$ is $\dot{C}^{\alpha}$. Let $n_j = n_j(x)$ satisfy (3.2) (for all $x \in \mathbf{R}$) and consider the inequality

$$\sup_x |\varphi(x+h) - \varphi(x)| \leq \sup_x |\varphi(x+h) - g^j_{n_j(x+h)}| + \sup_x |g^j_{n_j(x+h)} - g^j_{n_j(x)}| + \sup_x |g^j_{n_j(x)} - \varphi(x)|.$$

By (8.3), the first and third terms in the right-hand side of this inequality are bounded by $c\,2^{-j\alpha}$. Assume, for example, that $|h| < 1$. If $h > 0$, choose $n_j(x) = \lfloor x2^j \rfloor$, otherwise choose $n_j(x) = \lceil x2^j \rceil$. A simple calculation yields $|n_j(x+h) - n_j(x)| \leq |n_j(h)| + \varepsilon$, where $\varepsilon = 0$ or ± 1. Now, let j be such that $2^{-j} \leq |h| < 2^{-j+1}$. This gives $|n_j(h)| = 1$; hence we find, from (8.2), that $\sup_x |g^j_{n_j(x+h)} - g^j_{n_j(x)}| \leq c\,2^{-j\alpha}$. Putting all inequalities together yields $\sup_x |\varphi(x+h) - \varphi(x)| \leq c' 2^{-j\alpha} \leq c|h|^{\alpha}$, i.e., $\varphi(x)$ is $\dot{C}^{\alpha}$.

($\Leftarrow$) $G(1) = 2$, $G(-1) = 0$ result from Proposition 4.1. Since $\varphi(x)$ is $\dot{C}^{\alpha}$, we have $|\varphi((n+1)2^{-j}) - \varphi(n2^{-j})| \leq c\,2^{-j\alpha}$, i.e.,

$$\|(1-X)\Phi^j(X)\|_{\infty} = \|\Phi(X)(1-X)G^j(X)\|_{\infty} \leq c\,2^{-j\alpha}$$

(the first equality comes from (5.3)). Because $\varphi(x)$ is stable, we can apply (2.7) to obtain the inequality $\|(1-X)G^j(X)\|_{\infty} \leq c'\,2^{-j\alpha}$, which is (8.2). □

This theorem provides an intuitive interpretation of regularity of order $0 \leq \alpha < 1$ for binary subdivision schemes: regularity $\dot{C}^{\alpha}$ holds if and only if the absolute values of the "slopes" $(g^j_{n+1} - g^j_n)/2^{-j}$ of the discrete curves g^j_n's (see next section) grow less than $2^{j(1-\alpha)}$ when j indefinitely increases. For example, if the slopes of g^j_n are always bounded for all j's, then $\varphi(x)$ is $\dot{C}^1$. On the contrary, less regularity allows slopes to increase indefinitely and the resulting limit function, although continuous, may present a "fractal" structure as shown in Fig. 3. Note that in this case, (8.3) means that uniform convergence of the curves g^j_n is slower as slopes increase faster.

As an example, consider the convergence of binary subdivision schemes in the case of positive masks $g_n > 0$, $n = 0, \cdots, L-1$, as studied by Micchelli and Prautzsch in [17]. They found that

$$\sup_{0 \leq n-m \leq L-2} |g^j_m - g^j_n| \leq c\,(1 - \min g_n)^j;$$

hence any binary subdivision scheme with positive mask uniformly converges to a continuous function [17]. Theorem 8.1 immediately applies to show that the limit function is, in fact, $\dot{C}^{\alpha}$, where $\alpha = -\log_2(1 - \min g_n)$.

Since we have a characterization of regularity for stable $\varphi(x)$'s, it is easy to find a condition on g_n that states an exact regularity order $0 < \alpha < 1$.

COROLLARY 8.2. *Let g_n be a stable binary subdivision scheme such that $G(1) = 2$ and $G(-1) = 0$. If, for $0 < \alpha < 1$,*

$$\max_n |g^j_{n+1} - g^j_n| \text{ decreases as } 2^{-j\alpha} \quad \text{when } j \to \infty, \tag{8.4}$$

then the limit function $\varphi(x)$ is $\dot{C}^{\alpha}$ but is not $\dot{C}^{\alpha+\varepsilon}$, for any $\varepsilon > 0$.

Proof. This is an immediate consequence of Theorem 8.1. If $\varphi(x)$ were $\dot{C}^{\alpha+\varepsilon}$ (with $\varepsilon > 0$ small enough so that $\alpha + \varepsilon < 1$), we would have $|g^j_{n+1} - g^j_n| \leq c\,2^{-j(\alpha+\varepsilon)}$, which contradicts (8.4). □

Note that Corollary 8.2 does not hold if $\alpha = 1$, since $|g^j_{n+1} - g^j_n|$ cannot decrease faster than 2^{-j} as $j \to \infty$ when $\varphi(x)$ is more regular than C^1 (see §10). Otherwise, intuitively the derivative of $\varphi(x)$ would vanish identically, which would imply $\varphi(x) = 0$ since $\varphi(x)$ is compactly supported.

9. Continuously differentiable limit functions. In this section, we study the derivatives of the limit function $\varphi(x)$. We start by defining finite differences of the g^j_n's, which will be shown to converge to the derivatives of $\varphi(x)$. The first finite difference is

$$\Delta g^j_n = (g^j_n - g^j_{n-1})/2^{-j}, \quad \text{i.e., } \Delta G^j(X) = 2^j(1-X)G^j(X). \tag{9.1}$$

In other words, the Δg^j_n's are the *slopes* of the "discrete curve" g^j_n plotted against $n2^{-j}$ (see Figs. 2 and 3). Finite differences $\Delta^k g^j_n$ of order k are simply obtained by applying k times the difference operator Δ:

$$\Delta^k G^j(X) = 2^{jk}(1-X)^k G^j(X). \tag{9.2}$$

In order to study finite differences $\Delta^k g^j_n$ similarly, as for the g^j_n's, it is convenient to express them as binary subdivision schemes as well, associated to masks other than g_n. The following lemma shows that this is possible when $G(X)$ has enough zeros at $X = -1$.

LEMMA 9.1. *Assume $G(X)$ has at least k zeros at $X = -1$ and define $G_k(X)$ by*

$$G(X) = \left(\frac{1+X}{2}\right)^k G_k(X). \tag{9.3}$$

Then the finite differences $\Delta^k g^j_n$'s follow a binary subdivision scheme with the initial sequence's polynomial $(1-X)^k$ and polynomial mask $G_k(X)$.

Proof. This is an immediate generalization of the first part of Lemma 7.2. From (9.2), (9.3), we have

$$\Delta^k G^j(X) = 2^{jk}(1-X)^k \prod_{i=0}^{j-1} \left(\frac{1+X^{2^i}}{2}\right)^k G^j_k(X),$$

where $G^j_k(X) = (G_k)^j(X)$ is defined by (2.5). Using the identity $(1-Y)(1+Y) = 1-Y^2$ for $Y = X, X^2, X^4, \cdots$, we obtain

$$\Delta^k G^j(X) = G^j_k(X)(1-X^{2^j})^k, \tag{9.4}$$

which from (4.1) proves the lemma. □

Using the preceding sections we can extend the results of §7 to higher-order regularity C^N (N-times continuously differentiable functions).

THEOREM 9.2. *If the sequence of the Nth-order finite differences $\Delta^N g^j_{n_j}$ (where n_j satisfies (3.2)) uniformly converges as $j \to \infty$, then $\varphi(x)$ is C^N. The converse is true if $\varphi(x)$ is stable.*

In addition, $\Delta^k g^j_{n_j}$ (where n_j satisfies (3.2)) converges uniformly to $\varphi^{(k)}(x)$, the kth-order derivative of $\varphi(x)$, for $k = 0, \cdots, N$, and $G(X)$ has at least $N+1$ zeros at $X = -1$.

Proof. (⇒) Let us first prove uniform convergence of the kth-order finite differences ($k = 0, \cdots, N$) by backward induction on k. We show that if $\Delta^{k+1} g^j_n$ converges uniformly to some (continuous) function $h(x)$, then $\Delta^k g_n^j$ converges uniformly to the primitive of $h(x)$ defined by

$$H(x) = \int_{-\infty}^{x} h(u)\, du.$$

For simplicity we assume $k = 0$, the proof being identical for $k > 0$.

First we prove that $H(x)$ is compactly supported. The functions $\Delta g^j_{\lfloor x2^j \rfloor}$ are all Riemann-integrable and converge uniformly to the function $h(x)$ of compact support $[0; L-1]$ (where L is the length of g_n); therefore,

$$\int_0^{L-1} \Delta g^j_{\lfloor x2^j \rfloor}\, dx = \sum_n 2^{-j} \Delta g_n^j \text{ tends to } \int_0^{L-1} h(u)\, du \quad \text{as } j \to \infty.$$

But since $\Delta G^j(1) = 0$ (see (9.4)), these integrals vanish, which shows that $H(x)$ is compactly supported.

Now, since $H(x)$ is C^1 and has compact support, it is uniformly continuously differentiable and, therefore, $\sup_x |\Delta g^j_{n_j} - (H(n_j 2^{-j}) - H((n_j-1)2^{-j}))/2^{-j}|$ tends to zero as $j \to \infty$, where n_j are integers satisfying (3.2). This can be written

$$\|2^j(1-X)(G^j(X) - \Psi^j(X))\|_\infty \to 0,$$

where $\Psi^j(X)$ is the polynomial associated to the sequence $H(n2^{-j})$. But for any polynomial $U(X)$, we have

$$\|U(X)\|_\infty \le \sum_k |u_k - u_{k-1}| \le d\, \|(1-X)U(X)\|_\infty,$$

where d is the degree of the polynomial $U(X)$. Applying this to $U(X) = G^j(X) - \Psi^j(X)$ of degree $(L-1)(2^j-1)$, we obtain $\sup_{n_j} |g^j_{n_j} - H(n_j 2^{-j})| = \|G^j(X) - \Psi^j(X)\|_\infty \le (L-1)\|2^j(1-X)(G^j(X) - \Psi^j(X))\|_\infty$, which tends to zero; therefore, g_n^j converges uniformly to $\varphi(x) = H(x)$, and $h(x)$ is the derivative of $\varphi(x)$. By induction it follows that the kth-order finite differences converge uniformly to the kth-order derivatives of $\varphi(x)$ for $0 \le k \le N$.

In particular, the continuous uniform limit of $\Delta^N g_n^j$ is $\varphi^{(N)}(x) \in C^0$. Therefore, $\varphi(x)$ is C^N. The property that $G(X)$ has at least $N+1$ zeros at $X = -1$ follows easily by forward induction on the derivative order k as a consequence of Proposition 4.1 and Lemma 9.1.

($\Leftarrow$) We prove uniform convergence of the kth-order finite differences to the kth-order derivative of $\varphi(x)$ $(k = 0, \cdots, N)$, from the assumption that $\varphi(x)$ is stable and C^N, by forward induction on k. For $k = 0$, this is true by Theorem 7.1. It remains to prove that this implies $\sup_x |\varphi^{(k)}(x) - \Delta^k g^j_{n_j}| \to 0$ for $k = 1, \cdots, N$, where n_j satisfies (3.2). For simplicity, assume $k = 1$. The proof is identical for larger k's when one replaces Δ by Δ^k. Define $\Delta\Phi^j(X) = 2^j(1-X)\Phi^j(X)$, where $\Phi^j(X)$ is defined by (5.2), i.e., $\Delta\varphi(n_j 2^{-j}) = 2^j(\varphi(n_j 2^{-j}) - \varphi((n_j-1)2^{-j}))$. We have

$$\begin{aligned}\sup_x |\varphi'(x) - \Delta g^j_{n_j}| \le{} & \sup_x |\varphi'(x) - \varphi'(n_j 2^{-j})| \\ & + \sup_x |\varphi'(n_j 2^{-j}) - \Delta\varphi(n_j 2^{-j})| \\ & + \sup_x |\Delta\varphi(n_j 2^{-j}) - \Delta g^j_{n_j}|.\end{aligned} \tag{9.5}$$

The first term in the right-hand side of (9.5) tends to zero as $j \to \infty$ because $\varphi'(x)$ is continuous and compactly supported, hence uniformly continuous. The second term also tends to zero because $\varphi(x)$ is uniformly continuously differentiable on a compact support. Note that this implies

$$\sup_{n_j} |\Delta\varphi(n_j 2^{-j}) - \Delta\varphi((n_j - 1)2^{-j})| = \|(1 - X)\Delta\Phi^j(X)\|_\infty \to 0. \tag{9.6}$$

The third term in the right-hand side of (9.5) can be written as $\|\Delta\Phi^j(X)-\Delta G^j(X)\|_\infty$. But from (5.3) we have $\Phi(X)(\Delta\Phi^j(X) - \Delta G^j(X)) = (\Phi(X) - 1)\Delta\Phi^j(X)$. Since $\Phi(1) = 1$ (5.8), $X - 1$ divides $\Phi(X) - 1$ and we can write, using the norm inequality (2.6), $\|\Phi(X)(\Delta\Phi^j(X) - \Delta G^j(X))\|_\infty \le c\,\|(X-1)\Delta\Phi^j(X)\|_\infty$, which tends to zero by (9.6). Now we can use (2.7) with $V(X) = \Phi(X)$ because $\varphi(x)$ is stable. This yields $\|\Delta\Phi^j(X) - \Delta G^j(X)\|_\infty \to 0$ as $j \to \infty$, which ends the proof. □

The direct part of this theorem already appeared in [14], [15]. The converse part also appeared in [14], [15] for interpolatory subdivision schemes (we have seen in §6 that interpolatory subdivision schemes are stable.)

This theorem is useful because it allows us to estimate the regularity of the derivatives of a stable limit function $\varphi(x)$ the same way as for $\varphi(x)$ itself: if $G(X)$ has enough zeros at $X = -1$, the finite differences of the g_n^j's, which converge to the derivatives of $\varphi(x)$, all follow binary subdivision schemes.

Theorem 9.2 also provides an *upper bound* for regularity. Since it is necessary that $G(X)$ has $N+1$ zeros at $X = -1$ to obtain C^N stable limit functions $\varphi(x)$, the regularity order of $\varphi(x)$ is always bounded by the number of zeros at $X = -1$ in $G(X)$. We shall see that this upper bound may be attained.

However, it is important to note that imposing zeros at $X = -1$ in $G(X)$ does *not* ensure any regularity in general. It does not even ensure convergence, as in the example $G(X) = (1 + X^3)^{N+1}$, which has $N+1$ zeros at $X = -1$, although g_n^j does not converge for the same reason as for the choice $G(X) = 1 + X^3$ treated in §4. (Section 13 derives a sharp upper bound for regularity.)

Finally, note that the number of zeros of $G(X)$ at $X = -1$ is an upper bound for regularity only for *stable* limit functions. This upper bound may be exceeded for unstable limit functions, as shown in the following example [2], for which the converse part of Theorem 9.2 fails—as well as many other "optimality" results given in the rest of this paper.

Consider the polynomial mask $G(X) = 2^{-N}(1 + X)(1 + X^2)^N$. Setting $U_j(X) = 1 + X + X^2 + \cdots + X^{2^j-1}$ and applying (2.6) several times give

$$\begin{aligned}\|(1 - X)G^j(X)\|_\infty &\le 2^{-jN}\|(1 - X^{2^j})\|_1\|(U_j(X^2))^N\|_\infty\\ &\le 2^{-jN+1}\|U_j(X)\|_1^{N-1}\|U_j(X)\|_\infty\\ &\le 2^{-j+1};\end{aligned}$$

therefore, by Theorem 8.1 the limit function $\varphi(x)$ exists and is $\dot{C}^1$, hence continuous. Theorem 9.2 cannot improve this regularity order since $G(X)$ has only one zero at $X = -1$. However, $\varphi(x)$ is unstable since $1 + X^2$ divides $G(X)$ (see §6), so we might expect higher regularity for $\varphi(x)$.

Now consider another mask $\tilde{G}(X) = 2^{-N}(1 + X)^{N+1}$. It is easy to see that the subdivision scheme $\tilde{g}_n^j$ converges to a $\dot{C}^N$ limit function $\tilde{\varphi}(x)$, i.e., the $(N-1)$th derivative of $\tilde{\varphi}(x)$, for which the mask polynomial is $\tilde{G}_{N-1}(X) = (1+X)^2/2$, is $\dot{C}^1$. This comes from Theorems 8.1 and 9.2 since we have $\|(1 - X)\tilde{G}_{N-1}^j\| \le$

$2^{-j}\|(1-X^{2^j})\|_1\|U_j(X)\|_\infty \le 2^{-j+1}$. Now, since the two masks are related by $(1+X)^N G(X) = (1+X^2)^N\tilde{G}(X)$, we have by iteration $(1+X)^N G^j(X) = (1+X^{2^j})^N\tilde{G}^j(X)$, i.e.,

$$\sum_{k=0}^{N}\binom{N}{k} g^j_{n-k} = \sum_{k=0}^{N}\binom{N}{k}\tilde{g}^j_{n-2^jk}.$$

Letting $n=n_j$ and $j\to\infty$ gives, by Definition 3.1,

$$\varphi(x) = 2^{-N}\sum_{k=0}^{N}\binom{N}{k}\tilde{\varphi}(x-k),$$

which proves that $\varphi(x)$ is $\dot{C}^N$, hence C^{N-1}, even though $G(X)$ has only one zero at $X=-1$.

This example shows that an unstable binary subdivision scheme may converge to an arbitrary regular limit function while *all* finite differences diverge. Note that since $\tilde{\varphi}(x)$ can also be expressed as a sum of integer translates of $\varphi(x)$ (see the beginning of §4.1), both functions have the same regularity order. It is easy to check that the regularity estimate $\dot{C}^N$ is optimal for $\tilde{\varphi}(x)$ (which is, in fact, the B-spline of degree N [15]); hence it is also optimal for $\varphi(x)$.

Therefore, the argument used in this example has led to an optimal regularity estimate for an *unstable* limit function, while the rest of this paper derives regularity estimates that are optimal for all *stable* limit functions. This example can be easily generalized to the case where unstability is due to the fact that $G(X)$ is divisible by $X^2-e^{i\omega}$ (see §6). Note that if the conjecture mentioned in §6 is true, then this methods works for arbitrary unstable limit functions (in the sense of (6.1)).

10. Determining the exact Hölder regularity order. Recall the definition of Hölder regularity. The limit function $\varphi(x)$ is regular of order $r=N+\alpha$ $(0<\alpha\le 1)$, $\varphi(x)\in\dot{C}^r$, if it is C^N and its Nth derivative $\varphi^{(N)}(x)$ is Lipschitz of order α, $\varphi^{(N)}(x)\in\dot{C}^\alpha$, as defined earlier by (8.1). Hölder spaces $\dot{C}^r$ generalize the spaces C^N of N-times continuously differentiable functions. As already mentioned in the case $N=1$, $\dot{C}^N$ contains functions that are not C^N, such as spline functions of degree N. In fact "$\varphi(x)$ is $\dot{C}^N$" can be thought of as "$\varphi(x)$ is almost C^N," since if $\varphi(x)$ is $\dot{C}^{N+\varepsilon}$, for some $\varepsilon>0$, then $\varphi(x)$ is truly C^N. Other spaces, based on the Fourier transform of $\varphi(x)$, are sometimes used to define a regularity order $r\in R$ as well. They will be considered later in §17.

Using the results of the preceding sections, we can extend the characterization of Lipschitz limit functions $\dot{C}^\alpha$ $(0<\alpha\le 1)$, derived in §8, to any Hölder regularity order $r>0$.

THEOREM 10.1. *If* $G(1)=2$, $G(X)$ *has at least* $N+1$ *zeros at* $X=-1$ *and*

$$\max_n|\Delta^N g^j_{n+1}-\Delta^N g^j_n|\le c\,2^{-j\alpha} \tag{10.1}$$

for some $\alpha>0$, *then* $\varphi(x)$ *is* $\dot{C}^{N+\alpha}$. *The converse is true whenever* $\varphi(x)$ *is stable. Moreover,* (10.1) *implies* $\alpha\le 1$ (*if* $\varphi(x)\not\equiv 0$), *and*

$$\max_n|\Delta^N g^j_{n+1}-\Delta^N g^j_n| = \|(1-X)\Delta^N G^j(X)\|_\infty \tag{10.2}$$

can be replaced in (10.1) *by any of the following:*

$$\max_n|(g^j_N)_{n+1}-(g^j_N)_n| = \|(1-X)G^j_N(X)\|_\infty, \tag{10.3}$$

(10.4) $$\max_n |(f_N^j)_n| = \|F_N^j(X)\|_\infty,$$

(10.5) $$\max_{0\le n\le 2^j-1} \sum_k |(f_N^j)_{n+2^jk}|,$$

where we have set $G(X) = 2^{-N}(1+X)^N G_N(X) = 2^{-N}(1+X)^{N+1}F_N(X)$. *The iterated polynomials* $G_N^j(X)$, $F_N^j(X)$, *corresponding to the sequences* $(g_N^j)_n$, $(f_N^j)_n$, *are defined by* (2.5).

Proof. ($\Rightarrow$) Assume for the moment that $\alpha \le 1$. Since (10.1) implies, by Theorem 8.1, that $\Delta^N g_n^j$ converges uniformly to a $\dot{C}^\alpha$ function, it follows from Theorem 9.2 that all finite differences $\Delta^k g_n^j$ converge uniformly to $\varphi^{(k)}(x)$, for $k = 0, \cdots, N$. Hence $\varphi(x)$ is $\dot{C}^{N+\alpha}$.

($\Leftarrow$) If $\varphi(x)$ is stable and C^N, then by Theorem 9.2, $\Delta^N g_n^j$ converges uniformly to $\varphi^{(N)}(x) \in \dot{C}^\alpha$. Using (5.3) and the stability of $\varphi(x)$ we have $\|(1-X)\Delta^N G^j(X)\|_\infty \le c\,\|(1-X)\Delta^N\Phi^j(X)\|_\infty$, where $\Delta^N\Phi^j(X) = 2^{jN}(1-X)^N\Phi^j(X)$ corresponds to the coefficients $\Delta^N\varphi(n2^{-j})$. Now, we have

$$|\Delta^N\varphi(x) - \Delta^N\varphi(x-2^{-j})| = 2^j\left|\int_{x-2^{-j}}^{x} (\Delta^{N-1}\varphi'(y) - \Delta^{N-1}\varphi'(y-2^{-j}))\,dy\right|$$
$$\le \max_x |\Delta^{N-1}\varphi'(x) - \Delta^{N-1}\varphi'(x-2^{-j})|.$$

By backward induction on N, it follows that

$$\|(1-X)\Delta^N\Phi^j(X)\|_\infty \le \max_x |\varphi^{(N)}(x) - \varphi^{(N)}(x-2^{-j})| \le c\,2^{-j\alpha},$$

which proves (10.1).

We now prove that (10.2)–(10.5) are "equivalent" in the following sense. Two sequences u_j and v_j are equivalent if there exist two constants c_1 and c_2, independent of j, such that $c_1 v_j \le u_j \le c_2 v_j$. From Lemma 9.1, we then have $\Delta^N G^j(X) = (1-X^{2^j})^N G_N^j(X)$. Hence, using the norm inequality (2.6), $\|(1-X)\Delta^N G^j(X)\|_\infty \le 2^N\|(1-X)G_N^j(X)\|_\infty$. Now, since the degree of $(1-X)G_N^j(X)$ is less than 2^jL, where L is the length of the sequence $(g_N)_n$, we also have

$$\|(1-X)G_N^j(X)\|_\infty = \|(1-X^{2^jL})^N(1-X)G_N^j(X)\|_\infty$$
$$= \left\|\left(\frac{1-X^{2^jL}}{1-X^{2^j}}\right)^N (1-X)\Delta^N G^j(X)\right\|_\infty$$
$$\le c_N\,\|(1-X)\Delta^N G^j(X)\|_\infty.$$

This proves that (10.2) and (10.3) are equivalent. The proof of (10.3)$\Leftrightarrow$(10.4) is very similar, based on the relation $(1-X)G_N^j(X) = (1-X^{2^j})F_N^j(X)$, which comes from Lemma 9.1. The equivalence (10.4)$\Leftrightarrow$(10.5) is obvious.

We finally show that (10.1) implies $\alpha \le 1$. Since $G(1) = 2$, we have $F_N(1) = F_N^j(1) = 1$; therefore, $\|F_N^j(X)\|_\infty \ge 2^{-j}\|F_N^j(X)\|_1 \ge 2^{-j}|F_N^j(1)| = 2^{-j}$, which shows, from (10.1) written with (10.4), that $\alpha \le 1$. □

The "equivalent" sequences (10.2)–(10.5) allow useful flexibility in the formulation of Theorem 10.1. As in §8, the following corollary immediately results from Theorem 10.1.

COROLLARY 10.2. *Let g_n be a stable binary subdivision scheme such that $G(1) = 2$ and $G(X)$ has at least $N+1$ zeros at $X=-1$. If, for $0<\alpha<1$,*

$$\max_n |\Delta^N g^j_{n+1} - \Delta^N g^j_n| \quad \textit{decreases as } 2^{-j\alpha} \quad \textit{when } j \to \infty, \tag{10.6}$$

then the limit function $\varphi(x)$ is $\dot{C}^{N+\alpha}$, but is not $\dot{C}^{N+\alpha+\varepsilon}$, for any $\varepsilon>0$.

This does not hold for $\alpha = 1$, since by Theorem 10.1, (10.1) implies $\alpha \le 1$. Of course, in (10.6) we can choose either (10.2), (10.3), (10.4), or (10.5).

Note that the characterization (10.1), or the criterion (10.6), depends on the choice of N. Theorem 10.1 (or Corollary 10.2) therefore allows us to check whether the exact regularity order r (that is, the number such that $\varphi(x)$ is $\dot{C}^r$ but not $\dot{C}^{r+\varepsilon}$, for any $\varepsilon>0$) falls in the range $N \le r < N+1$.

Assume, for example, that (10.6) is tested for some $N = N_0$ larger than the *unknown* exact regularity order r. This test necessarily fails, which only ensures that $\varphi(x)$ is not $\dot{C}^{N_0}$. On the other hand, if the value of N is too small, i.e., $N = N_1 < r-1$, then necessarily (10.6) is satisfied with $\alpha = 1$. This shows that $\varphi(x)$ is $\dot{C}^{N_1+1}$, but does not tell whether $\varphi(x)$ is actually more regular or not. In both cases (under or overestimated N's), the criterion (10.6) has to be checked all over again for other values of N to determine r. It is only when it turns out that $N < r \le N+1$ that the criterion is really optimal and provides $N+\alpha = r$; therefore, the exact regularity order cannot be determined in general unless all possible values of N are tried.

However, if (10.1) can be extended to *negative* values of α, then the exact regularity order r is determined even if N is "too large," i.e., $N+1 \ge r$. That is, even if the criterion (10.1) for regularity order $r > N$ fails, it could be used to characterize lower regularity orders $0 < r \le N$. In particular, if we use all of the zeros at $X=-1$ in $G(X)$ (i.e., if $G(X)$ has no more than $N+1$ such zeros), then the characterization (10.1), extended to any $\alpha \le 1$, necessarily provides the exact regularity order r. This extension is provided by the following theorem.

THEOREM 10.3. *Theorem 10.1 and Corollary 10.2 hold for $-N < \alpha \le 1$, with the following slight restriction. If (10.1) holds for $\alpha = -n$, $n = 0, 1, \cdots, N-1$, then $\varphi(x)$ is only "almost" $\dot{C}^{N-n}$, i.e., its $(N-n-1)$th derivative satisfies*

$$|\varphi^{(N-n-1)}(x+h) - \varphi^{(N-n-1)}(x)| \le c.|h||\log|h|| \quad \textit{for all } x,\ h \in \mathbf{R}. \tag{10.7}$$

This theorem will be proven if we can simultaneously increase α and decrease N by 1 in (10.1). We, therefore, need the following lemma.

LEMMA 10.4. *Assume that $G(1) = 2$, $G(-1) = 0$, and that $G(X)$ has at least $N+1$ zeros at $X=-1$. The condition*

$$\max_n |\Delta^{N-1} g^j_{n+1} - \Delta^{N-1} g^j_n| \le c\, 2^{-j(\alpha+1)} \tag{10.8}$$

implies (10.1). The converse implication holds for $\alpha < 0$ only. When $\alpha = 0$, (10.1) implies

$$\max_n |\Delta^{N-1} g^j_{n+1} - \Delta^{N-1} g^j_n| \le c\, j 2^{-j}. \tag{10.9}$$

Proof. ($\Rightarrow$) We have

$$2^{-j} \max_n |\Delta^N g^j_n - \Delta^N g^j_{n-1}| = \max_n |\Delta^{N-1} g^j_n - 2\Delta^{N-1} g^j_{n-1} + \Delta^{N-1} g^j_{n-2}|$$

$$\leq \max_n(|\Delta^{N-1}g_n^j - \Delta^{N-1}g_{n-1}^j| + |\Delta^{N-1}g_{n-1}^j - \Delta^{N-1}g_{n-2}^j|);$$

therefore, (10.8) clearly implies (10.1).

($\Leftarrow$) Condition (10.8) implies $1+\alpha \leq 1$ by Theorem 10.1. We, therefore, assume $\alpha \leq 0$ to prove the converse implication. Rewrite (10.1) and (10.8) using (10.4), knowing that $F_{N-1}^j(X) = 2^{-j}F_N^j(X)(1-X^{2^j})/(1-X)$ by Lemma 9.1. We, therefore, have to prove that (10.1), that is, $\|F_N^j(X)\|_\infty \leq c\,2^{-j\alpha}$ implies (10.8), that is $\|F_N^j(X)(1-X^{2^j})/(1-X)\|_\infty \leq c\,2^{-j\alpha}$. There is a problem at $X=1$; we, therefore, subtract $F_N^j(1) = F_N(1) = 1$ to $F_N^j(X)$ as shown:

$$\left\|F_N^j(X)\frac{1-X^{2^j}}{1-X}\right\|_\infty \leq \left\|(F_N^j(X)-1)\frac{1-X^{2^j}}{1-X}\right\|_\infty + \left\|\frac{1-X^{2^j}}{1-X}\right\|_\infty.$$

The second term in the right-hand side equals 1. Denote the first one by $\|H^j(X)\|_\infty$. From (2.3) written for $F_N(X)$, we have $F_N^j(X) - 1 = (F_N^{j-1}(X^2)-1) + (F_N(X) - 1)F_N^{j-1}(X^2)$. But since $F_N(1) = 1$, $X-1$ divides $F_N(X)-1$; therefore, $H^j(X) = H^{j-1}(X^2)(1+X) + (X^{2^j}-1)F_N^{j-1}(X^2)(F_N(X)-1)/(X-1)$ and

$$\|H^j(X)\|_\infty \leq \|H^{j-1}(X)\|_\infty + c\,2^{-(j-1)\alpha}.$$

By induction on j, for $\alpha < 0$, $\|H^j(X)\|_\infty \leq c'\,2^{-j\alpha}$ follows, which implies (10.8). When $\alpha = 0$, we have $\|H^j(X)\|_\infty \leq c'\,j$, which implies (10.9). □

Proof of Theorem 10.3. If α is not a negative integer, the generalization of Theorem 10.1 to $-N < \alpha \leq 0$ follows from several applications of Lemma 10.4. When $\alpha = -n$, $n = 0, \cdots, N-1$, by n successive applications of Lemma 10.4, (10.1) implies $\max_n |\Delta^{N-n}g_{n+1}^j - \Delta^{N-n}g_n^j| \leq c$. Applying Lemma 10.4 again, we only obtain $|\Delta^{N-n-1}g_{n+1}^j - \Delta^{N-n-1}g_n^j| \leq c\,j2^{-j}$. By Theorem 10.1, this implies that $\varphi(x)$ is $\dot{C}^{N-n-\varepsilon}$ (for any $\varepsilon > 0$), but we have a little more: mimicking the proof of the direct part of Theorem 8.1, we have $|\varphi^{(N-n-1)}(x+h) - \varphi^{(N-n-1)}(x)| \leq c\,j2^{-j}$ for $2^{-j} \leq |h| < 2^{-j+1}$, which gives (10.7). □

11. A practical, optimal Hölder regularity estimate. Theorem 10.3 already provides an optimal regularity criterion (10.1) (with $-N < \alpha \leq 1$). However, it is not implementable on a computer as written since it needs to be verified for all j's and the order of magnitude of the constant c in (10.1) is unknown. The aim of this section is to transform this criterion into an easily implementable estimate [19] for Hölder regularity, computable with a finite number of operations.

The following theorem assumes some properties and notation we have already met:

- $G(1) = 2$;
- $G(X)$ has at least $N+1$ zeros at $X=-1$;
- $F_N(X)$ (corresponding to the sequence $(f_N)_n$) is, as defined in Theorem 10.1, $G(X)$ "without its $N+1$ zeros at $X=-1$," i.e.,

$$G(X) = \left(\frac{1+X}{2}\right)^N (1+X)\,F_N(X).$$

It generates iterated polynomials $F_N^j(X)$ and sequences $(f_N^j)_n$ by (2.5).

THEOREM 11.1. *With the above notation and assumptions, define the Hölder regularity estimate* $N + \alpha_N^j$ *by*

$$2^{-j\alpha_N^j} = \max_{0 \le n \le 2^j - 1} \sum_k |(f_N^j)_{n+2^j k}| \tag{11.1}$$

and let $\alpha_N = \sup_j \alpha_N^j$. *The sequence* α_N^j *converges to* $\alpha_N \le 1$ *as* $j \to \infty$. *If there exists* j *such that* $N + \alpha_N^j > 0$, *then* $\varphi(x)$ *is* $\dot{C}^{N+\alpha_N^j}$ *(almost* $\dot{C}^{N+\alpha_N^j}$ *if* $\alpha_N^j \in -\mathbf{N}$, *see (10.7)); therefore,* $\varphi(x)$ *is* $\dot{C}^{N+\alpha_N-\varepsilon}$ *for any* $\varepsilon > 0$.

In addition, if $\varphi(x)$ *is stable, then the regularity estimate is optimal: If* $\alpha_N \ne 1$, *or if* $\alpha_N = 1$, *and* $G(X)$ *has no more than* $N+1$ *zeros at* $X = -1$, *then* $\varphi(x)$ *is* $\dot{C}^{N+\alpha_N-\varepsilon}$, *but is not* $\dot{C}^{N+\alpha_N+\varepsilon}$ *for any* $\varepsilon > 0$. *Moreover, the rate of convergence of the estimates* $N + \alpha_N^j$ *to the exact regularity order* $N + \alpha_N$ *is given by*

$$0 \le \alpha_N - \alpha_N^j \le c/j. \tag{11.2}$$

Proof. From (11.1) and Theorem 10.3 rewritten with (10.5), we have $\alpha_N^j \le 1$ for all j; hence $\alpha_N \le 1$. Now, using the relation $F_N^{i+j}(X) = F_N^j(X) F_N^i(X^{2^j})$ or the matrix formulation (12.1) given in the next section, we easily find that $2^{-(i+j)\alpha_N^{i+j}} \le 2^{-i\alpha_N^i} 2^{-j\alpha_N^j}$, i.e.,

$$\alpha_N^{i+j} \ge \frac{i\alpha_N^i + j\alpha_N^j}{i+j}.$$

The following proof of convergence of the α_N^j is due to Cohen [3], [4]: Let $\varepsilon > 0$ be an arbitrary small number and J such that $\alpha_N^J \ge \alpha^N - \varepsilon$. For any j, write $j = nJ + i$, $0 \le i \le J - 1$. Applying the inequality above several times, we find $\alpha_N^j \ge ((j-i)\alpha_N^J + i\alpha_N^i)/j$; hence, when j is large enough, $\alpha_N^j \ge \alpha_N - 2\varepsilon$, which proves that $\alpha_N^j \to \alpha_N$ as $j \to \infty$.

We now prove the announced regularity order for $\varphi(x)$. Let $G_N(X) = (1 + X)F_N(X)$ be as in Theorem 10.1. By Lemma 7.2 applied to $G_N(X)$, we immediately obtain (10.1), written with (10.3) and $\alpha = \alpha_N^i$; therefore, Theorem 10.3 applies with $\alpha = \alpha_N^i$, for any i such that $\alpha_N^i > -N$. The limit function is thus $\dot{C}^{N+\alpha_N^i}$ (with the restriction (10.7)), and, therefore, $\varphi(x)$ is $\dot{C}^{N+\alpha_N-\varepsilon}$ for any $\varepsilon > 0$.

Now assume that $\varphi(x)$ is stable. From (11.1), the condition (10.1), rewritten with (10.5) is satisfied only when $\alpha \le \liminf_{j\to\infty} \alpha_N^j = \alpha_N$. Now if $\varphi(x)$ were $\dot{C}^{N+\alpha_N+\varepsilon}$, where $\alpha_N < 1$ and $\varepsilon > 0$, by Theorem 10.1 (10.1) would hold with $\alpha = \alpha_N + \varepsilon$, which contradicts $\alpha \le \alpha_N$; therefore, if $\varphi(x)$ is stable, $\alpha_N < 1$ implies that $\varphi(x)$ is not $\dot{C}^{N+\alpha_N+\varepsilon}$ for any $\varepsilon > 0$. In addition, $\varphi(x)$ cannot be $N + 1 + \varepsilon$ if $G(X)$ has no more than $N+1$ zeros at $X = -1$ because of Theorem 9.2.

We finally prove (11.2). When $\varphi(x)$ is stable and $\dot{C}^{N+\alpha_N-\varepsilon}$, by Theorem 10.3, (10.1), written with (10.5), holds for $\alpha = \alpha_N - \varepsilon$. By definition of α_N^j (11.1), we thus have $2^{-j\alpha_N^j} \le c\,2^{-j(\alpha_N-\varepsilon)}$ for any $\varepsilon > 0$, which implies (11.2). □

Of course, we can replace (10.5) in (11.1) by any other equivalent sequence (10.2), (10.3), (10.4). We would still obtain a sequence $N + \beta_N^j$, which converges to the optimal regularity order $N + \alpha_N$, however, $\varphi(x)$ may not be regular of order $N + \beta_N^j$ for any fixed j because β_N^j may be greater than α_N.

Let us make precise the practical outcomes of Theorem 11.1. For a given number of iterations j, and a given N, the computation of $N + \alpha_N^j$—with a finite number of

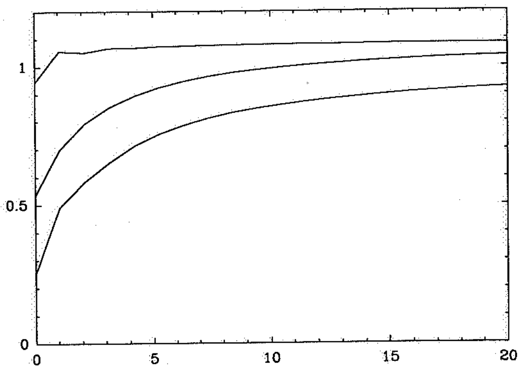


FIG. 4. *Program output of regularity estimates* $N + \alpha_N^j$ (11.1) ($N = 0, 1, 2$) *for* $j = 1$ *to* 20 *iterations. The corresponding limit function is the Daubechies "minimum phase" wavelet of length* 5 (*see* §14), *whose exact regularity order is* $r = 1.0878\cdots$. *For* $N = 0$, *the estimate is bounded by* 1 *and, therefore, does not converge to* r. *For* $N = 2$, *the estimate converges fairly rapidly to* r. *After* 20 *iterations we find* $2 + \alpha_2^{20} = 1.0831\cdots$.

operations—by (11.1) gives a Hölder regularity estimate for $\varphi(x)$ in all cases. Since $\lim_j \alpha_N^j = \sup_j \alpha_N^j$, the estimate is improved when the number of iterations j increases.

Figure 4 shows that N must be chosen large enough because the estimate $N + \alpha_N^j$ is bounded by $N + 1$, whereas the exact regularity order of $\varphi(x)$ might be greater than $N + 1$. If N is too small, $N + \alpha_N^j$, in fact, necessarily converges to $N + 1$. It is therefore recommended that N should be chosen maximal (i.e., such that $G(X)$ has exactly $N + 1$ zeros at $X = -1$). In this case Theorem 11.1 ensures that the regularity estimates $N + \alpha_N^j$ converge to $N + \alpha_N$, which, provided that $\varphi(x)$ is stable, gives the exact regularity order of $\varphi(x)$. In practice, Fig. 4 shows that the convergence rate of the estimates α_N^j is fairly high. When the scaling sequence length L is not too large (e.g., $L \leq 10$), the exact regularity order is numerically estimated to two decimal places after a few dozen iterations. However, it can be shown [19] that the computational load of an implementation of (11.1) is increasing exponentially with j (increasing j by one roughly doubles the number of operations required to compute (11.1)).

Note that from Theorem 9.2, finite differences $\Delta^k g_n^j$ converge uniformly to the derivatives of a stable limit function $\varphi(x)$ whenever these derivatives exist.

Theorem 11.1 is the main result of this paper. It permits us to estimate sharply Hölder regularity in most cases of interest. (See §9 for the derivation of the optimal regularity estimate on a particular example of an unstable limit function.) The remainder of this paper connects this result to related work on regularity estimates, and illustrates it with examples.

12. Relation to Daubechies and Lagarias estimates. In a recent paper [9], Daubechies and Lagarias determined sharp conditions for Hölder regularity based on matrix products. Although the approach in [9] relies on two-scale difference equations (5.1) rather than on limit functions (3.3), the above results, which were derived independently, are closely related to what can be found in [9]. In fact, (11.1) reads, in matrix notation,

$$2^{-j\alpha_N^j} = \max_{\varepsilon_i = 0 \text{ or } 1} \left\| \prod_{i=0}^{j-1} \mathbf{F}_N^{\varepsilon_i} \right\|_1, \tag{12.1}$$

where the matrices F_N^0 and F_N^1 of size $(L-1)\times(L-1)$ (where L is the length of the sequence $(f_N)_n$) are defined as

$$\mathbf{F}_N^0 = \begin{pmatrix} (f_N)_0 & 0 & 0 & 0 & \cdots \\ (f_N)_2 & (f_N)_1 & (f_N)_0 & 0 & \cdots \\ (f_N)_4 & (f_N)_3 & (f_N)_2 & (f_N)_1 & \cdots \\ (f_N)_6 & (f_N)_5 & (f_N)_4 & (f_N)_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{12.2}$$

$$\mathbf{F}_N^1 = \begin{pmatrix} (f_N)_1 & (f_N)_0 & 0 & 0 & \cdots \\ (f_N)_3 & (f_N)_2 & (f_N)_1 & (f_N)_0 & \cdots \\ (f_N)_5 & (f_N)_4 & (f_N)_3 & (f_N)_2 & \cdots \\ (f_N)_7 & (f_N)_6 & (f_N)_5 & (f_N)_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{12.3}$$

and $\|\mathbf{A}\|_1$ denotes the l^1-norm of a square matrix $\mathbf{A} = ((a_{i,j}))$, i.e.,

$$\|\mathbf{A}\|_1 = \max_i \sum_j |a_{i,j}|.$$

Formulation (12.1) can be proved as follows. Consider the operators of polynomial "biphase decomposition [22]" $\mathbf{D}^\varepsilon$ ($\varepsilon = 0$ or 1), defined by the relation $U(X) = U^0(X^2) + XU^1(X^2)$, where $U^\varepsilon(X) = \mathbf{D}^\varepsilon\{U(X)\}$. Clearly $\mathbf{F}_N^\varepsilon$, seen as an operator acting on polynomials of degree $\le L-2$, transforms $U(X)$ into $\mathbf{D}^\varepsilon\{F_N(X)U(X)\}$. Applying j times the identity $\mathbf{D}^\varepsilon\{U(X)V(X^2)\} = \mathbf{D}^\varepsilon\{U(X)\}V(X)$ gives the polynomial associated to the sequence $(f_N^j)_{n+2^jk}$ (where $n = \varepsilon_0\varepsilon_1\cdots\varepsilon_{j-1}$ in base 2) as

$$\left(\prod_{i=0}^{j-1} \mathbf{D}^{\varepsilon_i}\right)\{F_N^j(X)\} = \prod_{i=0}^{j-1}(\mathbf{F}^{\varepsilon_i}\{1\}),$$

where the polynomial 1 corresponds to the vector $(1\ 0\ 0\ \cdots\ 0)^t$. Therefore, $(f_N^j)_{n+2^jk}$, seen as a vector indexed by k, is equal to the first column of the matrix product in (12.1). To obtain the other columns, replace the initial polynomial 1 by X^m. This amounts to shifting the value of n in $(f_N^j)_{n+2^jk}$, hence changing the values of the ε_i. But since (11.1) involves the maximum over the values of n, the l^1-norm of the first column of the matrix product can be replaced by the l^1-norm of the whole matrix product, which gives (12.1).

Using (12.1) in place of (11.1) in Theorem 11.1, we easily recover the results on global Hölder regularity derived in [9]. Formulation (12.1) and that used in [9] differ only by some minor details: Daubechies and Lagarias use l^2-norms rather than l^1-norms, and the matrices they consider are a bit larger than (12.2), (12.3) because they correspond to $G(X) = 2^{-N}(1+X)^{N+1}F_N(X)$ rather than $F_N(X)$. Although regularity estimates are not proved to be optimal in general in [9], Daubechies and Lagarias prove optimality for several examples, such as those of §14.

Working with matrices is useful when we want to find optimal regularity estimates "by hand" [9], without implementing (11.1). Unfortunately, it seems difficult to derive a general method for determining the optimal regularity by matrix manipulation. As a result, unlike an implementation of (11.1) on a computer, each example has to be investigated separately and requires fastidious treatment. We here recall for completeness the basic method used in [9].

THEOREM 12.1 (Daubechies and Lagarias [9]). *The following method often provides a sharp Hölder regularity estimate for a limit function* $\varphi(x)$:

- *Compute the eigenvalues of* $\mathbf{F}_N^0$ *and* $\mathbf{F}_N^1$ *and let* ρ^0, ρ^1 *be their respective spectral radii (largest moduli of eigenvalues).*
- *Assume, for example, that* $\rho^0 > \rho^1$. *Compute matrix* $\mathbf{B}$, *whose columns are proportional to the eigenvectors of* $\mathbf{F}_N^0$. *The norm of the diagonal matrix* $\mathbf{B}^{-1}\mathbf{F}_N^0\mathbf{B}$ *is therefore* ρ^0.
- *Parameterize* $\mathbf{B}$ *by* $L-1$ *numbers, one for each column. If we can find a parameterization of* $\mathbf{B}$ *such that*

$$\|\mathbf{B}^{-1}\mathbf{F}_N^1\mathbf{B}\| \le \rho^0, \tag{12.4}$$

where $\|\cdot\|$ *is any matrix norm, then* $\varphi(x)$ *is regular of order* $N - \log_2 \rho^0$ *(and this Hölder regularity estimate is moreover optimal if* $\varphi(x)$ *is stable).*

Proof. First, specifying $\varepsilon_i = 0$ for all i in (12.1) gives $2^{-j\alpha_N^j} \ge \|(\mathbf{F}_N^0)^j\|$. Let λ be an eigenvalue of $\mathbf{F}_N^0$ and v an associated nonzero eigenvector. We have, on one hand, $\|(\mathbf{F}_N^0)^j v\| \le \|(\mathbf{F}_N^0)^j\| \cdot \|v\|$, and on the other hand, $\|(\mathbf{F}_N^0)^j v\| = |\lambda|^j \|v\|$. It follows that $(\rho^0)^j = \sup |\lambda|^j \le \|(\mathbf{F}_N^0)^j\| \le 2^{-j\alpha_N^j}$. Now, with the change of basis $\mathbf{B}$, we have

$$2^{-j\alpha_N^j} = \max_{\varepsilon_i} \left\| \mathbf{B} \left(\prod_{i=0}^{j-1} \mathbf{B}^{-1}\mathbf{F}_N^{\varepsilon_i}\mathbf{B} \right) \mathbf{B}^{-1} \right\| \le c \max_{\varepsilon_i} \prod_{i=0}^{j-1} \|\mathbf{B}^{-1}\mathbf{F}_N^{\varepsilon_i}\mathbf{B}\|.$$

But we have $\|\mathbf{B}^{-1}\mathbf{F}_N^0\mathbf{B}\| = \rho^0$ and (12.4); therefore, $2^{-j\alpha_N^j} \le c\,(\rho^0)^j$ follows. We, therefore, have proved that $(\rho^0)^j \le 2^{-j\alpha_N^j} \le c\,(\rho^0)^j$, which implies $\alpha^N = \lim_j \alpha_N^j = -\log_2 \rho^0$. The theorem therefore follows from Theorem 11.1. □

Note that this method is only optimal if (12.4) is met for at least one matrix norm, otherwise the obtained estimate, $N - \log_2 \|\mathbf{B}^{-1}\mathbf{F}_N^1\mathbf{B}\|$, is suboptimal. Whether (12.4) holds for a large class of masks g_n is an open problem [9].

13. A sharp upper bound for regularity. So far we have seen two types of Hölder regularity estimates: One is optimal in (almost) all cases (§11), but many iterations, performed on a computer, are necessary to determine the regularity order accurately. The other (§12) requires the calculation of two spectral radii of matrices, but is sometimes suboptimal. Based on the latter, it is nevertheless possible to obtain a (possibly sharp) *upper bound* for regularity of stable limit functions that only requires the computation of one spectral radius and gives the exact regularity order whenever condition (12.4) is satisfied:

Specifying $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for all i in (12.1) gives $2^{-j\alpha_N^j} \ge \max(\|(\mathbf{F}_N^0)^j\|, \|(\mathbf{F}_N^1)^j\|)$. We have seen that this is greater than $\max((\rho^0)^j, (\rho^1)^j)$; therefore, an upper bound for the Hölder regularity is $N - \log_2 \max(\rho^0, \rho^1)$. By Theorem 12.1, this upper bound is attained for stable limit functions if (12.4) holds.

The computation of this upper bound can be simplified to the search of the spectral radius of only one matrix $\mathbf{F}_N$, defined as the common submatrix of $\mathbf{F}_N^0$ and $\mathbf{F}_N^1$.

$$\mathbf{F}_N^0 = \left(\begin{array}{c|ccc} (f_N)_0 & 0 & \cdots & 0 \\ \hline (f_N)_2 & & & \\ (f_N)_4 & & \mathbf{F}_N & \\ \vdots & & & \end{array} \right) \quad \text{and} \quad \mathbf{F}_N^1 = \left(\begin{array}{ccc|c} & & & \vdots \\ & \mathbf{F}_N & & (f_N)_{L-3} \\ & & & (f_N)_{L-2} \\ \hline 0 & \cdots & 0 & (f_N)_{L-1} \end{array} \right).$$

We have

$$(13.1)\qquad N - \log_2 \max(\rho^0, \rho^1) = N - \log_2 \max(|(f_N)_0|, |(f_N)_{L-1}|, \rho(\mathbf{F}_N)),$$

where $\rho(\mathbf{F}_N)$ is the spectral radius of $\mathbf{F}_N$. Therefore the regularity order of a stable limit function is at most (13.1).

A similar upper bound can be computed using the inequality

$$2^{-j\alpha_N^j} \geq \max\left(\sum_k |(f_N^j)_{2^j k}|, \sum_k |(f_N^j)_{2^j k-1}|\right),$$

which yields a fast implementation [19]: the computational load is here linear in j (compare with §11). When $j \to \infty$, this gives an upper bound which may be greater than (13.1) but is still sharp. This result and Theorem 11.1 can be used to compute sharp lower and upper bounds for the Hölder regularity of $\varphi(x)$. Table 1 provides values of these bounds for the examples presented in the next section.

14. Examples: Daubechies orthonormal wavelets. A family of orthonormal wavelets with compact support has been constructed by Daubechies in [6]. The construction is based on binary subdivision schemes. The "mother wavelet" is defined as the limit function $\psi(x)$ of the scheme (1.2) with initial sequence $h_n = (-1)^n g_{L-1-n}$ (where L is the mask length). She showed that the regular functions $2^{-j/2}\psi(2^{-j}x-k)$, defined for all integers j and k, form an orthonormal basis of $\mathbf{L}^2(\mathbf{R})$ if L is even and

$$(14.1)\qquad G(X)\tilde{G}(X) - G(-X)\tilde{G}(-X) = 4X^{L-1},$$

where $\tilde{G}(X)$ is the polynomial associated to the sequence g_{L-1-n}. In [6], $G(X)$ is, moreover, required to have as many zeros at $X = -1$ as possible. This results in several possible solutions for $G(X)$ that have exactly $N+1 = L/2$ zeros at $X = -1$ [6], [7].

Examples of $G(X)$ in [6] have all zeros outside the unit circle ("minimum phase" choice in the signal processing terminology, since X corresponds to a delay). In [9], the optimal regularities of "minimum phase" Daubechies wavelets $\psi(x)$ for $L = 4$, 6, and 8 are obtained using the method described in the preceding section. It turns out that (12.4) holds for these lengths; therefore, the upper bound (13.1) is attained and actually equals $N - \log_2 |(f_N)_0|$. The estimated regularity of Daubechies "minimum phase" wavelets derived in [6] is, therefore, $-\log_2 |g_0|$ in this case. It can easily be checked that the convergent binary subdivision schemes involved are stable; hence this estimate is optimal. This can be checked directly [9] from Theorem 10.3 by noting that the first "slope" of $\Delta^N g_n^j$ is $|2g_0|^j = 2^{j(1-\alpha)}$, where $\alpha = -\log_2 |g_0|$. Table 1 lists these optimal regularities (for $L \leq 8$), the corresponding outputs of a program implementing (11.1), and upper bounds derived in §13.

There are other solutions g_n, derived for $L \geq 8$ in [7], which, unlike "minimum phase solutions," are close to being symmetric. Table 1 shows that the regularity estimates for these wavelets, based on Theorem 11.1, are lower than those of "minimum phase" wavelets. This will be justified in §17.

15. "Strictly linear phase" symmetric limit functions. In this section we apply the above results to a subclass of scaling sequences that is often encountered. This section is also a prerequisite for comparing Hölder regularity estimates to those determined using Fourier techniques (§17). The subclass considered here consists of

TABLE 1

Some regularity estimates for two types of Daubechies orthonormal wavelets: Minimum phase wavelets [6] and "more symmetric" ones [7] (for mask lengths $L \geq 8$). The upper bound for Hölder regularity in the right-most column is obtained by adding $\frac{1}{2}$ to the optimal Sobolev regularity estimate, derived in [6, Appendix] (see §17). These two apply to all Daubechies wavelets that differ only by their phase. The numbers r_{20} are the Hölder regularity estimates (11.1) obtained by computer program after $j = 20$ iterations. Note that more symmetry decreases regularity in general. For minimum phase wavelets, these estimates converge rapidly to the optimal Hölder regularity estimates r_∞ derived in [9] by using the method described in §12. The upper bounds for both types of wavelets are obtained from §13. They are sharper than the "Sobolev" upper bound and in fact give optimal Hölder regularity estimates in the "minimum phase" case for lengths $L \leq 8$.

L	Optimal Sobolev regularity estimate	More symmetric wavelets		Minimum phase wavelets			Upper bound
		r_{20}	Upper bound	r_{20}	r_∞	Upper bound	
4	0.4999	—	—	0.5500	0.5500	0.5500	0.9999
6	0.9150	—	—	1.0831	1.0878	1.0878	1.4150
8	1.2755	1.3960	1.4026	1.6066	1.6179	1.6179	1.7755
10	1.5967	1.7621	1.7759	1.9424	—	1.9689	2.0967
12	1.8883	2.1019	2.1223	2.1637	—	2.1891	2.3883
14	2.1586	2.4420	2.4681	2.4348	—	2.4604	2.6586
16	2.4147	2.7155	2.7500	2.7358	—	2.7608	2.9147
18	2.6616	2.9977	3.0393	3.0432	—	3.0736	3.1616
20	2.9027	3.2651	3.3110	3.3098	—	3.3813	3.4027

scaling sequences for which either $G(X)$ or $G(X)/(1+X)$ is "strictly linear phase," in the following sense.

DEFINITION 15.1. *A polynomial $U(X)$ (or its associated sequence u_n of finite length L) is* strictly linear phase *if it is symmetric, $u_n = u_{L-1-n}$, and if the trigonometric polynomial $U(e^{i\omega})e^{-i(L-1)\omega/2}$ does not change sign for any $w \in R$.*

Note that symmetry of u_n implies $U(e^{i\omega})e^{-i(L-1)\omega/2} \in \mathbf{R}$. This condition is called "linear phase" in signal processing [22]. The above definition requires more, namely that no discontinuities of the phase due to a change of sign in $U(e^{i\omega})e^{-i(L-1)\omega/2}$ occur. Therefore, complex zeros of the symmetric polynomial U(X) occur in pairs $(z, 1/\bar{z})$ not only for $|z| \neq 1$, but also *on* the unit circle. That is, roots on the unit circle have even order. It follows that $U(X)$ has an even number of roots, hence L is odd.

If $G(X)$ or $G(X)/(1+X)$ is strictly linear phase, then for N odd (even, respectively), the sequence $(f_N)_n$ in (11.1) is also strictly linear phase. The following theorem shows that in this case, the determination of the exact regularity order of a (stable) limit function $\varphi(x)$ only requires the computation of the spectral radius of one matrix. This is to be compared with §§12 and 13, where it is shown that two matrices are involved in the general case, and the computation of one matrix's spectral radius only provides an upper bound for regularity.

The following regularity estimate has been derived independently, by other means, and on particular examples of strictly linear phase scaling sequences, in [6] and [10] (see §§16 and 17).

THEOREM 15.2. *Assume $G(1) = 2$, $G(X)$ has at least $N+1$ zeros at $X = -1$: $G(X) = 2^{-N}(1+X)^{N+1}F_N(X)$, and $F_N(X)$ is strictly linear phase. Define $(\hat{f}_N)_n = (f_N)_{((L-1)/2)\pm n}$ (where L is the length of $(f_N)_n$) and the $(L-1)/2 \times (L-1)/2$ matrix*

$\hat{\mathbf{F}}_N$ *obtained by "folding" the following* $(L-1)/2 \times (L-2)$ *matrix*

$$\begin{pmatrix} \cdots & (\hat{f}_N)_3 & (\hat{f}_N)_2 & (\hat{f}_N)_1 & (\hat{f}_N)_0 & (\hat{f}_N)_1 & (\hat{f}_N)_2 & (\hat{f}_N)_3 & \cdots \\ \cdots & (\hat{f}_N)_1 & (\hat{f}_N)_0 & (\hat{f}_N)_1 & (\hat{f}_N)_2 & (\hat{f}_N)_3 & (\hat{f}_N)_4 & (\hat{f}_N)_5 & \cdots \\ \cdots & (\hat{f}_N)_1 & (\hat{f}_N)_2 & (\hat{f}_N)_3 & (\hat{f}_N)_4 & (\hat{f}_N)_5 & (\hat{f}_N)_6 & (\hat{f}_N)_7 & \cdots \\ & & & & \vdots & & & & \end{pmatrix}$$

around its middle column, i.e.,

$$\hat{\mathbf{F}}_N = \begin{pmatrix} (\hat{f}_N)_0 & 2(\hat{f}_N)_1 & 2(\hat{f}_N)_2 & \cdots \\ (\hat{f}_N)_2 & (\hat{f}_N)_1 + (\hat{f}_N)_3 & (\hat{f}_N)_0 + (\hat{f}_N)_4 & \cdots \\ (\hat{f}_N)_4 & (\hat{f}_N)_3 + (\hat{f}_N)_5 & (\hat{f}_N)_2 + (\hat{f}_N)_6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15.1}$$

Let ρ *be its spectral radius. One has* $\rho \geq \frac{1}{2}$. *If* $\rho < 2^N$, *then the limit function* $\varphi(x)$ *is* $\dot{C}^{N-\log_2 \rho}$ *(almost* $\dot{C}^{N-\log_2 \rho}$ *in the sense of* (10.7) *if* $\rho \geq 1$ *is an integer power of two). In addition, if* $\varphi(x)$ *is stable, and if either* $\rho > \frac{1}{2}$ *or* $\rho = \frac{1}{2}$ *and* $G(X)$ *has no more than* $N+1$ *zeros at* $X = -1$, *then the estimate is optimal:* $\varphi(x)$ *is not* $\dot{C}^{N-\log_2 \rho+\varepsilon}$ *for any* $\varepsilon > 0$.

Proof. Define $(\hat{f}_N^j)_n = (f_N^j)_{(2^{j-1}-2^{-1})(L-1)+n}$. This noncausal, symmetric sequence is strictly linear phase. We first prove that $\|F_N^j(X)\|_\infty = \max_n |(\hat{f}_N^j)_n| = |(\hat{f}_N^j)_0|$. Using Fourier coefficients, we have $(\hat{f}_N^j)_n = \frac{1}{2\pi}\int_0^{2\pi} \hat{F}_N^j(e^{i\omega})e^{in\omega}\, d\omega$, where $\hat{F}_N^j(e^{i\omega}) = \sum_n (\hat{f}_N^j)_n e^{in\omega} = \pm|\hat{F}_N^j(e^{i\omega})|$. Hence

$$\max_n |(\hat{f}_N^j)_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |\hat{F}_N^j(e^{i\omega})|\, d\omega = |(\hat{f}_N^j)_0|.$$

The theorem, therefore, results from Theorem 10.3 if we prove that $|(\hat{f}_N^j)_0|$ is equivalent to ρ^j as $j \to \infty$. From (2.4) written for $F_N(X)$, we have, for$0 \leq m \leq 2^j - 1$, $(f_N^{j+1})_{2^{j+1}n+m+2^j} = \sum_k (f_N)_{k+1}(f_N^j)_{2^j(2n-k)+m}$. This means, in matrix notation,

$$((f_N^{j+1})_{2^{j+1}n+m+2^j})_n = \mathbf{F}_N^1((f_N^j)_{2^j n+m})_n,$$

where $\mathbf{F}_N^1$ is defined by (12.3). Let $m = 2^j - (L-1)/2$ (for sufficiently large j's to ensure $m \geq 0$). The above equation is then rewritten, in terms of the $(\hat{f}_N^j)_n$, as $((\hat{f}_N^{j+1})_{2^{j+1}(n-(L-3)/2)})_n = \mathbf{F}_N^1((\hat{f}_N^j)_{2^j(n-(L-3)/2)})_n$ By symmetry, this equation can be restricted to $n = 0, \cdots, (L-3)/2$, in which case the action of $\mathbf{F}_N^1$ is exactly that of $\hat{\mathbf{F}}_N$. It follows by induction on j that $|(\hat{f}_N^j)_0|$ is equivalent to $\|(\hat{\mathbf{F}}_N)^j\|_\infty$, hence to ρ^j, when $j \to \infty$. □

16. Examples: Deslauriers and Dubuc interpolatory schemes. Deslauriers and Dubuc [10]–[12] studied the regularity of limit functions of a special family of interpolatory subdivision schemes based on Lagrangian interpolation. Recall that for interpolatory schemes the iterated points g_n^j are carried unchanged at each iteration. Here, we simply insert between $g_n^j = g_{2n}^{j+1}$ and $g_{n+1}^j = g_{2n+2}^{j+1}$ the value g_{2n+1}^{j+1} of the Lagrangian polynomial interpolation of the K consecutive values $g_{n+1-K/2}^j, \cdots, g_n^j, g_{n+1}^j, \cdots, g_{n+K/2}^j$, where K is even. This corresponds to a mask g_n of length

TABLE 2

Optimal Hölder regularity estimates r of interpolatory subdivision schemes of Deslauriers and Dubuc [10], [11], [12] for several Lagrangian interpolation orders K corresponding to mask lengths $L = 2K - 1$ (see §16). These estimates are also optimal in the "Fourier sense," and the numbers $(r-1)/2$ give the optimal Sobolev regularity estimates listed in Table 1 (see §17).

K	L	r	K	L	r
2	3	1.	12	23	4.7767
4	7	2.	14	27	5.3173
6	11	2.8300	16	31	5.8294
8	15	3.5511	18	35	6.3233
10	19	4.1935	20	39	6.8054

$L = 2K - 1$, which reads (when made causal by shifting)

$$\begin{cases} g_{2n} = \delta_{n-K/2}, \\ g_{2n+1} = L_n(\frac{K-1}{2}), \end{cases} \tag{16.1}$$

where $L_n(X)$ is the Lagrangian polynomial $L_n(X) = \prod_{k\neq n}(X-k)/(n-k)$ associated to the interpolation points $k = 0, \cdots, K-1$.

Shensa has shown [21] that $G(X)$ of length $L = 2K - 1$ is exactly $G(X) = G_W(X)\tilde{G}_W(X)$, where $G_W(X)$ is the polynomial mask of Daubechies wavelets of compact support $[0, K-1]$ (see §14—this fact will be useful in §17). From §14 it follows that $G(X)$ has exactly K zeros at $X = -1$. Moreover, it is strictly linear phase because $G(e^{i\omega}) = G_W(e^{i\omega})\tilde{G}_W(e^{i\omega}) = |G_W(e^{i\omega})|^2 e^{i(K-1)\omega}$. Thus Theorem 15.2 applies with $N = K - 1$. Moreover, since all interpolatory subdivision schemes are stable (§6), Theorem 15.2 will provide the *exact* regularity order of $\varphi(x)$.

The matrices $\hat{\mathbf{F}}^{K-1}$ (15.1) needed by Theorem 15.2 can be easily determined using the formula

$$(f_{K-1})_n = c \begin{pmatrix} K-2 \\ n \end{pmatrix}^{-1} \sum_{i=0}^{n} (-1)^i \begin{pmatrix} K-1 \\ i \end{pmatrix}^2, \qquad n = 0, \cdots, K-2,$$

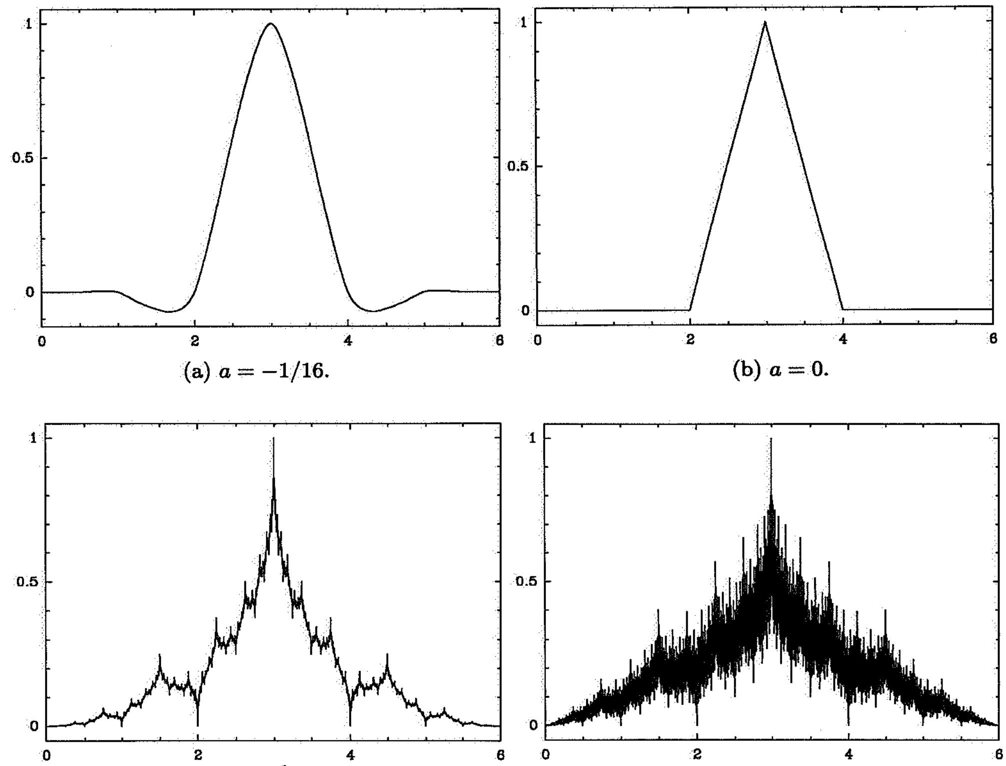
which results from (16.1) after some calculation. Determination of their spectral radii yields to the optimal regularities listed in Table 2. For $L = 7$ (i.e., $K = 4$), using 4 zeros at $X = -1$ in $G(X)$, we find that the limit function is almost $\dot{C}^2$ in the sense of (10.7), which was first proven by Dubuc in [12]. However, when only 2 zeros at $X = -1$ in $G(X)$ are used ($N = 1$), we find that the spectral radius of Theorem 15.2 is $\rho = \frac{1}{2}$, hence the limit function is in fact $\dot{C}^2$.

In [10], Deslauriers and Dubuc extended the study of the previous subdivision scheme for $L = 7$ (i.e., $K = 4$) to the following interpolatory mask (here defined for $n = -3, \cdots, 3$):

$$g_0 = 1, \quad g_{\pm 1} = 1/2 - a, \quad g_{\pm 3} = a, \quad g_n = 0 \quad \text{elsewhere},$$

where $a \in R$. The case $a = -1/16$ corresponds to the previous example, for which the limit function is $\dot{C}^2$.

The simplicity and usefulness of Theorem 15.2 is well illustrated through this example. The mask g_n is easily seen to be strictly linear phase for $-1/16 \leq a \leq \frac{1}{2}$; therefore, Theorem 15.2 applies in this case. (For other values of a we have to use more general theorems such as Theorem 11.1.) Now, for $a \neq -1/16$, $G(X)$ has exactly



(a) $a = -1/16$. (b) $a = 0$.

(c) $a = \frac{1}{4}$. (d) $a = 0.4$.

FIG. 5. *Plots of Deslauriers and Dubuc limit functions corresponding to* $g_0 = 1$, $g_{\pm 1} = 0.5 - a$, $g_{\pm 3} = a$, *and* $g_n = 0$ *elsewhere. The successive values of* a *are* $a = -1/16$ *(regularity order* 2*),* $a = 0$ *(regularity order* 1*),* $a = \frac{1}{4}$ *(regularity order* $\log_2(\sqrt{5} - 1) = 0.305\cdots$*) and* $a = 0.4$ *(regularity order* $0.104\cdots$*).*

two zeros at $X = -1$, and we can, therefore, apply Theorem 15.2 with $N = 1$. We have $(\hat{f}_1)_0 = 1 + 4a$, $(\hat{f}_1)_{\pm 1} = -4a$, and $(\hat{f}_1)_{\pm 2} = 2a$; hence

$$\hat{\mathbf{F}}_1 = \begin{pmatrix} 1 + 4a & -8a \\ 2a & -4a \end{pmatrix}.$$

Its spectral radius is $\rho = (1 + \sqrt{1 + 16a})/2$. From Theorem 15.2, the exact regularity order of $\varphi(x)$ is $r = 2 - \log_2(1 + \sqrt{1 + 16a})$, which decreases from 2 to zero when a increases from $-1/16$ to $\frac{1}{2}$. Figure 5 illustrates this through several examples corresponding to various values of a.

17. Comparison with Fourier-based regularity estimates. This paper has developed a direct approach based on the definition of Hölder regularity. But several other approaches for estimating regularity based on the Fourier transform $\hat{\varphi}(\omega)$ of the (compactly supported) limit function $\varphi(x)$ have also been considered [3]–[5], [10], [11], [23]. Note that we have easy access to $\hat{\varphi}(\omega)$ from mask g_n by [3]–[6]

$$\hat{\varphi}(\omega) = \lim_{j \to \infty} G^j(e^{i\omega}). \tag{17.1}$$

The idea is here to estimate the decay of $\hat{\varphi}(\omega)$ as $|\omega| \to \infty$. To do this, several functional spaces (other than $\dot{C}^r$) can be used to interpolate the spaces C^N of N-times continuously differentiable functions. We generally consider one of the following spaces: $\mathbf{H}_1^r$, $\mathbf{H}_2^r$, $\mathbf{H}_\infty^r$, defined by the conditions $|\omega|^r\hat{\varphi}(\omega) \in \mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty$, respectively. (The spaces $\mathbf{H}_2^r$ are the Sobolev spaces of order r.) Estimations of the parameter r for these spaces ensure some Hölder regularity, since we have, for any $\varepsilon > 0$,

$$\mathbf{H}_\infty^{r+1+2\varepsilon} \subset \mathbf{H}_2^{r+1/2+\varepsilon} \subset \mathbf{H}_1^r \subset \dot{C}^r. \tag{17.2}$$

(These inclusions are easily proven. The second one uses the Cauchy–Schwarz inequality and [11] contains a proof of the last one.)

In [6], Daubechies has derived an estimate for $\varphi(x) \in \dot{C}^{r-\varepsilon}$ based on $\mathbf{H}_\infty^{r+1}$. This estimate is easily recovered from the results derived in this this paper. We have, using the notation of Theorem 10.1,

$$\|F_N^j(X)\|_\infty = \max_n |(f_N^j)_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |F_N^j(e^{i\omega})|\, d\omega \leq \max_{\omega\in\mathbf{R}} |F_N^j(e^{i\omega})|.$$

Define the number β^j such that $2^{-j\beta^j} = \max_{\omega\in\mathbf{R}} |F_N^j(e^{i\omega})|$. Then, by Theorem 10.3 and 11.1, $\varphi(x)$ is $\dot{C}^{N+\beta-\varepsilon}$, where $\beta = \limsup_{j\to\infty}\beta_j$. Cohen [3], [4] has shown that the sequence β^j actually converges to β (the proof is the same as in Theorem 11.1) and that, under some weak conditions on $G(X)$, the optimal regularity order r based on $\mathbf{H}_1^r$ lies between $N+\beta-\varepsilon$ and $N+1+\beta+\varepsilon$. In the case of Daubechies orthonormal wavelets (§14), Cohen and Daubechies [3]–[5] found that β is equivalent to $(\frac{1}{2} - \frac{1}{4}\log_2 3)L \approx 0.10376L$ as $L \to \infty$. It follows (from the following theorem) that the optimal Hölder regularity order of Daubechies orthonormal wavelets is also asymptotically equivalent to $(0.10376\cdots)L$ as $L \to \infty$. However, for small values of L ($L \leq 20$), the estimates derived in this paper, listed in Table 1, are much sharper than the asymptotic result of Cohen and Daubechies.

Daubechies has also derived [6, Appendix] other regularity estimates for the special case of her orthonormal wavelets described in §14. It turns out that her estimates are optimal for the Sobolev spaces $\mathbf{H}_2^r$. This is due to Theorem 15.2 and the property, already mentioned in §14, that the polynomial mask $G(X)$ of a Daubechies wavelet is such that $G(X)\tilde{G}(X)$ is the polynomial mask of a Deslauriers and Dubuc interpolatory scheme [21]. We have $G(e^{i\omega})\tilde{G}(e^{i\omega}) = |G(e^{i\omega})|^2$; therefore, from (17.1) the Fourier transform of the limit function of a Deslauriers and Dubuc scheme is $|\hat{\varphi}(\omega)|^2$, where $\hat{\varphi}(\omega)$ is the Fourier transform of the limit function corresponding to the wavelet. The following theorem shows that since the Deslauriers and Dubuc limit functions are strictly linear phase, their optimal Hölder regularity estimates r, provided by Theorem 15.2 and listed in Table 2, are also optimal for the spaces $\mathbf{H}_1^r$. This implies $\varphi(x) \in \mathbf{H}_2^{r/2}$; therefore, $\varphi(x) \in \dot{C}^{(r-1)/2-\varepsilon}$, which is optimal for spaces $\mathbf{H}_2^{r/2}$. This regularity order is exactly the one derived by Daubechies in [6]. Table 1 lists these optimal Sobolev regularity orders for several lengths.

The above discussion shows that if $G(X)\tilde{G}(X)$ is strictly linear phase, then Theorem 15.2 applied on $G(X)\tilde{G}(X)$ provides the optimal Sobolev regularity of the limit function corresponding to the polynomial mask $G(X)$. This result has been derived independently by Daubechies and Cohen [5] using the Littlewood–Paley theory. Recently, Villemoes [23] has shown that this holds more generally under the weak conditions on $G(X)$ of Cohen [3].

But are these "Fourier-optimal" estimates optimal for Hölder regularity? The following theorem shows that the answer is *no*. The basic reason for this is that the exact Hölder regularity order of $\varphi(x)$ depends on the *phase* of $\hat{\varphi}(\omega)$, i.e., on the phase of $G(e^{i\omega})$ by (17.1), whereas Fourier-based regularity estimates only depend on the *modulus* of $\hat{\varphi}(\omega)$ (or $G(e^{i\omega})$). This theorem also shows that in the framework of §15 (the "strictly linear phase" case), optimal Fourier-based estimates are, in fact, also optimal for Hölder regularity. This is natural since the strictly linear phase case corresponds to limit functions that can be made zero-phase by shifting, i.e., $\hat{\varphi}(\omega) \geq 0$.

THEOREM 17.1. *For strictly linear phase masks, optimal regularity estimates based on* $\mathbf{H}_1^r$ *are also optimal for Hölder regularity.*

Optimal regularity estimates based on $\mathbf{H}_2^r$ *are not optimal for Hölder regularity in general. Nonetheless, they are off by* $\frac{1}{2}$ *at most compared to optimal Hölder regularity estimates.*

Proof. We first prove optimality in the strictly linear phase case. From (17.1), the framework of §15 can easily be reduced to the case $\hat{\varphi}(\omega) \geq 0$. Optimality for spaces $\mathbf{H}_1^r$ and $\dot{C}^r$ coincide if we prove that in this case $\varphi(x) \in \dot{C}^\alpha$ implies $\varphi(x) \in \mathbf{H}_1^{\alpha-\varepsilon}$, for any $\varepsilon > 0$. We may restrict to $0 < \alpha \leq 1$, otherwise just consider a derivative of $\varphi(x)$. The integral $I(w) = \int \sin(\omega h/2)|h|^{-1-\alpha+\varepsilon}\, dh$ absolutely converges for $0 < \alpha \leq 1$; making a change of variable yields $I(w) = |w|^{\alpha-\varepsilon} I(1)$; therefore,

$$\begin{aligned} \int \hat{\varphi}(\omega)|w|^{\alpha-\varepsilon}\, dw &= c \iint \hat{\varphi}(\omega)\sin(\omega h/2)|h|^{-1-\alpha+\varepsilon}\, dh d\omega \\ &= c \int (\varphi(h/2) - \varphi(-h/2))|h|^{-1-\alpha+\varepsilon}\, dh \end{aligned}$$

absolutely converges because $\varphi(x)$ is compactly supported and $\dot{C}^\alpha$. This proves that $\varphi(x) \in \mathbf{H}_1^{\alpha-\varepsilon}$.

Table 1 shows that regularity orders of Daubechies orthonormal wavelets that are optimal for Sobolev spaces $\mathbf{H}_2^r$ are not optimal for Hölder regularity.

The fact that optimal Fourier-based estimates are greater than or equal to $r - \frac{1}{2}$, where r is the exact Hölder regularity estimate, results from the well-known inclusion $\dot{C}^r \subset \mathbf{H}_2^{r-\varepsilon}$, which holds for compactly supported functions [16]. □

Trivial extensions of this theorem can be derived for other "Fourier-based" spaces, using inclusions like (17.2).

Note that Table 1 shows that the Hölder regularity estimates of "more symmetric" wavelets are numerically found to be less than those of minimum phase wavelets (the ones that are "nonsymmetric" the most). That is, more symmetry (for the same modulus of $G(e^{i\omega})$) decreases regularity. In addition, both regularity estimates are greater than the optimal Sobolev regularity order that constitutes a lower-bound for the exact Hölder regularity order. In fact, Theorem 17.1 shows that this lower bound is attained for strictly linear phase masks.

Acknowledgments. I would like to thank Pierre Duhamel and Thierry Blu of Centre National d'Études des Télécommunications, Paris, for fruitful discussions and for carefully reading the manuscript, Ingrid Daubechies of AT&T Bell Labs for letting me use her unpublished material [7], [9] and for pointing out the results of Albert Cohen [3]–[5]; and Lars F. Villemoes of the Technical University of Denmark for suggesting reference [16], which led to an improvement of Theorem 17.1.

I am also grateful to the anonymous referees for pointing out several deficiencies, which led to §3, and for suggesting the references of Dyn and Levin.

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