Linear Strategies for the Gaussian MAC With User Cooperation

Michèle Wigger
Telecom ParisTech
46 rue Barrault
75014 Paris, France
michele.wigger@telecom-paristech.fr

Anand D. Sarwate
ITA Center, CALIT2
University of California, San Diego
La Jolla, CA 92093-0447
asarwate@ucsd.edu

***Abstract*—The capacity region of the two-user additive white Gaussian noise (AWGN) multiple-access channel (MAC) with user cooperation is studied. This channel differs from the classical AWGN MAC in that here each transmitter observes a noisy version of the channel inputs sent by the other transmitter. A new achievable region is presented based on a coding scheme where each transmitter sends a linear combination of these observations and of the codeword it produced to encode its message. For certain choices of the parameters our scheme can be viewed as a scheme where the messages are encoded into stationary processes and where the encoders apply linear time-invariant filters to their observations. The rates achieved by this latter scheme can be expressed in terms of the power spectra of the filters and input processes. For most choices of filters, the optimal input spectra are given by frequency-division and can be found using a water-filling algorithm.**

It is shown that when the transmitters' observations are very noisy our scheme with general parameters outperforms the best previously known schemes. Moreover, the scheme allows to conclude that user cooperation strictly improves the capacity region, irrespective of how noisy the transmitters' observations are. In contrast, when the observations are almost noise-free, then a scheme previously proposed by Carleial improves on our scheme. It is shown that for symmetric setups—i.e., equal power constraints at the two transmitters and equal variances of the noises corrupting the transmitters' observations—and in the asymptotic regime when the observations become noise-free Carleial's lower bound on the sum-capacity and Tandon and Ulukus's recent upper bound on the sum-capacity are tight in the sense that they both tend to the sum-capacity of the setup with noise-free observations with identical slopes.

I. Introduction

Cooperation through different mechanisms such as feedback, conferencing, or eavesdropping provides gains in the capacity of multiple-access channels (MACs). In this paper we study the gains provided by eavesdropping for the two-user additive white Gaussian noise (AWGN) MAC.

Thus, we consider a setup where two transmitters simultaneously wish to communicate with a common receiver who observes the sum of the two transmitted signals corrupted by AWGN, and where each transmitter observes the signal sent by the other transmitter also corrupted by AWGN. This setup is also called the AWGN MAC with user cooperation, because through their observations each transmitter learns about the other transmitter's message allowing the transmitters to cooperate in future transmissions. The MAC with user cooperation was first considered by Willems in [1], but only in the extreme case where each transmitter observes a perfect (noise-free) version of the signal sent by the other transmitter and only for discrete memoryless channels.

The AWGN MAC with user cooperation has previously been studied in [2], [3], and [4]. Carleial [2] and Willems et al. [3] proposed achievable regions based on block-Markov strategies where after each block each transmitter decodes part of the other transmitter's message. Tandon and Ulukus [4] recently presented an outer bound on the capacity. These inner and outer bounds on capacity coincide only in the two extreme cases where the observations are noise-free and where the observations are completely useless or missing. In this paper we analyze these existing bounds for a symmetric setup with equal power constraints at the two transmitters and equal variances of the noises corrupting the transmitters' observations. We show that for this setup the sum-rate achieved by Carleial's scheme and the upper bound on the sum-rate by Tandon and Ulukus are asymptotically tight in the regime where the transmitters' observations are almost noise free in the following sense: as the variances of the noises corrupting these observations tend to 0 these lower and upper bounds on the sum-capacity tend to the sum-capacity for the case with noise-free observations with identical slopes.

In contrast, when the observations are very noisy then there is a significant gap between Carleial's or Willems et al.'s achievable sum-rate and Tandon and Ulukus' upper bound on the sum-capacity. In fact, in this regime the known achievable sum-rates are suboptimal. This is proved in the present paper by proposing new achievable regions which strictly improve on these lower bounds when the cooperation links are very noisy. Our achievable regions moreover have the property that for symmetric setups, no matter how noisy the transmitters' observations are, our regions always include rate pairs that lie outside the no-cooperation capacity region. (This is unlike the schemes by Willems et al. and by Carleial which do not improve over the no-cooperation capacity when the cooperation links exceed a certain threshold.) This result allows us to prove that also for non-symmetric setups the capacity region

The research of Anand D. Sarwate was supported by the California Institute for Telecommunications and Information Technology (CALIT2) at the University of California, San Diego.

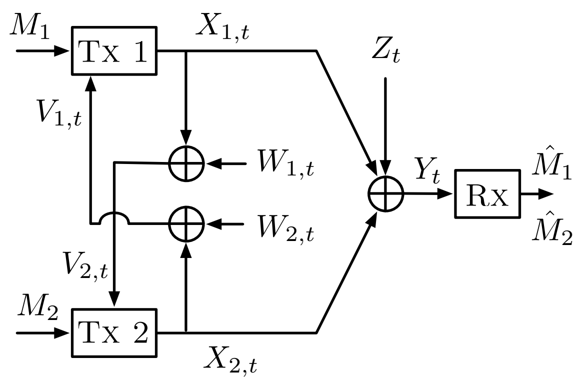

Fig. 1. The two-user multiple-access channel with user cooperation. Each transmitter can overhear a noisy version of the other transmitter's symbol.

with user cooperation is always strictly larger than without cooperation, no matter how noisy the cooperation-links are.

The coding strategies leading to our achievable regions are very simple: each user forms a linear combination of the overheard signal of the other user and its own codeword. The linear combinations are chosen in a way that the original channel is transformed into a new virtual memoryless Gaussian vector channel. We show that this scheme achieves rates strictly larger than the no-cooperation capacity region. We then turn to evaluating a subset of the achievable rate region. These rates correspond to mutual informations induced by a situation in which each user applies a strictly causal linear time-invariant (LTI) filter to the signal of the other user. The achievable rates are integrals of functions of the spectra of the associated filter. Furthermore, we show that for most filters, the optimal spectra of the input processes can be found by frequency division and water-filling.

II. Channel Model

Our communication scenario is depicted in Figure 1: two transmitters wish to simultaneously communicate with a common receiver over a AWGN MAC and each transmitter observes a noisy version of the other transmitter's signal.

Specifically, given that at (the discrete) time t Transmitter 1 sends the symbol $x_{1,t}$ and Transmitter 2 sends the symbol $x_{2,t}$, the receiver observes

$$Y_t = x_{1,t} + x_{2,t} + Z_t,$$

Transmitter 1 observes

$$V_{1,t} = x_{2,t} + W_{2,t},$$

and Transmitter 2 observes

$$V_{2,t} = x_{1,t} + W_{1,t},$$

where $\{Z_t\}$, $\{W_{1,t}\}$, and $\{W_{2,t}\}$ are independent sequences of independent and identically distributed (i.i.d) zero-mean Gaussian random variables of variances $N > 0$, $\sigma_1^2 \geq 0$, and $\sigma_2^2 \geq 0$, respectively.

The goal of the communication is that Transmitter 1 conveys a Message M_1 and Transmitter 2 an independent Message M_2 to the receiver, where each Message M_ν, for $\nu \in \{1,2\}$, is assumed to be uniformly distributed over the discrete finite set $\mathcal{M}_\nu$.

Each Transmitter $\nu \in \{1,2\}$ can compute its channel inputs as a function of its message M_ν and of the previously observed symbols $V_{\nu,1},\ldots,V_{\nu,t-1}$. Thus, each Transmitter ν uses a sequence of encoding functions

$$\psi_{\nu,t}^{(n)} \,:\, \mathcal{M}_\nu \times \mathbb{R}^{t-1} \to \mathbb{R}, \qquad t \in \{1,\ldots,n\}, \tag{1}$$

to compute the inputs

$$X_{\nu,t} = \psi_{\nu,t}^{(n)}\left(M_\nu, V_{\nu,1},\ldots,V_{\nu,t-1}\right), \quad t \in \{1,\ldots,n\}, \tag{2}$$

where n denotes the blocklength of the scheme. The channel inputs $X_{\nu,1},\ldots,X_{\nu,n}$ have to satisfy an average block power constraint P_ν. Thus, we only allow for encoding functions that for $\nu \in \{1,2\}$ satisfy

$$\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[\left(\varphi_{\nu,t}^{(n)}\left(M_\nu, V_{\nu,1},\ldots,V_{\nu,t-1}\right)\right)^2\right] \leq P_\nu, \tag{3}$$

where the expectation is over the messages and the realizations of the noise sequences $\{Z_t\}$, $\{W_{1,t}\}$, and $\{W_{2,t}\}$.[1]

The receiver guesses the pair of messages M_1 and M_2 based on its observed n outputs $Y^n \triangleq (Y_1,\ldots,Y_n)$, i.e., as

$$\left(\hat{M}_1, \hat{M}_2\right) = \phi^{(n)}(Y^n) \tag{4}$$

where $\phi^{(n)}$ denotes an encoding function of the form:

$$\phi^{(n)} \,:\, \mathbb{R}^n \to \mathcal{M}_1 \times \mathcal{M}_2. \tag{5}$$

An error occurs in the communication whenever the pair $\left(\hat{M}_1, \hat{M}_2\right)$ is not equal to the pair (M_1, M_2).

We define a *blocklength n, powers P_1 and P_2 code* of rate pair $\left(\frac{1}{n}\log(|\mathcal{M}_1|), \frac{1}{n}\log(|\mathcal{M}_2|)\right)$ as a triple

$$\left(\left\{\varphi_{1,t}^{(n)}\right\}_{t=1}^{n}, \left\{\varphi_{2,t}^{(n)}\right\}_{t=1}^{n}, \phi^{(n)}\right),$$

where $\left\{\varphi_{1,t}^{(n)}\right\}$ and $\left\{\varphi_{2,t}^{(n)}\right\}$ are of the form (1) and satisfy (3), and where $\phi^{(n)}$ is of the form in (5). In the following we say that a rate pair (R_1, R_2) is achievable if for every $\delta > 0$ and every sufficiently large n there exists a blocklength n, powers P_1 and P_2 code of rates exceeding $R_1 - \delta$ and $R_2 - \delta$ such that the average probability of a decoding error tends to 0 as the blocklength $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\left[\phi^{(n)}\left(Y_1,\ldots,Y_n\right) \neq (M_1, M_2)\right] = 0.$$

The set of all achievable rate pairs is called the capacity region and will be denoted by $C_{\text{UserCoop}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$. The supremum of the sum of rates $R_\Sigma = R_1 + R_2$ over all achievable rate pairs (R_1, R_2) is called the sum-capacity and will be denoted by $C_{\Sigma,\text{UserCoop}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$. In a symmetric setup where $P_1 = P_2 = P$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$

[1]The achievability results in this paper remain valid also when the expected average block-power constraints (3) are replaced by average block-power constraints that hold with probability 1.

the capacity region is also denoted by $C_{\text{UserCoop}}(P, N, \sigma^2)$ and the sum-capacity by $C_{\Sigma,\text{UserCoop}}(P, N, \sigma^2)$. Thus, $C_{\text{UserCoop}}(P, N, \sigma^2) \triangleq C_{\text{UserCoop}}(P, P, N, \sigma_1^2, \sigma_2^2)$ and $C_{\Sigma,\text{UserCoop}}(P, N, \sigma^2) \triangleq C_{\Sigma,\text{UserCoop}}(P, P, N, \sigma_1^2, \sigma_2^2)$.

The setup includes as special cases the setup without cooperation which we model as $\sigma_1^2 = \sigma_2^2 = \infty$, the setup with one-sided cooperation which we model as $\sigma_1^2 = \infty$ and $\sigma_2^2 = 0$, and the special case with noise-free cooperation links which corresponds to $\sigma_1^2 = \sigma_2^2 = 0$. In the case without cooperation we denote the capacity region by $C_{\text{NoCoop}}(P_1, P_2, N)$ and the sum-capacity by $C_{\Sigma,\text{NoCoop}}(P_1, P_2, N)$; in a symmetric setup where $P_1 = P_2 = P$, we also use $C_{\text{NoCoop}}(P, N) \triangleq C_{\text{NoCoop}}(P, P, N)$ and $C_{\Sigma,\text{NoCoop}}(P, N) \triangleq C_{\Sigma,\text{NoCoop}}(P, P, N)$. When the cooperation links are noise-free, then the capacity region coincides with the capacity region of the AWGN MAC with *full cooperation* where there are no cooperation links between the two transmitters but where the transmitters perfectly know each other's message even before the communication starts. Therefore, in this case we denote the capacity region by $C_{\text{FullCoop}}(P_1, P_2, N)$ (or $C_{\text{FullCoop}}(P, N)$ in a symmetric setup) and the sum-capacity by $C_{\Sigma,\text{FullCoop}}(P_1, P_2, N)$ (or $C_{\Sigma,\text{FullCoop}}(P, N)$ in a symmetric setup).

III. Relationships between existing bounds

The described channel has previously been studied by Carleial [2] and by Willems et al. [3]. They presented achievable regions based on block-Markov strategies where after each block the transmitters decode parts of the other transmitter's message. In this paper we focus on Carleial's achievability result; the result in [3] is very similar to the result in [2], and ignored in the following. Carleial's inner bound in [2] is generally characterized by 13 inequalities and thus hard to evaluate. But when either $\sigma_1^2 \le N$ or $\sigma_2^2 \le N$ the bounds simplify to only three bounds.

Definition 1: Given channel parameters $P_1, P_2, N > 0$ and $\sigma_1^2, \sigma_2^2 \ge 0$ such that $\min\{\sigma_1^2, \sigma_2^2\} \le N$, define the region $R_{\text{Carleial}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ as the set of all nonnegative rate pairs (R_1, R_2) that simultaneously satisfy the following three inequalities:

$$R_1 \le \frac{1}{2}\log\left(1 + \frac{\alpha_1 P_1}{\min\{\sigma_1^2, N\}}\right)$$

$$R_2 \le \frac{1}{2}\log\left(1 + \frac{\alpha_2 P_2}{\min\{\sigma_2^2, N\}}\right)$$

$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\sqrt{\bar{\alpha}_1\bar{\alpha}_2 P_1 P_2}}{N}\right)$$

where $\bar{\alpha}_1 \triangleq 1 - \alpha_1$ and $\bar{\alpha}_2 \triangleq 1 - \alpha_2$.

Theorem 1 (Carleial [2]): For channel parameters P_1, P_2, N and σ_1^2, σ_2^2 such that $\min\{\sigma_1^2, \sigma_2^2\} \le N$ the rate region $R_{\text{Carleial}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ is achievable for the AWGN MAC with user cooperation, i.e.,

$$R_{\text{Carleial}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2) \subseteq C_{\text{UserCoop}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2). \tag{6}$$

Tandon and Ulukus recently proved the following outer bound on the capacity [4].

Definition 2: Given channel parameters $P_1, P_2, N > 0$ and $\sigma_1^2, \sigma_2^2 \ge 0$, define the region $R_{\text{Tandon}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ as the set of all nonnegative rate pairs (R_1, R_2) that simultaneously satisfy the following three inequalities:

$$R_1 \le \frac{1}{2}\log\left(1 + \frac{\alpha_1 P_1 (N + \sigma_1^2)}{N\sigma_1^2}\right)$$

$$R_2 \le \frac{1}{2}\log\left(1 + \frac{\alpha_2 P_2 (N + \sigma_2^2)}{N\sigma_2^2}\right)$$

$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{\alpha_1\alpha_2 P_1 P_2 (N + \sigma_1^2 + \sigma_2^2)}{N\sigma_1^2\sigma_2^2} + \frac{\alpha_1 P_1 (N + \sigma_1^2)\sigma_2^2 + \alpha_2 P_2 (N + \sigma_2^2)\sigma_1^2}{N\sigma_1^2\sigma_2^2}\right)$$

$$R_1 + R_2 \le \frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\sqrt{\bar{\alpha}_1\bar{\alpha}_2 P_1 P_2}}{N}\right)$$

where as before $\bar{\alpha}_1 = 1 - \alpha_1$ and $\bar{\alpha}_2 = 1 - \alpha_2$.

Theorem 2 (Tandon and Ulukus [4]): The capacity region $C_{\text{UserCoop}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$ is included in the region $R_{\text{Tandon}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2)$:

$$C_{\text{UserCoop}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2) \subseteq R_{\text{Tandon}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2).$$

The inner bound in [2] and the outer bound in [4] coincide only in the extreme case of no cooperation, i.e.,

$$\begin{aligned} R_{\text{Carleial}}(P_1, P_2, N, 0, 0) &= R_{\text{Tandon}}(P_1, P_2, N, 0, 0) \\ &= C_{\text{FullCoop}}(P_1, P_2, N), \end{aligned}$$

and in the extreme case of *noise-free* cooperation, i.e.,

$$\begin{aligned} R_{\text{Carleial}}(P_1, P_2, N, \infty, \infty) &= R_{\text{Tandon}}(P_1, P_2, N, \infty, \infty) \\ &= C_{\text{NoCoop}}(P_1, P_2, N). \end{aligned}$$

We can show however, that in the symmetric setup (i.e., $P_1 = P_2 = P$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$) and in the asymptotic regime of vanishing $\sigma^2 \downarrow 0$, the sum-rate achieved by Carleial's scheme and the upper bound on the sum-capacity derived by Tandon and Ulukus are asymptotically tight in the sense that they tend to the full-cooperation sum-capacity with identical slopes. They thus also establish the slope of the sum-capacity in this regime.

Before making these observations precise in Proposition 1 ahead, we introduce two definitions. Let $R_{\Sigma,\text{Carleial}}(P, N, \sigma^2)$ denote the maximum sum-rate that is achievable with Carleial's scheme in a symmetric setup, i.e.,

$$R_{\Sigma,\text{Carleial}}(P, N, \sigma^2) \triangleq \max_{(R_1,R_2)\in R_{\text{Carleial}}(P,P,N,\sigma^2,\sigma^2)} (R_1 + R_2).$$

Also, let $R_{\Sigma,\text{Tandon}}(P, N, \sigma^2)$ denote Tandon and Ulukus' upper bound on the sum-capacity in a symmetric setup, i.e.,

$$R_{\Sigma,\text{Tandon}}(P, N, \sigma^2) \triangleq \max_{(R_1,R_2)\in R_{\text{Tandon}}(P,P,N,\sigma^2,\sigma^2)} (R_1 + R_2).$$

Proposition 1: For a symmetric setup where $P_1 = P_2 = P$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, Tandon and Ulukus's upper bound and

Carleial's lower bound on the sum-capacity are asymptotically tight for small σ^2 in the sense that

$$\lim_{\sigma^2 \downarrow 0} \frac{R_{\Sigma,\text{Tandon}}(P,N,\sigma^2) - C_{\Sigma,\text{FullCoop}}(P,N)}{R_{\Sigma,\text{Carleial}}(P,N,\sigma^2) - C_{\Sigma,\text{FullCoop}}(P,N)} = 1. \tag{7}$$

Moreover, they establish the asymptotic behavior of the sum-capacity for small σ^2:

$$\lim_{\sigma^2 \downarrow 0} \frac{C_{\Sigma,\text{UserCoop}}(P,N,\sigma^2) - C_{\Sigma,\text{FullCoop}}(P,N)}{\sigma^2} = \frac{1-\sqrt{1+\frac{4P}{N}}}{N}. \tag{8}$$

Proof: The proof follows by analyzing the maximum sum-rate achieved by Carleial's scheme and the upper bound on the sum-capacity by Tandon and Ulukus. The analysis is simplified by first proving that in Tandon and Ulukus's upper bound, without loss in optimality, one can restricting attention to symmetric parameters $\alpha_1 = \alpha_2$, and by (possibly sub-optimally) also restricting attention to symmetric choices of parameters $\alpha_1 = \alpha_2$ when evaluating Carleial's achievable sum-rate. ■

When the cooperation links are very noisy, there is a significant gap between Carleial's and Willems et al.'s lower bound and Tandon and Ulukus's upper bound on the sum-capacity. In particular, when the cooperation link noise-variances σ_1^2, σ_2^2 exceed a certain threshold (depending on the channel parameters P_1, P_2, N), these lower bounds collapse to the sum-capacity in the case without user-cooperation (see [8]). Tandon and Ulukus' upper bound in contrast suggests that for all values of the cooperation link noise-variances $\sigma_1^2, \sigma_2^2 < \infty$ the sum-capacity with user cooperation exceeds the sum-capacity without user cooperation. In the next section we present achievable regions that strictly exceed the no-cooperation sum-capacity for all finite cooperation link noise-variances $0 \le \sigma_1^2, \sigma_2^2 < \infty$.

IV. Block Scheme and a New Achievable Region

A. Block Scheme

We propose a *linear* scheme that is similar to the schemes for the single-user channel in [5], [6], and for the MAC with noiseless or noisy output feedback in [7], [8]. In contrast to the schemes proposed by Carleial and Willems, in our scheme the transmitters do not decode any parts of the other transmitter's message. Our scheme is described as follows.

Fix a positive integer η, two strictly lower-triangular $\eta \times \eta$ matrices B_1 and B_2, and two positive semi-definite $\eta \times \eta$ matrices K_{U1} and K_{U2} that satisfy the constraints (9) and (10) shown on top of the next page.

We then split the blocklength n into $\frac{n}{\eta}$ subblocks of length η. (That is, we assume that n is an integer multiple of η.) The idea of our scheme is that each Transmitter ν first encodes its Message M_ν into a codeword $U_\nu^n(M_\nu) \triangleq (U_{\nu,1}, \ldots, U_{\nu,n})$—in a way that we describe later on—and then in each subblock $\tau \in \{1, \ldots, n/\eta\}$ it transmits a linear combination of the symbols $U_{\nu,(\tau-1)\eta+1}, \ldots, U_{\nu,\tau\eta}$ and its observations in this same subblock τ. Thus, the transmitter's inputs in block τ do not depend on the transmitter's observations in the previous blocks. To describe Transmitter ν's channel inputs in subblock τ in more detail, we define the η-dimensional column-vectors

$$\begin{aligned}
\mathbf{U}_{\nu,\tau} &\triangleq (U_{\nu,(\tau-1)\eta+1}, \ldots, U_{\nu,\tau\eta})^\mathsf{T} \\
\mathbf{V}_{\nu,\tau} &\triangleq (V_{\nu,(\tau-1)\eta+1}, \ldots, V_{\nu,\tau\eta})^\mathsf{T} \\
\mathbf{X}_{\nu,\tau} &\triangleq (X_{\nu,(\tau-1)\eta+1}, \ldots, X_{\nu,\tau\eta})^\mathsf{T}.
\end{aligned}$$

Transmitter ν's channel inputs in subblock τ are then:

$$\mathbf{X}_{\nu,\tau} = \mathbf{U}_{\nu,\tau} + \mathsf{B}_\nu \mathbf{V}_{\nu,\tau},$$

where B_ν is the strictly lower-triangular matrix we fixed at the beginning of the description of the scheme. The matrix B_ν needs to be strictly lower-triangular because Transmitter ν learns the observation $V_{\nu,t}$ only after producing its time-t channel input $X_{\nu,t}$. Notice that the same matrix B_ν is chosen for all the subblocks.

We next describe the generation of the two codebooks $\mathcal{C}_1$ and $\mathcal{C}_2$ that the transmitters use to encode their messages. Codebook $\mathcal{C}_\nu$, for $\nu \in \{1,2\}$, is generated by randomly and independently drawing the $|\mathcal{M}_\nu|$ codewords in the codebook in the following way. Each codeword $U_\nu(m_\nu)$ is generated by first drawing n/η independent η-dimensional zero-mean Gaussian vectors $\mathbf{U}_{\nu,1}(m_\nu), \ldots, \mathbf{U}_{\nu,n/\eta}(m_\nu)$ of covariance matrix $\mathsf{K}_{U\nu}$ (which we fixed at the beginning of the description), and then choosing the ℓ-th symbol, for $\ell \in \{1, \ldots, \eta\}$, of the vector $\mathbf{U}_{\nu,\tau}$ as the $t = (\nu-1)\eta + \ell$-th entry of the codeword $U_\nu(m_\nu)$. Notice that because our chosen matrices $\mathsf{B}_1, \mathsf{B}_2, \mathsf{K}_{U1}, \mathsf{K}_{U2}$ satisfy (9) and (10), the constructed channel inputs satisfy the average block-power constraints in (3).

The decoding at the receiver is performed in two steps. The receiver first forms the η-dimensional column-vectors $\mathbf{Y}_1, \ldots, \mathbf{Y}_{n/\eta}$:

$$\mathbf{Y}_\tau \triangleq (Y_{(\tau-1)\eta+1}, \ldots, Y_{\tau\eta})^\mathsf{T}, \qquad \tau \in \{1, \ldots, n/\eta\}.$$

Notice that the vectors $\mathbf{Y}_1, \ldots, \mathbf{Y}_{n/\eta}$ can be seen as the outputs of an η-dimensional Gaussian vector MAC with inputs $\mathbf{U}_{1,1}, \ldots, \mathbf{U}_{1,n/\eta}$ and $\mathbf{U}_{2,1}, \ldots, \mathbf{U}_{2,n/\eta}$ and channel law as described in (11) on top of the next page. The receiver therefore decodes the desired messages using an optimal decoding rule for a Gaussian multi-antenna MAC with noises that are correlated across antennas but uncorrelated in time.

B. Achievable Region

The scheme described in the previous subsection achieves the following rate region.

Definition 3: Given a positive integer η, two positive semidefinite $\eta \times \eta$-matrices K_{U1} and K_{U2}, two strictly lower-triangular $\eta \times \eta$-matrices B_1 and B_2, the region $\mathcal{R}_\text{B}(\eta, \mathsf{K}_{U1}, \mathsf{K}_{U2}, \mathsf{B}_1, \mathsf{B}_2)$ is defined as the set of all nonnegative rate pairs (R_1, R_2) satisfying:

$$\begin{aligned}
R_1 &\le I(\mathbf{U}_1; \mathbf{Y} | \mathbf{U}_2) \\
R_2 &\le I(\mathbf{U}_2; \mathbf{Y} | \mathbf{U}_1) \\
R_1 + R_2 &\le I(\mathbf{U}_1, \mathbf{U}_2; \mathbf{Y}).
\end{aligned}$$

$$(I-\mathsf{B}_1\mathsf{B}_2)^{-1}\left(\mathsf{K}_{U1}+\mathsf{B}_1\mathsf{K}_{U2}\mathsf{B}_1^{\intercal}+\mathsf{B}_1\mathsf{B}_2\mathsf{B}_2^{\intercal}\mathsf{B}_1^{\intercal}\sigma_2^2+\mathsf{B}_1\mathsf{B}_1^{\intercal}\sigma_1^2\right)(I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}\leq\eta P_1 \tag{9}$$

$$(I-\mathsf{B}_2\mathsf{B}_1)^{-1}\left(\mathsf{K}_{U2}+\mathsf{B}_2\mathsf{K}_{U1}\mathsf{B}_2^{\intercal}+\mathsf{B}_2\mathsf{B}_1\mathsf{B}_1^{\intercal}\mathsf{B}_2^{\intercal}\sigma_1^2+\mathsf{B}_2\mathsf{B}_2^{\intercal}\sigma_2^2\right)(I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}\leq\eta P_2 \tag{10}$$

$$\begin{aligned}\mathbf{Y}_\tau &= \left((I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}+(I-\mathsf{B}_2\mathsf{B}_1)^{-\intercal}\mathsf{B}_2\right)\mathbf{U}_{1,\tau}+\left((I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}\mathsf{B}_1+(I-\mathsf{B}_2\mathsf{B}_1)^{-\intercal}\right)\mathbf{U}_{2,\tau}\\ &\quad+\left((I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}\mathsf{B}_1\mathsf{B}_2+(I-\mathsf{B}_2\mathsf{B}_1)^{-\intercal}\mathsf{B}_2\right)\mathbf{W}_{1,\tau}+\left((I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}\mathsf{B}_1+(I-\mathsf{B}_1\mathsf{B}_2)^{-\intercal}\mathsf{B}_2\mathsf{B}_1\right)\mathbf{W}_{2,\tau}+\mathbf{Z}_\tau\end{aligned} \tag{11}$$

Here, $\mathbf{U}_1$ and $\mathbf{U}_2$ are independent centered Gaussian η-dimensional vectors of covariance matrices K_{U1} and K_{U2}, and $\mathbf{Y}$ is an η-dimensional vector defined through:

$$\mathbf{X}_1=(\mathsf{I}_\eta-\mathsf{B}_1\mathsf{B}_2)^{-1}\left(\mathbf{U}_1+\mathsf{B}_1\mathbf{U}_2+\mathsf{B}_1\mathsf{B}_2\mathbf{W}_1+\mathsf{B}_1\mathbf{W}_2\right) \tag{12}$$

$$\mathbf{X}_2=(\mathsf{I}_\eta-\mathsf{B}_2\mathsf{B}_1)^{-1}\left(\mathbf{U}_2+\mathsf{B}_2\mathbf{U}_1+\mathsf{B}_2\mathsf{B}_1\mathbf{W}_2+\mathsf{B}_2\mathbf{W}_1\right) \tag{13}$$

$$\mathbf{Y}=\mathbf{X}_1+\mathbf{X}_2+\mathbf{Z}, \tag{14}$$

where $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{Z}$ are η-dimensional zero-mean Gaussian vectors of covariance matrices $\sigma_1^2\mathsf{I}_\eta$, $\sigma_2^2\mathsf{I}_\eta$, and $N\mathsf{I}_\eta$, where I_η denotes the η-by-η identity matrix.

Definition 4: Define the two-dimensional region

$$\mathcal{R}_{\text{Block}}(P_1,P_2,N,\sigma_1^2,\sigma_2^2)\triangleq\text{cl}\left(\bigcup_{\eta\in\mathbb{N},\mathsf{K}_{U1},\mathsf{K}_{U2},\mathsf{B}_1,\mathsf{B}_2}\mathcal{R}_{\text{B}}(\eta,\mathsf{K}_{U1},\mathsf{K}_{U2},\mathsf{B}_1,\mathsf{B}_2)\right), \tag{15}$$

where the union is taken over all positive integers η, all positive semidefinite $\eta\times\eta$-matrices K_{U1} and K_{U2}, and all strictly-lower triangular $\eta\times\eta$ matrices B_1 and B_2 such that the power constraints (9) and (10) are satisfied.
Notice that the power constraints (9) and (10) assure that the input vectors as defined in (12) and (13) satisfy

$$\frac{1}{\eta}\|\mathbf{X}_1\|^2\leq P_1 \qquad\text{and}\qquad \frac{1}{\eta}\|\mathbf{X}_2\|^2\leq P_2. \tag{16}$$

Theorem 3: The region $\mathcal{R}_{\text{Block}}$ is achievable for the two-user AWGN MAC with user cooperation, i.e.,

$$\mathcal{R}_{\text{Block}}(P_1,P_2,N,\sigma_1^2,\sigma_2^2)\subseteq C_{\text{UserCoop}}(P_1,P_2,N,\sigma_1^2,\sigma_2^2).$$

In contrast to the regions in [2] and [3], our achievable region in Theorem 3 improves on the no-cooperation capacity for all symmetric setups. The following more general result can be derived.

Proposition 2: Irrespective of the powers $P_1,P_2,N>0$ and the (finite) cooperation link noise-variances $\sigma_1^2,\sigma_2^2\geq 0$, the capacity region of the AWGN MAC with user cooperation is strictly larger than the capacity region of the AWGN MAC without user cooperation, i.e.,

$$C_{\text{UserCoop}}(P_1,P_2,N,\sigma_1^2,\sigma_2^2)\supsetneq C_{\text{NoCoop}}(P_1,P_2,N,\sigma_1^2,\sigma_2^2). \tag{17}$$

Proof: For symmetric setups where $P_1=P_2=P$ and $\sigma_1^2=\sigma_2^2=\sigma^2$ the proposition follows directly from Theorem 3. In fact, for the choice of parameters $\eta=2$, $\mathsf{K}_{U1}=\begin{pmatrix}P & \frac{P}{\sqrt{D}}\\ \frac{P}{\sqrt{D}} & \frac{P^2}{D}\end{pmatrix}$, $\mathsf{K}_{U2}=\begin{pmatrix}P & -\frac{P}{\sqrt{D}}\\ -\frac{P}{\sqrt{D}} & \frac{P^2}{D}\end{pmatrix}$, $\mathsf{B}_1=\begin{pmatrix}0 & 0\\ -\sqrt{\frac{P}{D}}\gamma & 0\end{pmatrix}$ and $\mathsf{B}_2=\begin{pmatrix}0 & 0\\ \sqrt{\frac{P}{D}}\gamma & 0\end{pmatrix}$, where $\gamma=\frac{N}{2\sigma^2+N+\sigma^2\frac{N}{P}}$ and $D=P+\gamma^2(P+\sigma^2)$, the region $\mathcal{R}_{\text{B}}(\eta,\mathsf{K}_{U1},\mathsf{K}_{U2},\mathsf{B}_1,\mathsf{B}_2)$ is strictly larger than the no-cooperation capacity $C_{\text{NoCoop}}(P_1,P_2,N)$. The proof for asymmetric setups is based on the idea of using a time-sharing/rate-splitting strategy to combine our linear scheme with a no-cooperation scheme, similar to the proof in [8, SectionV-E1]. ■

V. Linear Filtering Region

The multi-letter achievable region in Theorem 3 is generally difficult to evaluate and to analyze. In the following we present a single-letter expression characterizing a subset of $\mathcal{R}_{\text{Linear}}$, and thus an achievable region. This new (possibly smaller) achievable region is defined as the limiting region obtained when in Definition 4 the union is taken only over matrices $\mathsf{B}_1,\mathsf{B}_2,\mathsf{K}_{U1}$, and K_{U2} that are Toeplitz and the parameter $\eta\to\infty$. Notice that while in this setup we do not know whether this ”stationary” choice is optimal, it has been shown [6] that in a Gaussian single-user channel with noise-free feedback such a stationary choice is optimal for a similar class of linear schemes.

The resulting single-letter achievable region is characterized as a union of regions where the union is over pairs (H_1,H_2) of strictly causal filters and power spectral densities $\mathcal{S}_{U1}$ and $\mathcal{S}_{U2}$. It still seems hard to evaluate this achievable region even numerically, because the optimal choice of the filters H_1 and H_2 is not known. However, given a choice of the filters H_1 and H_2 it is possible to determine the optimal power spectra $\mathcal{S}_{U1}$ and $\mathcal{S}_{U2}$ which simplifies the evaluation of the region. In particular, it can be shown that a frequency-division strategy is optimal for almost all filters and available transmit powers. The solutions resemble the solutions in [9] for the MAC with inter-symbols interference.

A. An Achievable Region

Definition 5: Let H_1,H_2 be the Fourier transforms of the impulse response of two strictly-causal filters, and let $\mathcal{S}_{U1}$ and $\mathcal{S}_{U2}$ be two power-spectral densities. Define the region $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1},\mathcal{S}_{U2},H_1,H_2)$ as the set of all nonnegative rate pairs

(R_1, R_2) that satisfy

$$R_1 \leq \frac{1}{4\pi}\int_0^{2\pi} \log\left(1 + \frac{|1+H_2(\omega)|^2 \mathcal{S}_{U1}(\omega)}{\mathcal{N}(\omega)}\right) \mathrm{d}\omega \tag{18}$$

$$R_2 \leq \frac{1}{4\pi}\int_0^{2\pi} \log\left(1 + \frac{|1+H_1(\omega)|^2 \mathcal{S}_{U2}(\omega)}{\mathcal{N}(\omega)}\right) \mathrm{d}\omega \tag{19}$$

$$R_1 + R_2 \leq \frac{1}{4\pi}\int_0^{2\pi} \log\left(1 + \frac{|1+H_2(\omega)|^2 \mathcal{S}_{U1}(\omega)}{\mathcal{N}(\omega)} + \frac{|1+H_1(\omega)|^2 \mathcal{S}_{U2}(\omega)}{\mathcal{N}(\omega)}\right), \tag{20}$$

where

$$\begin{aligned}\mathcal{N}(\omega) \triangleq\ & |H_2(\omega) + H_1(\omega)H_2(\omega)|^2\sigma_1^2 \\ & + |H_1(\omega) + H_2(\omega)H_1(\omega)|^2\sigma_2^2 \\ & + |1 - H_1(\omega)H_2(\omega)|^2 N. \end{aligned} \tag{21}$$

Definition 6: Define the two-dimensional region

$$\mathcal{R}_{\text{Filter}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2) \triangleq \text{cl}\left(\bigcup_{\mathcal{S}_{U1},\mathcal{S}_{U2},H_1,H_2} \mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)\right),$$

where the union is over all power spectral densities $\mathcal{S}_{U1}$ and $\mathcal{S}_{U2}$ and over all transfer functions H_1, H_2 of strictly-causal filters such that the power constraints (22) and (23) (expressed in the Fourier-domain) shown on top of the next page are satisfied.

Theorem 4: The region $\mathcal{R}_{\text{Filter}}$ is achievable over the AWGN MAC with user cooperation, i.e.,

$$\mathcal{R}_{\text{Filter}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2) \subseteq C_{\text{UserCoop}}(P_1, P_2, N, \sigma_1^2, \sigma_2^2).$$

Proof: Fix two finite, strictly-causal filters with transfer functions

$$H_1(\omega) = \sum_{k=1}^{m} a_{1,k} e^{-\mathrm{i}\omega k} \tag{24}$$

$$H_2(\omega) = \sum_{k=1}^{m} a_{2,k} e^{-\mathrm{i}\omega k}. \tag{25}$$

and two power spectral densities $\mathcal{S}_{U1}$ and $\mathcal{S}_{U2}$ such that the tuple $(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$ satisfies the power constraints (22) and (23) with strict inequality. In the following we sketch a proof of achievability of the region $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$.

By continuity arguments and because by definition the capacity region is closed, then the region $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$ is also achievable if H_1 and H_2 are the transfer functions of *infinite* strictly causal filters, or if the the tuple $(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$ satisfies the power constraints (22) and (23) with equality.

The achievability of the region $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$ is proved as follows. We first express the region in terms of mutual informations of stationary processes (Subsection V-A1). We then present a sequence of achievable regions $\{\mathcal{R}(\eta)\}_{\eta=1}^{\infty}$ (Subsection V-A2) using Theorem 3. Finally, using the representation of $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$ derived in Subsection V-A1, we prove that every rate pair in the interior of $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$ is contained in the region $\mathcal{R}(\eta)$ for all sufficiently large η. Since the capacity region is closed, this then proves the achievability of the entire region $\mathcal{R}_{\text{F}}(\mathcal{S}_{U1}, \mathcal{S}_{U2}, H_1, H_2)$, i.e., also of its boundary points.

In the following subsections let $\{U_{1,t}\}$ and $\{U_{2,t}\}$ be independent discrete-time zero-mean stationary Gaussian sequences of power spectral densities $\mathcal{S}_{U1}$ and $\mathcal{S}_{U2}$ Also, let U_1^η denote the tuple $(U_{1,1}, \ldots, U_{1,n})$, and U_2^η the tuple $(U_{2,1}, \ldots, U_{2,n})$.

1) Formulation of $\mathcal{R}_{\text{F}}$ using stationary processes.: For all $-\infty < t < \infty$ define

$$\tilde{X}_{1,t} = U_{1,t} + \sum_{k=1}^{m} a_{1,k}\left(\tilde{X}_{2,t-k} + W_{2,t-k}\right) \tag{26}$$

$$\tilde{X}_{2,t} = U_{2,t} + \sum_{k=1}^{m} a_{2,k}\left(\tilde{X}_{1,t-k} + W_{1,t-k}\right) \tag{27}$$

$$\tilde{Y}_t = \tilde{X}_{1,t} + \tilde{X}_{2,t} + Z_t. \tag{28}$$

Notice that the inputs $\tilde{X}_{1,t}$ and $\tilde{X}_{2,t}$ here are stationary stochastic processes. Therefore, by Parseval's theorem and because by our assumption the tuple $(H_1, H_2, \mathcal{S}_{U1}, \mathcal{S}_{U2})$ satisfies the power constraints (22) and (23) with strict inequality, for each integer t and $\nu \in \{1, 2\}$:

$$\mathsf{E}\left[\tilde{X}_{\nu,t}^2\right] < P_\nu. \tag{29}$$

Furthermore, by Szegö's Theorem [10], [6], constraints (18)–(20) are equivalent to the constraints

$$R_1 \leq \lim_{\eta\to\infty} \frac{1}{\eta} I\left(\tilde{Y}^\eta\ ;\ U_1^\eta \middle| U_2^\eta\right) \tag{30}$$

$$R_2 \leq \lim_{\eta\to\infty} \frac{1}{\eta} I\left(\tilde{Y}^\eta\ ;\ U_2^\eta \middle| U_1^\eta\right) \tag{31}$$

$$R_1 + R_2 \leq \lim_{\eta\to\infty} \frac{1}{\eta} I\left(\tilde{Y}^\eta\ ;\ U_1^\eta, U_2^\eta\right) \tag{32}$$

where $\tilde{Y}^\eta$ denotes the tuple $(\tilde{Y}_1, \ldots, \tilde{Y}_n)$. Therefore, it suffices to prove that every nonnegative rate-pair (R_1, R_2) that simultaneously satisfies the three constraints (30)–(32) is achievable.

2) Sequence of Achievable Regions with Block Scheme: For each positive integer η, define $\mathsf{B}_1(\eta)$ as the $\eta \times \eta$ strictly-lower triangular matrix whose column-j, row-i entry equals $a_{1,i-j}$ if $0 < i - j \leq m$ and equals 0 otherwise, and similarly, $\mathsf{B}_2(\eta)$ as the $\eta \times \eta$ strictly-lower triangular matrix whose column-j, row-i entry equals $a_{2,i-j}$ if $0 < i - j \leq m$ and equals 0 otherwise. Further, for each positive integer η define $\mathsf{K}_{U1}(\eta)$ as the covariance matrix of the vector $\mathbf{U}_1(\eta) \triangleq (U_{1,1}, \ldots, U_{1,\eta})^\intercal$ and $\mathsf{K}_{U2}(\eta)$ as the covariance matrix of the vector $\mathbf{U}_2(\eta) \triangleq (U_{2,1}, \ldots, U_{2,\eta})^\intercal$.

Also, for each η let $\mathbf{Y}(\eta), \mathbf{X}_1(\eta)$, and $\mathbf{X}_2(\eta)$ be η-dimensional vectors defined through (12), (13), (14) when B_1 is replaced with $\mathsf{B}_1(\eta)$, B_2 with $\mathsf{B}_2(\eta)$, K_{U1} with $\mathsf{K}_{U1}(\eta)$, and K_{U2} with $\mathsf{K}_{U2}(\eta)$.

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathcal{S}_{U1}(\omega)+\mathcal{S}_{U2}(\omega)|H_1(\omega)|^2+|H_1(\omega)H_2(\omega)|^2\sigma_1^2+|H_1(\omega)|^2\sigma_2^2}{|1-H_1(\omega)H_2(\omega)|^2}d\omega\leq P_1 \tag{22}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\mathcal{S}_{U2}(\omega)+\mathcal{S}_{U1}(\omega)|H_2(\omega)|^2+|H_1(\omega)H_2(\omega)|^2\sigma_2^2+|H_2(\omega)|^2\sigma_1^2}{|1-H_1(\omega)H_2(\omega)|^2}d\omega\leq P_2 \tag{23}$$

It can be shown that for all sufficiently large positive integers η the vectors $\mathbf{X}_1(\eta)$ and $\mathbf{X}_2(\eta)$ have power at most ηP_1 and ηP_2, and that thus the tuple $(\mathsf{B}_1(\eta),\mathsf{B}_2(\eta),\mathsf{K}_{U1}(\eta),\mathsf{K}_{U2}(\eta))$ satisfies the power constraints (9) and (10). Therefore, by Theorem 3, for each positive integer η the region $\mathcal{R}_{\mathrm{B}}(\mathsf{B}_1(\eta),\mathsf{B}_2(\eta),\mathsf{K}_{U1}(\eta),\mathsf{K}_{U2}(\eta))$ is achievable, i.e., for every positive integer η:

$$\begin{aligned}&\mathcal{R}_{\mathrm{B}}(\eta,\mathsf{B}_1(\eta),\mathsf{B}_2(\eta),\mathsf{K}_{U1}(\eta),\mathsf{K}_{U2}(\eta))\\&\qquad\subseteq C_{\text{UserCoop}}(P_1,P_2,N,\sigma_1^2,\sigma_2^2),\end{aligned} \tag{33}$$

where recall that the region $\mathcal{R}_{\mathrm{B}}(\mathsf{B}_1(\eta),\mathsf{B}_2(\eta),\mathsf{K}_{U1}(\eta),\mathsf{K}_{U2}(\eta))$ is defined as the set of all nonnegative rate pairs (R_1,R_2) that satisfy

$$R_1\leq\frac{1}{\eta}I(\mathbf{Y}(\eta);\mathbf{U}_1(\eta)|\mathbf{U}_2(\eta)) \tag{34}$$

$$R_2\leq\frac{1}{\eta}I(\mathbf{Y}(\eta);\mathbf{U}_2(\eta)|\mathbf{U}_1(\eta)) \tag{35}$$

$$R_1+R_2\leq\frac{1}{\eta}I(\mathbf{Y}(\eta);\mathbf{U}_1(\eta),\mathbf{U}_2(\eta)). \tag{36}$$

For every η we then choose the region $\mathcal{R}(\eta)$ as $\mathcal{R}_{\mathrm{B}}(\mathsf{B}_1(\eta),\mathsf{B}_2(\eta),\mathsf{K}_{U1}(\eta),\mathsf{K}_{U2}(\eta))$, which as shown above is achievable.

3) Equivalence of Regions: To prove that every rate pair on the interior of the region $\mathcal{R}_F(H_1,H_2,\mathcal{S}_{U1},\mathcal{S}_{U2})$ is contained in the region $\mathcal{R}(\eta)$ for all sufficiently large η, by (30)–(32) and (34)–(36) it suffices to prove the following three limits:

$$\lim_{\eta\to\infty}\frac{1}{\eta}\Big(I(\mathbf{Y}(\eta);\mathbf{U}_1(\eta)|\mathbf{U}_2(\eta))-I(\tilde{Y}^\eta;U_1^\eta|U_2^\eta)\Big)=0$$

$$\lim_{\eta\to\infty}\frac{1}{\eta}\Big(I(\mathbf{Y}(\eta);\mathbf{U}_2(\eta)|\mathbf{U}_1(\eta)-I(\tilde{Y}^\eta;U_2^\eta|U_1^\eta)\Big)=0$$

$$\lim_{\eta\to\infty}\frac{1}{\eta}\Big(I(\mathbf{Y}(\eta);\mathbf{U}_1(\eta),\mathbf{U}_2(\eta))-I(\tilde{Y}^\eta;U_1^\eta,U_2^\eta)\Big)=0$$

The proof is lengthy, and therefore omitted. ■

B. Structural Results

In this section we derive the power spectral densities $\mathcal{S}_{U1}^*$ and $\mathcal{S}_{U2}^*$ that maximize the sum-rate (20) for given transfer functions H_1 and H_2.

To this end, we fix two transfer functions H_1 and H_2, and denote by R_{sum} the right-hand side of (20), by ρ_1 the right-hand side of (22), and by ρ_2 the right-hand side of (23). The Lagrangian is then given by

$$J(\mathcal{S}_{U1},\mathcal{S}_{U2},\lambda_2,\lambda_2)=R_{\text{sum}}+\lambda_1(\rho_1-P_1)+\lambda_2(\rho_2-P_2).$$

Differentiating with respect to $\mathcal{S}_{U1}(\omega)$ and $\mathcal{S}_{U2}(\omega)$ yields the KKT conditions which show that for ω such that the optimal $\mathcal{S}_{U1}^*(\omega)>0$:

$$\begin{aligned}&\frac{|1+H_2(\omega)|^2}{\mathcal{N}(\omega)+|1+H_2(\omega)|^2S_{U1}^*(\omega)+|1+H_1(\omega)|^2S_{U2}^*(\omega)}\\&\qquad=\frac{\lambda_1+\lambda_2|H_2(\omega)|^2}{|1-H_1(\omega)H_2(\omega)|^2},\end{aligned} \tag{37}$$

and for ω such that the optimal $\mathcal{S}_{U2}^*(\omega)>0$:

$$\begin{aligned}&\frac{|1+H_1(\omega)|^2}{\mathcal{N}(\omega)+|1+H_2(\omega)|^2S_{U1}^*(\omega)+|1+H_1(\omega)|^2S_{U2}^*(\omega)}\\&\qquad=\frac{\lambda_1|H_1(\omega)|^2+\lambda_2}{|1-H_1(\omega)H_2(\omega)|^2}.\end{aligned} \tag{38}$$

Therefore, for all ω such that both $\mathcal{S}_{U1}^*(\omega)>0$ and $\mathcal{S}_{U2}^*(\omega)>0$, we must have

$$\frac{\lambda_1+\lambda_2|H_2(\omega)|^2}{|1+H_2(\omega)|^2}=\frac{\lambda_1|H_1(\omega)|^2+\lambda_2}{|1+H_1(\omega)|^2}. \tag{39}$$

However, for most filters $H_1\neq H_2$ of interest this condition holds only on a set of measure 0, which implies that for these filters there is no loss in optimality in considering only power spectral densities where for every ω either $\mathcal{S}_{U1}(\omega)=0$ or $\mathcal{S}_{U2}(\omega)=0$. Thus, for these filters it is optimal that the two transmitters apply a frequency division strategy.

When the optimal $\mathcal{S}_{U1}^*(\omega)>0$ and $\mathcal{S}_{U2}^*(\omega)=0$, we obtain from (37) that

$$\mathcal{S}_{U1}^*(\omega)=\frac{|1-H_1(\omega)H_2(\omega)|^2|}{\lambda_1+\lambda_2|H_2(\omega)|^2}-\frac{\mathcal{N}(\omega)}{|1-H_2(\omega)|^2},$$

and a similarly, when the optimal $\mathcal{S}_{U2}^*(\omega)>0$ and $\mathcal{S}_{U1}^*(\omega)=0$, we obtain from (38) that

$$\mathcal{S}_{U2}^*(\omega)=\frac{|1-H_1(\omega)H_2(\omega)|^2|}{\lambda_1|H_1(\omega)|^2+\lambda_2}-\frac{\mathcal{N}(\omega)}{|1-H_1(\omega)|^2}.$$

We thus obtain the following Theorem.

Theorem 5: Given filters H_1 and H_2 such that (39) holds only over a set of measure 0, the optimal power spectra $\mathcal{S}_{U1}^*(\omega)$ and $\mathcal{S}_{U2}^*(\omega)$ have disjoint supports. For ω where the left-hand side of (39) is smaller than its right-hand side the optimal power spectral densities are:

$$\mathcal{S}_{U1}^*(\omega)=\frac{|1-H_1(\omega)H_2(\omega)|^2|}{\lambda_1+\lambda_2|H_2(\omega)|^2}-\frac{\mathcal{N}(\omega)}{|1-H_2(\omega)|^2}, \tag{40}$$

$$\mathcal{S}_{U2}^*(\omega)=0, \tag{41}$$

for ω where the let-hand side of (39) is larger than its right-hand side:

$$\mathcal{S}_{U1}^*(\omega) = 0, \tag{42}$$

$$\mathcal{S}_{U2}^*(\omega) = \frac{|1 - H_1(\omega)H_2(\omega)|^2|}{\lambda_1|H_1(\omega)|^2 + \lambda_2} - \frac{\mathcal{N}(\omega)}{|1 - H_1(\omega)|^2}. \tag{43}$$

The constants λ_1 and λ_2 must be chosen such that $\rho_1 = P_1$ and $\rho_2 = P_2$.

Following an idea in [11], the above optimal power spectral densities can be obtained using a waterfilling algorithm. We define:

$$\tilde{\mathcal{S}}_1(\omega) \triangleq \frac{\lambda_1 + \lambda_2|H_2(\omega)|^2}{|1 - H_1(\omega)H_2(\omega)|^2}\mathcal{S}_{U1}(\omega) \tag{44}$$

$$\tilde{\mathcal{S}}_2(\omega) \triangleq \frac{\lambda_1|H_1(\omega)|^2 + \lambda_2}{|1 - H_1(\omega)H_2(\omega)|^2}\mathcal{S}_{U2}(\omega) \tag{45}$$

$$\mathcal{N}_1(\omega) \triangleq \frac{\lambda_1 + \lambda_2|H_2(\omega)|^2}{|1 - H_1(\omega)H_2(\omega)|^2|1 + H_2(\omega)|^2}\mathcal{N}(\omega) \tag{46}$$

$$\mathcal{N}_2(\omega) \triangleq \frac{\lambda_1|H_1(\omega)|^2 + \lambda_2}{|1 - H_1(\omega)H_2(\omega)|^2|1 + H_1(\omega)|^2}\mathcal{N}(\omega). \tag{47}$$

Plugging these definitions into (20) results in

$$R_{\text{sum}} = \frac{1}{4\pi}\int_0^{2\pi} \log\left(1 + \frac{\tilde{\mathcal{S}}_1(\omega)}{\mathcal{N}_1(\omega)} + \frac{\tilde{\mathcal{S}}_2(\omega)}{\mathcal{N}_2(\omega)}\right) d\omega. \tag{48}$$

Using Theorem 5, it is easily verified that the power spectral densities $\tilde{\mathcal{S}}_{U1}^*$ and $\tilde{\mathcal{S}}_{U2}^*$ that maximize (48) can be obtained by waterfilling over the modified noise-spectra $\mathcal{N}_1$ and $\mathcal{N}_2$ as described in the following corollary.

Corollary 1: Given filters H_1 and H_2 such that (39) holds only over a set of measure 0, the optimal power spectra $\tilde{\mathcal{S}}_1^*(\omega)$ and $\tilde{\mathcal{S}}_2^*(\omega)$ have disjoint supports. For ω where $\mathcal{N}_1(\omega) < \mathcal{N}_2(\omega)$:

$$\tilde{\mathcal{S}}_1^*(\omega) = [1 - \mathcal{N}_1(\omega)]^+, \qquad \tilde{\mathcal{S}}_2^*(\omega) = 0, \tag{49}$$

and for ω where $\mathcal{N}_1(\omega) > \mathcal{N}_2(\omega)$:

$$\tilde{\mathcal{S}}_1^*(\omega) = 0, \qquad \tilde{\mathcal{S}}_2^*(\omega) = [1 - \mathcal{N}_2(\omega)]^+. \tag{50}$$

The constants λ_1 and λ_2 in the definitions (44)–(47) must be chosen such that $\rho_1 = P_1$ and $\rho_2 = P_2$.

VI. Example and Extensions

We now turn to a brief example of the rates achievable by simple filters. Figure 2 shows the rates achievable by 5 different strategies for a symmetric setup where $P_1 = P_2 = 3$ and $N = 1$. The horizontal axis is the cross-link noise power and the vertical axis is the rate in nats. The "one tap" region is given by optimizing filters of the form $H_1(z) = a_1 z^{-1}$ and $H_2(z) = -a_1 z^{-1}$ over a_1 and "two taps" plots are given by optimizing $H_1(z) = a_1 z^{-1} + a_2 z^{-1}$ and $H_2(z) = -a_1 z^{-1} + a_2 z^{-2}$ over (a_1, a_2). The plots show that for low cross-link SNR, or high σ^2, the linear filtering strategies outperform the Carleial, Willems, and No-Cooperation regions.

The schemes here raise interesting questions. The filtering-achievable region is a subset of the block achievable region, but are they in fact equal? How should we optimally choose filters for the filtering-achievable region? What happens if there is only one cooperation link? Is there a cut-off σ^2 for which linear strategies are provably superior to the Carleial and Willems strategies?

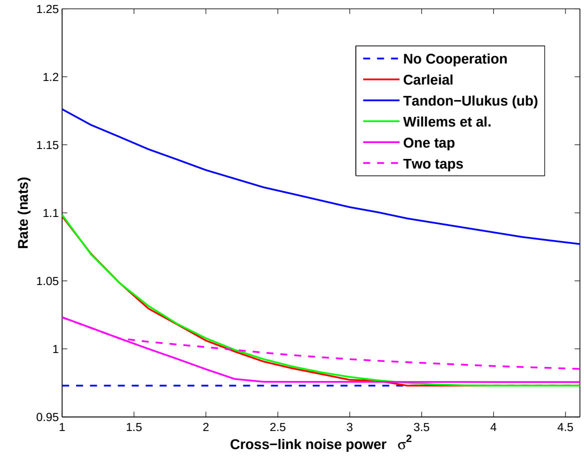

Fig. 2. Achievable rates versus cross-link noise under different strategies for $P = 3$ and $N = 1$.

Acknowledgements

The authors would like to acknowledge many helpful conversations with Young-Han Kim.

References

[1] F. M. Willems and E. C. Van der Meulen, "The discrete memoryless multiple-access channel with cribbing encoders," *IEEE Transactions on Information Theory*, vol. 31, no. 3, pp. 313–327, May 1985.

[2] A. Carleial, "Multiple-access channels with different generalized feedback signals," *IEEE Transactions on Information Theory*, vol. 28, no. 6, pp. 841–850, November 1982.

[3] F. Willems, E. van der Meulen, and J. Schalkwijk, "Generalized feedback for the discrete memoryless multiple access channel," in *Proceedings of the 21st Allerton Conference on Communication, Control, and Computing*, Allerton House, Monticello, IL, October 5-7 1983, pp. 284–292.

[4] R. Tandon and S. Ulukus, "Dependence balance based outer bounds for Gaussian networks with cooperation and feedback," Submitted to IEEE Transactions on Information Theory, Tech. Rep. arXiv:0812.1857v2 [cs.IT], December 19 2008. [Online]. Available: http://arxiv.org/pdf/0812.1857

[5] S. Butman, "A general formulation of linear feedback communication systems with solutions," *IEEE Transactions on Information Theory*, vol. 15, no. 3, pp. 392–400, May 1969.

[6] Y. Kim, "Feedback capacity of stationary Gaussian channels," *IEEE Transactions on Information Theory*, vol. 56, no. 1, pp. 57–85, January 2010.

[7] S. Pombra and T. M. Cover, "Non white Gaussian multiple access channels with feedback," *IEEE Transactions on Information Theory*, vol. 40, no. 3, pp. 885–892, May 1994.

[8] A. Lapidoth and M. Wigger, "On the AWGN MAC with imperfect feedback," *IEEE Transactions on Information Theory*, vol. 56, no. 11, November 2010.

[9] S. Verdú, "On channel capacity per unit cost," *IEEE Transactions on Information Theory*, vol. 5, no. 36, pp. 1019–1030, September 1990.

[10] R. Gray, "Toeplitz and circulant matrices : A review," *Foundations and Trends in Communications and Information Theory*, vol. 2, no. 3, 2005.

[11] R. S. Cheng and S. Verdti, "Gaussian multiaccess channels with isi: Capacity region and multiuser water-filling," *IEEE Transactions on Information Theory*, vol. 39, pp. 773–785, 1993.