Dependence Balance in Multiple Access Channels with Correlated Sources

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***Abstract*—A necessary condition is established for the lossy transmission of correlated sources over a memoryless multiple-access channel (MAC). It is used to derive lower bounds on the symmetric distortions that are achievable over Gaussian and binary adder MACs.**

When specialized to symmetric Gaussian MACs and Gaussian sources, the new lower bound recovers Lapidoth and Tinguely's max-correlation lower bound (2010) when the channel bandwidth is equal to the source bandwidth, and it improves on it when the channel bandwidth is higher.

An analogous condition is also derived for the MAC with correlated sources and feedback.

I. Introduction

We present new necessary conditions for lossy transmission of correlated sources over a multiple-access channel (MAC) without and with feedback. The main challenge is to bound the dependence that can be established between the MAC's inputs in terms of the dependence between the sources. Necessary conditions as well as sufficient conditions for this problem have appeared before [1]–[8]. Cover, El Gamal, and Salehi [1] presented conditions that are both necessary and sufficient, but those have a multi-letter form that is incomputable. For the symmetric Gaussian MAC, Lapidoth and Tinguely [2] provided a necessary condition using the Hirschfeld-Gebelein-Rényi (HGR) maximal correlation. A similar necessary condition for lossless transmission over the discrete memoryless MAC was proved by Kang and Ulukus [3]. More recently, Lapidoth and Wigger [4] derived a set of necessary conditions for lossy reconstructions by introducing an auxiliary random sequence that is generated from the sources and that renders the sources conditionally independent. A related necessary condition by Güler, Gündüz, and Yener [5] improves on the previous conditions for asymmetric distortions.

Here we derive a necessary condition that implicitly relates the mutual information between the two sources to the input distribution of the MAC. The condition is reminiscent of the Dependence-Balance (DB) bound of Hekstra and Willems [9] for the MAC with feedback. For the symmetric Gaussian MAC and the binary adder MAC, we use it to lower bound the symmetric achievable distortion. When the source and channel bandwidths are equal, we recover the necessary condition of Lapidoth and Tinguely [2] for the Gaussian MAC with jointly Gaussian sources. Also, for binary adder channels and doubly symmetric binary sources, numerical simulations suggest that we recover Kang and Ulukus's necessary condition [3]. The bound is tighter than [2] and [3] when there is more than one channel use for each source-pair generated by the source. We finally also derive a DB condition for the transmissibility of correlated sources over a MAC with feedback.

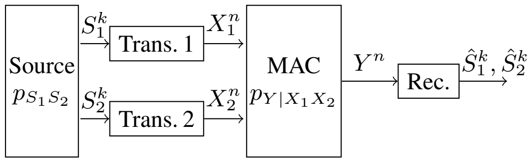

Fig. 1: MAC without feedback

II. Problem Setup

A source generates sequences $S_1^k := (S_{1,1}, \ldots, S_{1,k})$ and $S_2^k := (S_{2,1}, \ldots, S_{2,k})$ by drawing the pairs $\{(S_{1,t}, S_{2,t})\}$ independently and identically distributed (i.i.d.) according to some joint probability mass function (pmf) $p_{S_1S_2}$ over the source alphabets $\mathcal{S}_1 \times \mathcal{S}_2$. The source sequence S_1^k is observed by Transmitter 1 and the source sequence S_2^k by Transmitter 2. The two transmitters communicate with a common receiver over a discrete-time memoryless MAC characterized by input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, an output alphabet $\mathcal{Y}$, and a conditional pmf $p_{Y|X_1X_2}$. (For brevity we will sometimes write $p(s_1, s_2)$ and $p(y|x_1, x_2)$ instead of $p_{S_1S_2}(s_1, s_2)$ and $p_{Y|X_1X_2}(y|x_1, x_2)$.)

Each transmitter $j \in \{1, 2\}$ produces its channel input $X_j^n := (X_{j,1}, \ldots, X_{j,n})$ based on the sequence S_j^k it observes using some encoding function $f_j^{(n)} \colon \mathcal{S}_j^k \to \mathcal{X}_j^n$:

$$X_j^n = f_j^{(n)}(S_j^k).$$

The receiver observes the channel output sequence $Y^n := (Y_1, \ldots, Y_n)$ and reconstructs both source sequences

$$\hat{S}_j^k = h_j^{(n)}(Y^n), \quad j \in \{1, 2\}$$

by means of decoding functions $h_j^{(n)} \colon \mathcal{Y}^n \to \hat{\mathcal{S}}_j^k$, where $\hat{\mathcal{S}}_j$ is the reconstruction alphabet of the source symbol S_j.

The distortions between the sources and the reconstructions are measured by

$$d_j^{(k)}(S_j^k, \hat{S}_j^k) := \frac{1}{k} \sum_{i=1}^{k} \delta_j(S_{ji}, \hat{S}_{ji}),$$

where $\delta_1 \colon \mathcal{S}_1 \times \hat{\mathcal{S}}_1 \to \mathbb{R}^+$ and $\delta_2 \colon \mathcal{S}_2 \times \hat{\mathcal{S}}_2 \to \mathbb{R}^+$ are given per-letter distortion functions.

The triple (κ, D_1, D_2) is *achievable* for the source-channel pair $(p_{S_1S_2}, p_{Y|X_1X_2})$ if, for every $\epsilon > 0$, there exist encoding and decoding functions satisfying the following conditions whenever k and n are sufficiently large:

$$\frac{k}{n} = \kappa \tag{1a}$$

$$\mathbb{E}\left[d_j^{(k)}(S_j^k, \hat{S}_j^k)\right] \leq D_j + \epsilon, \qquad j \in \{1,2\}. \tag{1b}$$

III. Results for MACs Without Feedback

Our main result is a necessary condition for achievability.[1]

Theorem 1. *The triple (κ, D_1, D_2) is achievable only if there exist $p(\hat{s}_1, \hat{s}_2|s_1, s_2)$ and $p(x_1, x_2|q)$ such that for all $p(u|x_1, x_2, y, q)$ and all $p(w|s_1, s_2)$ satisfying the factorization*

$$p(w|s_1, s_2)p(s_1, s_2) = p(w)(s_1|w)p(s_2|w) \tag{2}$$

the following inequalities hold:

$$\kappa I(S_1; S_2) \geq I(X_1; X_2|Q) - I(X_1; X_2|UQ) \tag{3a}$$
$$\kappa I\big(S_1, S_2; \hat{S}_1, \hat{S}_2\big) \leq I(X_1X_2; Y) \tag{3b}$$
$$\kappa I(S_1S_2; \hat{S}_1\hat{S}_2) \leq I(X_1X_2; YU|Q) \tag{3c}$$
$$\kappa I(S_1; \hat{S}_1|S_2) \leq I(X_1; YU|X_2Q) \tag{3d}$$
$$\kappa I(S_2; \hat{S}_2|S_1) \leq I(X_2; YU|X_1Q) \tag{3e}$$
$$\kappa I\big(S_1S_2; \hat{S}_1\hat{S}_2\big|W\big) \leq I(X_1X_2; Y|V) \tag{3f}$$
$$\kappa I\big(S_1; \hat{S}_1\big|S_2W\big) \leq I(X_1; Y|X_2V) \tag{3g}$$
$$\kappa I\big(S_2; \hat{S}_2\big|S_1W\big) \leq I(X_2; Y|X_1V) \tag{3h}$$
$$\mathbb{E}\big[d(S_1, \hat{S}_1)\big] \leq D_1 \tag{3i}$$
$$\mathbb{E}\big[d(S_2, \hat{S}_2)\big] \leq D_2 \tag{3j}$$

for some $p(q)$ and $p(v|x_1, x_2)$ satisfying the factorization

$$\sum_q p(q)p(x_1, x_2|q)p(v|x_1, x_2) = p(v)p(x_1|v)p(x_2|v). \tag{4}$$

The mutual informations and expectations are calculated with respect to the pmfs:

$$p(w, s_1, s_2, \hat{s}_1, \hat{s}_2) = p(s_1, s_2)p(w|s_1, s_2)p(\hat{s}_1, \hat{s}_2|s_1, s_2) \tag{5a}$$
$$p(q, u, v, x_1, x_2, y) = p(q)p(x_1, x_2|q)p(y|x_1, x_2) \cdot p(u|x_1, x_2, y, q)p(v|x_1, x_2). \tag{5b}$$

Moreover, the cardinality $|\mathcal{Q}|$ of the alphabet of Q may be restricted to $|\mathcal{Q}| \leq |\mathcal{X}_1| \cdot |\mathcal{X}_2| + 3$.

Proof: The proof is sketched in Appendix A. ∎

The random variables W and U are formed by augmenting the system with the auxiliary channels $p(w|s_1, s_2)$ and $p(u|x_1, x_2, y, q)$. This technique was introduced in [10] for the Gaussian multiple description problem and in [9] for interactive communications. It has since been used in source-coding setups [11]–[14] as well as in channel-coding setups [15], [16].

Constraints (3b), (3f), (3g), and (3h) are related to the necessary conditions of [4, Theorem 1] and [5, Theorem 5]. New are Constraints (3a) and (3c)–(3e) that resemble the Dependence-Balance bound of Hekstra and Willems [9] for channels with user interaction (e.g. feedback or two-way communication). But in (3a), the channel output Y does not show up. This is because in our setting there is no feedback so communication is only one-way. The DB constraint (6a) limits the correlation between the inputs X_1 and X_2 that is not built on the time-sharing random variable Q, as a function of the source pmf $P_{S_1S_2}$. The constraint becomes stronger as the bandwidth mismatch factor κ increases.

To highlight the role of these new constraints, the rest of the paper focuses on the following corollary.

Corollary 2. *The triple (κ, D_1, D_2) is achievable only if there exist pmfs $p(\hat{s}_1, \hat{s}_2|s_1, s_2)$ and $p(x_1, x_2|q)$ such that for every auxiliary channel $p(u|x_1, x_2, y, q)$ the following inequalities hold for some $p(q)$:*

$$\kappa I(S_1; S_2) \geq I(X_1; X_2|Q) - I(X_1; X_2|UQ) \tag{6a}$$
$$\kappa I(S_1S_2; \hat{S}_1\hat{S}_2) \leq I(X_1X_2; YU|Q) \tag{6b}$$
$$\kappa I(S_1; \hat{S}_1|S_2) \leq I(X_1; YU|X_2\, Q) \tag{6c}$$
$$\kappa I(S_2; \hat{S}_2|S_1) \leq I(X_2; YU|X_1\, Q) \tag{6d}$$
$$\mathbb{E}\big[d(S_1, \hat{S}_1)\big] \leq D_1 \tag{6e}$$
$$\mathbb{E}\big[d(S_2, \hat{S}_2)\big] \leq D_2 \tag{6f}$$

If (6a) is relaxed, then Corollary 2 results in the standard cut-set necessary condition.

Next, we apply Corollary 2 to several symmetric examples, where we find lower bounds on the symmetric achievable distortion D.

A. The Gaussian MAC

We begin with the equal-power Gaussian MAC under the symmetric distortion constraint $D_1 = D_2 = D$. The input and output alphabets of the MAC are the reals, and the time-t channel output is

$$Y_t = X_{1t} + X_{2t} + Z_t$$

where $\{Z_t\}$ is an i.i.d. Gaussian noise sequence with zero mean and unit variance. We impose an average block power constraint P on both transmitters

$$\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}[X_{jt}^2] \leq P, \qquad j \in \{1,2\}.$$

Define

$$\mu := \sqrt{1 - 2^{-2\kappa I(S_1;S_2)}}. \tag{7}$$

Let

$$R_{S_1S_2}(D, D) := \min_{\substack{p_{\hat{S}_1\hat{S}_2|S_1S_2} \\ \mathbb{E}\delta(S_j;\hat{S}_j)\leq D,\ j=1,2}} I(S_1S_2; \hat{S}_1\hat{S}_2) \tag{8}$$

be the *symmetric joint rate-distortion function* under individual distortion constraints and

$$R^{\text{sym}}_{S_1|S_2}(D) = \min_{\substack{p_{\hat{S}_1\hat{S}_2|S_1S_2} \\ \mathbb{E}\delta(S_j;\hat{S}_j)\leq D,\ j=1,2}} \max\{I(S_1; \hat{S}_1|S_2), I(S_2; \hat{S}_2|S_1)\} \tag{9}$$

be the *symmetric worst-case conditional rate-distortion function.* We remark that $R^{\text{sym}}_{S_1|S_2}(D)$ is in general larger than the standard conditional rate distortion functions $R_{S_1|S_2}(D)$ and

[1] Alternatively, Theorem 1 can be stated as a max-min-max optimization problem with an objective function that is 1 if the constraints are satisfied and 0 otherwise. Here the left-most max is on the pairs $p(\hat{s}_1, \hat{s}_2|s_1, s_2)$ and $p(x_1, x_2|q)$, the min is on the pairs $p(u|x_1, x_2, y, q)$ and $p(w|s_1, s_2)$, and the right-most max is on the pairs $p(q)$ and $p(v|x_1, x_2)$.

$R_{S_2|S_1}(D)$ (i.e., the rate distortion function for S_1 when S_2 is available at both the encoder and decoder and vice versa).

Theorem 3. *For any source $p_{S_1S_2}$, the tuple (κ, D, D) is achievable for the symmetric Gaussian MAC with input power P only if the following two conditions hold:*

$$\kappa R_{S_1S_2}(D,D) \leq \frac{1}{2}\log\left(1+2P(1+\hat{\rho})\right) \tag{10}$$

$$\kappa R^{sym}_{S_1|S_2}(D) \leq \frac{1}{2}\log\left(1+P(1-\hat{\rho}^2)\right) \tag{11}$$

for some $\hat{\rho} \leq \bar{\rho}$ where

$$\bar{\rho} = \begin{cases} \mu & \text{if } P \geq \frac{\mu}{1-\mu^2} \\ 1 & \text{otherwise.} \end{cases} \tag{12}$$

Proof: The proof is sketched in Appendix B. ■

Remark 1. *From* (7) *and* (12) *we conclude that as $I(S_1;S_2)$ tends to zero, μ approaches zero, and hence so does $\bar{\rho}$. Consequently, when the sources are "almost-independent", the MAC inputs must be "almost-independent" in the sense that the RHS of* (10) *is "almost" upper bounded by the mutual information corresponding to independent Gaussians. Similarly, μ and hence also $\bar{\rho}$ approach zero as κ tends to zero.*

We now compare Theorem 3 with [2, Remark IV.1] where $\hat{\rho}$ in (10)–(11) is shown to be restricted to

$$\hat{\rho} \leq \rho_{\max} \tag{13}$$

where $\rho_{\max}$ is the Hirschfeld-Gebelein-Rényi (HGR) maximal correlation of the source. We refer to this set of necessary conditions as the *max-correlation condition*. The quantities $\bar{\rho}$ and $\rho_{\max}$ are not comparable in general. For example, for small κ, $\bar{\rho}$ is often smaller than $\rho_{\max}$ and for large κ, it is often larger. An improved necessary condition can thus be obtained by combining the two conditions.

Remark 2. *Theorem 3 can be strengthened by replacing* (12) *with*

$$\bar{\rho} = \begin{cases} \min\{\mu, \rho_{\max}\} & \text{if } P \geq \frac{\mu}{1-\mu^2} \\ \rho_{\max} & \text{otherwise.} \end{cases} \tag{14}$$

We now specialize Theorem 3 to jointly Gaussian sources.

Example 1 (Jointly Gaussian Sources)**.** Consider a jointly Gaussian source with zero mean and covariance matrix

$$\mathbf{K}_{S_1S_2} = \begin{bmatrix} \sigma^2 & \sigma^2\rho \\ \sigma^2\rho & \sigma^2 \end{bmatrix},$$

and suppose that distortion is measured by the squared-error distortion functions $\delta_j(\hat{s}, s) = (s-\hat{s})^2$ for $j = 1, 2$. For this Gaussian source, μ is given by

$$\mu = \sqrt{1-(1-\rho^2)^\kappa}, \tag{15}$$

and we have (see [17]):

$$R(D) = \frac{1}{2}\log\left(\frac{\sigma^2}{D}\right)$$

$$R_{S_1|S_2}(D) = R_{S_2|S_1}(D) = \frac{1}{2}\log\left(\frac{\sigma^2(1-\rho^2)}{D}\right)$$

$$R_{S_1S_2}(D,D) = \begin{cases} \frac{1}{2}\log\left(\frac{\sigma^4(1-\rho^2)}{D^2}\right) & D \leq \sigma^2(1-\rho) \\ \frac{1}{2}\log\left(\frac{\sigma^4(1-\rho^2)}{D^2-(D-\sigma^2(1-\rho))^2}\right) & D \geq \sigma^2(1-\rho). \end{cases}$$

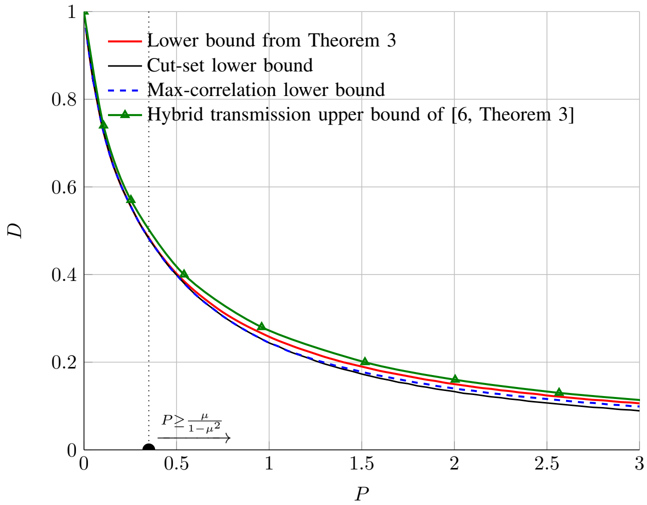

Fig. 2: Bounds on the minimum achievable distortion D as a function of P for $\sigma^2 = 1$, $\rho = 0.4$, and $\kappa = 0.5$.

Moreover, we find the following closed-form expression for $R^{\text{sym}}_{S_1|S_2}(D)$:

$$R^{\text{sym}}_{S_1|S_2}(D) = \begin{cases} \frac{1}{2}\log\left(\frac{\sigma^2(1-\rho^2)}{D}\right) & D \leq \sigma^2(1-\rho) \\ \frac{1}{2}\log\left(\frac{\sigma^2(1-\rho^2)}{2\sigma^2(1-\rho)-\frac{\sigma^4(1-\rho)^2}{D}}\right) & D \geq \sigma^2(1-\rho). \end{cases}$$

Notice that for symmetric jointly Gaussian sources, $R^{\text{sym}}_{S_1|S_2}(D)$ is larger than $R_{S_1|S_2}(D)$ when $D > \sigma^2(1-\rho)$. Interestingly, for all σ^2, D, ρ we have

$$R_{S_1S_2}(D,D) = R^{\text{sym}}_{S_1|S_2}(D) + R(D). \tag{16}$$

Fig. 2 depicts lower and upper bounds on the smallest distortion $D \geq 0$ as a function of the transmit power P so that $(\kappa = 0.5, D, D)$ is achievable for a source with variance $\sigma^2 = 1$ and correlation factor $\rho = 0.4$. For $P > \frac{\mu}{1-\mu^2}$ on the plot, the lower bound implied by Theorem 3 is strictly tighter than the max-correlation lower bound.

More generally, the following can be proved:

- For $\kappa \leq 1$ and small powers $P \leq \frac{\mu}{1-\mu^2}$, the lower bound implied by Theorem 3 coincides with the cut-set lower bound and the max-correlation lower bound. As shown in [2], for $\kappa = 1$ they further coincide with the upper bound achieved by uncoded transmission.
- For large powers $P > \frac{\mu}{1-\mu^2}$, the lower bound from Theorem 3 coincides with the max-correlation lower bound when $\kappa = 1$, is tighter for $\kappa < 1$ and looser for $\kappa > 1$.

B. The binary adder MAC

Suppose that the MAC is a binary adder channel defined by $\mathcal{X}_1 = \{0,1\}$, $\mathcal{X}_2 = \{0,1\}$, $\mathcal{Y} = \{0,1,2\}$, and

$$Y = X_1 + X_2.$$

Using Corollary 2 (and disregarding (11)), we have the following result for any source and any $\kappa > 0$.

Theorem 4. *Given a binary adder MAC and a source p_{S_1,S_2}, the triple (κ, D, D) is acheivable only if*

$$\kappa R_{S_1S_2}(D,D) \leq 1 + h_2(q^\star) - q^\star, \tag{17}$$

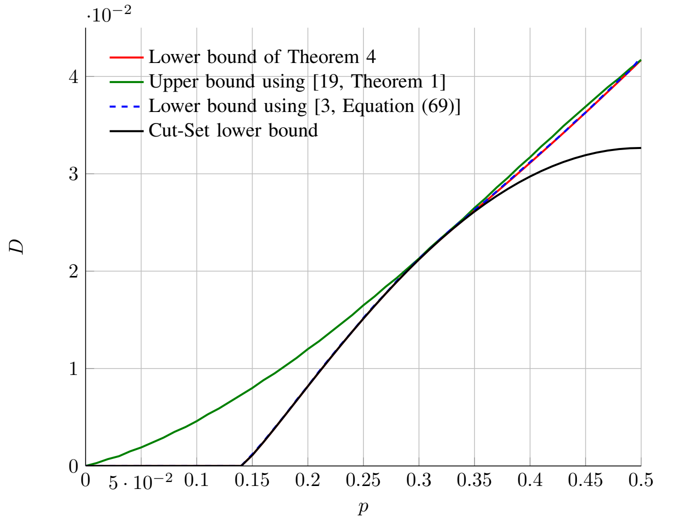

Fig. 3: Distortion upper and lower bounds as functions of p.

where

$$q^\star := \max\left\{\frac{1}{3},\ h_2^{-1}\big(\max\{0,\ 1-\kappa I(S_1;S_2)\}\big)\right\} \quad (18)$$

and $h_2(\cdot)$ *is the binary entropy function with the inverse* $h_2^{-1}(\cdot)$ *on* $[0,1/2]$.

Proof: Omitted. ∎

Notice that when $I(S_1;S_2)$ tends to 0 then the right-hand side of (18) approaches 1.5 which is the sum-capacity of the binary adder MAC with independent inputs.

Example 2 (DSBS Source)**.** Consider a DSBS (S_1,S_2) of parameter p and Hamming distortion functions $d_j(\hat{s},s) = \mathbb{1}(\hat{s} \neq s)$ for $j=1,2$. The symmetric joint rate-distortion function for this source is (see [18, Example 2.7.2])

$$R_{S_1S_2}(D,D) = \begin{cases} 1+h_2(p)-2h_2(D) & D \le D^\star \\ 1-p-(1-p)h_2\left(\frac{2D-p}{2(1-p)}\right) & D \ge D^\star \end{cases}$$

where $D^\star = \frac{1-\sqrt{1-2p}}{2}$. By Corollary 4, the triple $(\kappa = 1, D, D)$ is achievable only if

$$D \ge \begin{cases} h_2^{-1}\left(\frac{p}{2}\right) & p \ge \frac{1}{3} \\ h_2^{-1}\Big(\max\left\{0,\ \frac{h_2(p)-h_2(\frac{1}{3})+\frac{1}{3}}{2}\right\}\Big) & p \le \frac{1}{3}. \end{cases} \quad (19)$$

Fig 3 plots this lower bound and compares it with the upper bound of [19, Theorem 2] evaluated for the choice $Q=\emptyset$, $X_i = U_i = \hat{S}_i$, $i=1,2$, and $U_i = S_i \oplus Z_i$ where for $i=1,2$ the random variable Z_i is Bernoulli with parameter D. The lower bound shown by the dashed curve uses the data processing inequality of Kang and Ulukus (see [3, Equation (69)]). Numerical simulations suggest that the lower bound in (19) is similar to Kang-Ulukus' lower bound for $\kappa = 1$, and that it is tighter for $\kappa < 1$ and looser for $\kappa > 1$.

IV. MAC With Feedback

Consider now the MAC with feedback in Fig. 4. Here, each Encoder $j \in \{1,2\}$ can produce its channel inputs also in function of the past outputs:

$$X_{jt} = f_{jt}^{(n)}(S_j^k, Y^{t-1}), \qquad t \in \{1,\dots,n\},$$

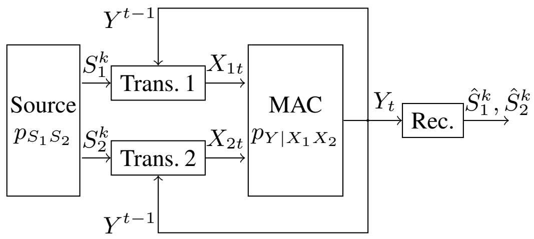

Fig. 4: MAC with feedback

for some feedback encoding function $f_{jt}^{(n)} \colon \mathcal{S}_j^k \times \mathcal{Y}^{t-1} \to \mathcal{X}_j$.

Theorem 5. *The tuple* (κ, D_1, D_2) *is achievable only if there exists a channel* $p(\hat{s}_1,\hat{s}_2|s_1,s_2)$ *and a pmf* $p(x_1,x_2|q)$ *so that for every auxiliary channel* $p(u|x_1,x_2,y,q)$ *the following inequalities hold for some* $p(q)$*:*

$$\kappa I(S_1;S_2) \ge I(X_1;X_2|Q) - I(X_1;X_2|UYQ) \quad (20a)$$
$$\kappa I(S_1S_2;\hat{S}_1\hat{S}_2) \le I(X_1X_2;YU|Q) \quad (20b)$$
$$\kappa I(S_1;\hat{S}_1|S_2) \le I(X_1;YU|X_2Q) \quad (20c)$$
$$\kappa I(S_2;\hat{S}_2|S_1) \le I(X_2;YU|X_1Q) \quad (20d)$$
$$\mathbb{E}\left[d(S_1,\hat{S}_1)\right] \le D_1 \quad (20e)$$
$$\mathbb{E}\left[d(S_2,\hat{S}_2)\right] \le D_2. \quad (20f)$$

The alphabet set $\mathcal{Q}$ *may be chosen to satisfy* $|\mathcal{Q}| \le |\mathcal{X}_1|\cdot|\mathcal{X}_2|+3$.

Proof: The proof is similar to the proof in Appendix A for MACs without feedback, but U is replaced by (U,Y) in (6a). This makes the set of admissible (κ, D_1, D_2) larger. ∎

Remark 3. *If* $I(S_1;S_2) = 0$ *and* $D_1 = D_2 = 0$*, then we recover the DB bound of Hekstra and Willems for MACs with independent messages [9, Theorem 3] (adapted for the MAC with feedback). As shown in [20], the DB bound can be strictly tighter than the cut-set bound for the MAC with feedback.*

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Appendix A
Outline of Proof of Theorem 1

Fix k, n and encoding and reconstruction functions so that the conditions in (1) hold. Constraints (5a), (5b), (3b), (3f), (3g), and (3h) can be proved following the steps in [4] and by noting the Markov chain $W - (S_1, S_2) - (\hat{S}_1, \hat{S}_2)$. Constraints (6e) and (6f) can be proved using standard steps.

We now prove the DB constraint (6a). Assume that for each time t, the MAC produces an additional virtual output U_t by passing the time-t inputs and output, and the past sequence U^{t-1} through an auxiliary channel

$$p(u_t|x_{1t}, x_{2t}, y_t, u^{t-1}). \tag{21}$$

Let $U^n := (U_1, \ldots, U_n)$. We have

$$\begin{aligned}
&kI(S_1; S_2) = I(S_1^k; S_2^k) \\
&\overset{(a)}{=} I(S_1^k U^n; S_2^k) - I(U^n; S_2^k|S_1^k) \\
&\overset{(a)}{=} I(U^n; S_2^k) - I(U^n; S_2^k|S_1^k) + I(S_1^k; S_2^k|U^n) \\
&\overset{(a)}{=} I(U^n; S_1^k S_2^k) - I(U^n; S_1^k|S_2^k) - I(U^n; S_2^k|S_1^k) \\
&\quad + I(S_1^k; S_2^k|U^n) \\
&\overset{(b)}{\geq} I(U^n; S_1^k S_2^k) - I(U^n; S_1^k|S_2^k) - I(U^n; S_2^k|S_1^k) \\
&\overset{(a)}{=} \sum_{t=1}^{n} I(S_1^k S_2^k; U_t|U^{t-1}) - \sum_{t=1}^{n} I(S_1^k; U_t|S_2^k U^{t-1}) \\
&\quad - \sum_{t=1}^{n} I(S_2^k; U_t|S_1^k U^{t-1}) \\
&\overset{(c)}{\geq} \sum_{t=1}^{n} I(X_{1t} X_{2t}; U_t|U^{t-1}) - \sum_{t=1}^{n} I(X_{1t}; U_t|X_{2t} U^{t-1}) \\
&\quad - \sum_{t=1}^{n} I(X_{2t}; U_t|X_{1t} U^{t-1}) \\
&\overset{(d)}{=} nI(X_{1T} X_{2T}; U_T|U^{T-1}T) - nI(X_{1T}; U_T|X_{2T} U^{T-1}T) \\
&\quad - nI(X_{2T}; U_T|X_{1T} U^{T-1}T) \\
&\overset{(e)}{=} nI(X_1 X_2; U|Q) - nI(X_1; U|X_2 Q) - nI(X_2; U|X_1 Q) \\
&= nI(X_1; X_2|Q) - nI(X_1; X_2|UQ)
\end{aligned} \tag{22}$$

where T is a uniform random variable over $\{1, \ldots, n\}$ independent of all previously defined random variables. In the above chain of inequalities, (a) holds by the chain rule of mutual information; (b) holds because mutual information is nonnegative; (c) holds because $S_1^k S_2^k - X_{1t} X_{2t} U^{t-1} - U_t$ forms a Markov chain; (d) holds by the definition of T, and (e) holds by defining $X_1 := X_{1T}$, $X_2 := X_{2T}$, $Y := Y_T$, $U := U_T$, and $Q := (U^{T-1}T)$.

Note that since a different auxiliary channel (21) can be chosen for each time t, and thus for each realization q of Q, the choice of $p(u|x_1, x_2, y, q)$ can depend on $p(x_1, x_2|q)$.

Finally, constraints (6b)–(6d) can be proved using standard steps but considering the augmented output (U_t, Y_t).

Appendix B
Proof Sketch of Theorem 3

Consider Corollary 2 and choose $U = Y + N$ where N is an independent zero-mean Gaussian of variance $\sigma_N^2 = \max\{0,\ P\frac{1-\mu^2}{\mu} - 1\}$.

Fix the correlation coefficient of X_1 and X_2 to $\hat{\rho} \in [0, 1]$. We will weaken the constraints in (3) to obtain a necessary condition in terms of $\hat{\rho}$:

$$\kappa R_{S_1 S_2}(D, D) \leq I(U; Y|Q) \leq \frac{1}{2} \log\left(1 + 2P(1 + \hat{\rho})\right) \tag{23}$$

$$\kappa R_{S_1|S_2}^{\text{sym}}(D, D) \leq \frac{1}{2} \log\left(1 + P(1 - \hat{\rho}^2)\right). \tag{24}$$

The DB constraint (6a) can be relaxed as follows:

$$\begin{aligned}
&\kappa I(S_1; S_2) \\
&\geq I(X_1; X_2|Q) - I(X_1; X_2|UQ) \\
&\geq h(U|Q) - h(U|X_1) - h(U|X_2) + h(U|X_1 X_2) \\
&\overset{(a)}{\geq} h(U|Q) + \frac{1}{2} \log\left(\frac{(1 + \sigma_N^2)}{2\pi e(1 + \sigma_N^2 + P(1 - \hat{\rho}^2))^2}\right) \\
&\overset{(b)}{\geq} \frac{1}{2} \log\left(\frac{(1 + \sigma_N^2)(\sigma_N^2 + 2^{2h(Y|Q)} \cdot (2\pi e)^{-1})}{(1 + \sigma_N^2 + P(1 - \hat{\rho}^2))^2}\right) \\
&\overset{(c)}{\geq} \frac{1}{2} \log\left(\frac{(1 + \sigma_N^2)(\sigma_N^2 + 2^{2\kappa R_{S_1 S_2}(D,D)})}{(1 + \sigma_N^2 + P(1 - \hat{\rho}^2))^2}\right)
\end{aligned} \tag{25}$$

where (a) holds by the (conditional) maximum entropy lemma [21]; (b) by the entropy-power inequality [22]; and (c) by (23), because $\log$ is a monotonically increasing function, and by $h(Y|X_1 X_2 Q) = \frac{1}{2} \log(2\pi e)$.

Thus, the triple (κ, D, D) is achievable, only if (23)–(25) hold for some $\hat{\rho} \in [0, 1]$. For $P < \frac{\mu}{1-\mu^2}$, the desired result is obtained simply by disregarding (25). For $P \geq \frac{\mu}{1-\mu^2}$, the constraints in (23) and (25) can be used to show that it suffices to consider $\hat{\rho} \in [0, \mu]$. (Details omitted.)