Information-Theoretic Tradeoffs in Distributed Hypothesis Testing

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Example: Distributed Control-System for Smart Cars

- Smart cars measuring speed, distance, road conditions
- Fixed road-side sensors measuring same parameters

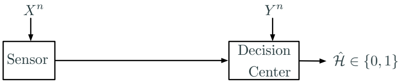
- Intact car system: measurements highly correlated
- Erroneous car system: measurements independent

Task of Distributed Control-System

Decide on joint distribution underlying the observations

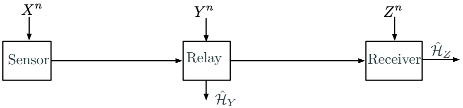
Outline of the Talk

- Simple single-sensor system



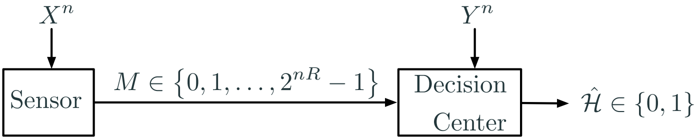
  - Infinitely many communication bits
  - Single communication bit
  - *nR* communication bits

- Multihop System



- Restriction only on expected communication rate
  - Simple single-sensor system
  - Multi-hop system

Distributed Hypothesis Testing



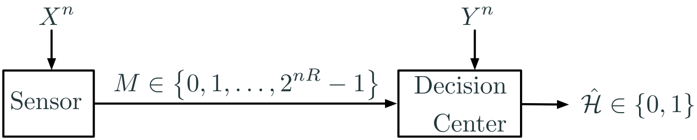
- "Normal situation" $\mathcal{H} = 0$: $(X^n, Y^n) \sim$ i.i.d. P_{XY}
- "Hazardous event" $\mathcal{H} = 1$: $(X^n, Y^n) \sim$ i.i.d. Q_{XY}

- Constraints on type-I and type-II error probabilities:

$$\overline{\lim_{n\to\infty}}\, \alpha_n = \overline{\lim_{n\to\infty}}\, \mathbb{P}\big[\hat{\mathcal{H}} = 1 | \mathcal{H} = 0\big] \leq \epsilon$$

$$-\underline{\overline{\lim_{n\to\infty}}}\, \frac{1}{n} \log \beta_n = -\overline{\lim_{n\to\infty}}\, \frac{1}{n} \log \mathbb{P}\big[\hat{\mathcal{H}} = 0 | \mathcal{H} = 1\big] \geq \theta$$

Distributed Hypothesis Testing



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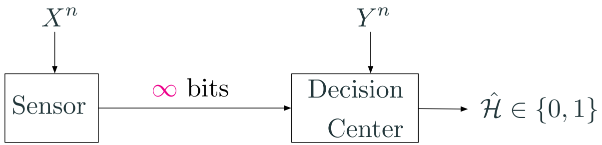
Rate-Exponent Tradeoff $\theta_\epsilon^*(R)$

Given $R > 0$, largest exponent θ that is ϵ-achievable.

- Ahlswede & Csiszár'86: $\theta_\epsilon^*(R)$ does not depend on $\epsilon \in [0, 1/2]$

Centralized Hypothesis Testing — The Ideal Case

Centralized Hypothesis Testing



- Rate R is so large that sensor can send *all* X^n to decision center

- Optimal decision: raise alarm unless statistics $\text{tp}\,(X^n, Y^n) \approx P_{XY}$

- Intuition: If system perfectly fits $\mathcal{H} = 0$, decide on $\hat{\mathcal{H}} = 0$, otherwise decide on $\hat{\mathcal{H}} = 1$

Example of Doubly-Symmetric Binary Sources

$$P_{XY}(x,y) = \begin{cases} 1/3 & x = y \\ 1/6 & x \neq y \end{cases} \qquad \text{and} \qquad Q_{XY}(x,y) = \frac{1}{2} \cdot \frac{1}{2}$$

- Ex. 1: if observations

$$\begin{aligned} X^n &= (0,1,1,0,1,0,1,0,1,1,0,1,0,0,1,1,0,0) \\ Y^n &= (0,0,1,0,1,0,1,0,1,0,1,1,1,0,1,0,1,0) \end{aligned}$$

$\text{tp}\left(X^n, Y^n\right) = P_{XY} \rightarrow$ decision center decides on $\hat{\mathcal{H}} = 0$

Example of Doubly-Symmetric Binary Sources

$$P_{XY}(x,y) = \begin{cases} 1/3 & x = y \\ 1/6 & x \neq y \end{cases} \qquad \text{and} \qquad Q_{XY}(x,y) = \frac{1}{2} \cdot \frac{1}{2}$$

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tp $(X^n, Y^n) = P_{XY} \rightarrow$ decision center decides on $\hat{\mathcal{H}} = 0$

- Ex. 2: if observations

$$\begin{aligned} X^n &= (0,1,0,1,1,1,0,1,0,1,1,0,1,0,0,1,1,0,0) \\ Y^n &= (1,0,0,1,0,1,0,0,0,1,0,1,1,1,1,1,0,1,0) \end{aligned}$$

tp (X^n, Y^n) far from $P_{XY} \rightarrow$ decision center decides on $\hat{\mathcal{H}} = 1$

Analysis of Proposed Centralized Scheme

- Type-I error probability (by the weak law of large numbers)

$$\alpha_n = 1 - \mathbb{P}\big[\hat{\mathcal{H}} = 0 | \mathcal{H} = 0\big] = 1 - P_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) \approx P_{XY}\Big)$$
$$\to 0 \quad \text{as} \quad n \to \infty$$

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- Probability of miss detection:

$$\begin{aligned} \beta_n &= Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) \approx P_{XY}\Big) \\ &= \dots \end{aligned}$$

Mathematical Preliminaries: Sanov's Theorem

- sequences $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ of type $\pi_{\mathbf{xy}}$

$$Q_{XY}^{\otimes n}(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{n} Q_{XY}(x_i, y_i) = \prod_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \big(Q_{XY}(x, y)\big)^{n\cdot\pi_{\mathbf{xy}}(x,y)}$$

Theorem (Sanov's theorem)

$$Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) = \pi\Big) \approx \exp\big(-n \cdot D(\pi||Q_{XY})\big)$$

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&= \exp\Big(n \cdot \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \pi_{\mathbf{xy}}(x,y) \cdot \log Q_{XY}(x,y)\Big) \\
&= \exp\Big(-n \cdot \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \pi_{\mathbf{xy}}(x,y)\Big(\log \frac{\pi_{\mathbf{xy}}(x,y)}{Q_{XY}(x,y)} + \log \frac{1}{\pi_{\mathbf{xy}}(x,y)}\Big)\Big) \\
&= \exp(-n \cdot (D(\pi_{\mathbf{xy}}||Q_{XY}) + H(\pi_{\mathbf{xy}})))
\end{aligned}
$$

Theorem (Sanov's theorem)

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\end{aligned}
$$

Theorem (Sanov's theorem)

$$
Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) = \pi\Big) \approx \exp\big(-n \cdot D(\pi||Q_{XY})\big)
$$

- There are $\approx 2^{nH(\pi)}$ sequences of a given type π

Use Sanov's Theorem to Finalize the Analysis

- Type-I error probability (by the weak law of large numbers)

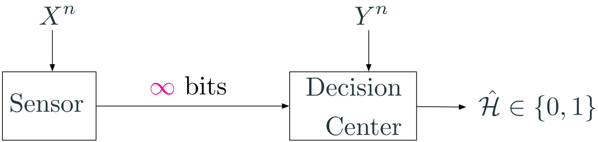
$$\alpha_n = 1 - P_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) \approx P_{XY}\Big) \to 0 \quad \text{as} \quad n \to \infty$$

- Type-II error probability

$$\begin{aligned} \beta_n &= Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) \approx P_{XY}\Big) \\ &= \sum_{\pi \approx P_{XY}} Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) = \pi\Big) \\ &\approx \exp\big(-n \cdot D(P_{XY} \| Q_{XY})\big) \end{aligned}$$

- N.B. The number of types π is polynomial in n

Optimal Exponent for Centralized Hypothesis Testing



- Rate R is so large that sensor can send *all* X^n to decision center

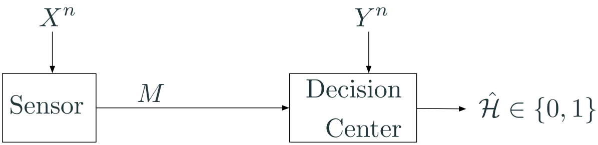
Theorem (Stein's exponent)

Largest achievable error exponent is:

$$\theta^\star(R=\infty) = D(P_{XY} \| Q_{XY}) := \sum_{x,y} P_{XY}(x,y) \log \frac{P_{XY}(x,y)}{Q_{XY}(x,y)}$$

Distributed Hypothesis Testing with Zero Communication Rate $R = 0$

Distributed Hypothesis Testing with $R = 0$ (Han'87)



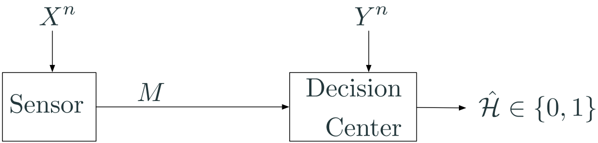
- Send message *M*, where number of bits representing *M* is sublinear in *n*

Theorem (Han'87)

Largest achievable error exponent is:

$$\theta^{\star}(R = 0) = \min_{\substack{\pi_{XY}: \\ \pi_X = P_X \\ \pi_Y = P_Y}} D(\pi_{XY} || Q_{XY})$$

Scheme for $\underline{R = 0}$ (Han'87)



- $M \in \{0, 1\}$ (1 bit) suffices
- Sensor sends its own local decision:
  If $\text{tp}(X^n) \approx P_X \to$ send $M = 0$, otherwise send $M = 1$
- Decision center:
  1. Local decision: $L = 0$ if $\text{tp}\left(Y^n\right) \approx P_Y$ and $L = 1$ else
  2. Final decision: $\hat{\mathcal{H}} = 0$ if $M = L = 0$, and $\hat{\mathcal{H}} = 1$ else.

Example of Doubly-Symmetric Binary Sources

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Ex.:

$$\begin{aligned} X^n &= (0,1,0,1,1,1,0,1,0,1,1,0,1,0,0,1,1,0,0) \\ Y^n &= (1,0,0,1,0,1,0,0,0,1,0,1,1,1,1,1,0,1,0) \end{aligned}$$

$\text{tp}\,(X^n) = P_X$ and $\text{tp}\,(Y^n) = P_Y \to$ decision center decides $\hat{\mathcal{H}} = 0$

For centralized setting ($R = \infty$), decision center decides $\hat{\mathcal{H}} = 1$!

Analysis of 1-bit Communication Scheme

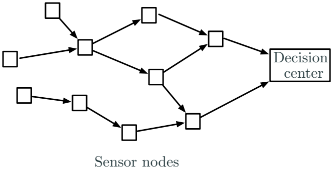
- Type-I error probability (by the weak law of large numbers)

$$\alpha_n = 1 - P_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n) \approx P_X \quad \text{and} \quad \mathrm{tp}\,(Y^n) \approx P_Y\Big) \to 0 \quad \text{as} \quad n \to \infty$$

- Type-II error probability:

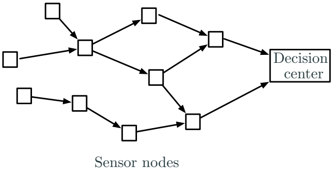
$$\begin{aligned}
\beta_n &= Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n) \approx P_X \quad \text{and} \quad \mathrm{tp}\,(Y^n) \approx P_Y\Big) \\
&= \sum_{\substack{\pi_{xy}:\ \pi_x \approx P_X \\ \pi_y \approx P_Y}} Q_{XY}^{\otimes n}\Big(\mathrm{tp}\,(X^n, Y^n) = \pi_{xy}\Big) \\
&\approx \exp\Big(- n \cdot \big(\min_{\substack{\pi_{xy}:\ \pi_x = P_X \\ \pi_y = P_Y}} D(\pi_{xy} \| Q_{XY}) \big) \Big)
\end{aligned}$$

Many-Sensors and One Detector



- “Normal situation” $\mathcal{H} = 0$: $(X_1^n, \dots, X_K^n, Y^n) \sim$ i.i.d. $P_{X_1 \cdots X_K Y}$
- “Hazardous event” $\mathcal{H} = 1$: $(X_1^n, \dots, X_K^n, Y^n) \sim$ i.i.d. $Q_{X_1 \cdots X_K Y}$

Many-Sensors and One Detector



- Each sensor k produces local decision
- Unanimous decision forwarding: sensor sends $M_k = 0$ only if all incoming messages 0 and local decision $\mathcal{H} = 0$
- Decision center: raises alarm unless all incoming messages 0 and local decision $\mathcal{H} = 0$
- Optimal exponent (independent of ϵ)

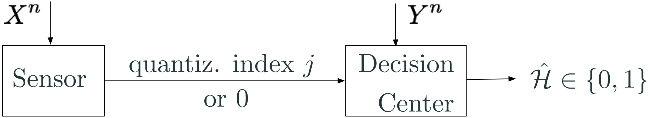
$$\theta_\epsilon^* = \min_{\substack{\pi_{X_1\cdots X_K Y}: \\ \pi_{X_k}=P_{X_k},\ \forall k \\ \pi_Y = P_Y}} D(\pi_{X_1\cdots X_K Y} || Q_{X_1\cdots X_K Y})$$

When is a single-bit transmission optimal for $R = 0$?

- Single-bit transmission (unanimous decision forward) optimal if
  - Two hypotheses $\mathcal{H} = 0$ and $\mathcal{H} = 1$
  - All decision centers interested in same exponent
  - Interactive communication allowed [Katz-Piantanida-Debbah-2016]
- Sending a single bit is not sufficient if
  - Exponents under both decisions need to be maximized
  - Different decision centers interested in different exponents
  - More than $K > 2$ hypotheses

Distributed Hypothesis Testing under Positive Rates $R > 0$

Distributed Hypothesis Testing with $R > 0$ (Han'87)



- Random codebook $\mathcal{C}_S = \{S^n(1), \ldots, S^n(2^{nR} - 1)\}$
- Quantization: Send $M = j$ if $\mathrm{tp}\,(S^n(j), X^n) \approx P_{SX}$; else $M = 0$
- Alarm $\hat{\mathcal{H}} = 1$ unless $M \geq 1$ and $\mathrm{tp}\,(S^n(M), Y^n) \approx P_{SY}$

- If $R \geq I(S;X)$ quantization succeeds and $\alpha_n \to 0$ as $n \to \infty$
- Achievable type-II error exponent

$$\theta_\epsilon^*(R) \geq \max_{\substack{P_{S|X}: \\ R \geq I(S;X)}} \min_{\substack{\pi_{SXY}: \\ \pi_{SX}=P_{SX} \\ \pi_{SY}=P_{SY}}} D(\pi_{SXY} || P_{S|X} Q_{XY})$$

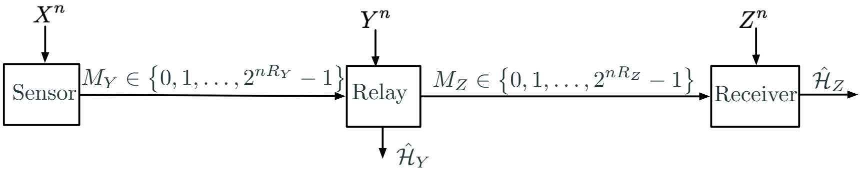
Testing Against Independence (Csiszar-Ahlswede '86)

- $\mathcal{H} = 0 :\ (X^n, Y^n) \sim$ i.i.d. P_{XY}
- $\mathcal{H} = 1 :\ (X^n, Y^n) \sim$ i.i.d. $P_X P_Y$

Optimal Rate-Exponent Tradeoff (does not depend on ϵ)

$$\theta^*_\epsilon(R) = \max_{\substack{P_{S|X}: \\ R \geq I(S;X)}} I(S;Y) =: \eta_{XY}(R)$$

Testing Against Independence over Two Hops



- Two decision centers (relay and receiver)
- Markov chain $X \to Y \to Z$ under both hypotheses
- $\mathcal{H} = 0 : (X^n, Y^n, Z^n)$ i.i.d. $\sim P_X \cdot P_{Y|X} \cdot P_{Z|Y}$
- $\mathcal{H} = 1 : (X^n, Y^n, Z^n)$ i.i.d. $\sim P_X \cdot P_Y \cdot P_Z$

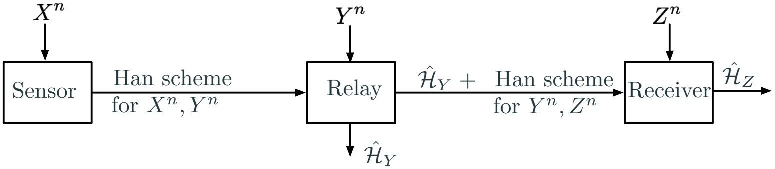
Definition of Exponents Region

- Error Constraints at the Relay
  - $\alpha_{Y,n} \triangleq \mathbb{P}[\hat{\mathcal{H}}_Y = 1|\mathcal{H} = 0] \leq \epsilon_Y$
  - $\beta_{Y,n} \triangleq \mathbb{P}[\hat{\mathcal{H}}_Y = 0|\mathcal{H} = 1] \leq 2^{-n\theta_Y}$
- Error Constraints at the Receiver
  - $\alpha_{Z,n} \triangleq \mathbb{P}[\hat{\mathcal{H}}_Z = 1|\mathcal{H} = 0] \leq \epsilon_Z$
  - $\beta_{Z,n} \triangleq \mathbb{P}[\hat{\mathcal{H}}_Z = 0|\mathcal{H} = 1] \leq 2^{-n\theta_Z}$

Definition

$\mathcal{E}^*_{\epsilon_Y,\epsilon_Z}(R_Y, R_Z)$ is the closure of the set of all (ϵ_Y, ϵ_Z)-achievable exponent pairs (θ_Y, θ_Z)

Optimal Scheme and Exponents Region



- Independent Han-scheme over each link
- Unanimous Decision-Forwarding
- $\hat{\mathcal{H}}_Z = 0$ only if $\hat{\mathcal{H}}_Y = 0$ *and* Han scheme for (Y^n, Z^n) indicates $\mathcal{H} = 0$

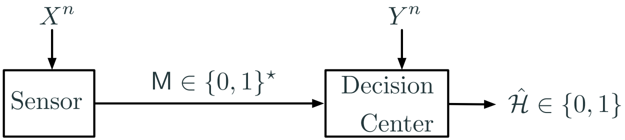
Theorem (Salehkalaibar, W', Wang 2017, Cao, Zhu, Tan'2019)

$\mathcal{E}_{\epsilon_Y,\epsilon_Z}(R_Y, R_Z)$ *is the set of all pairs* (θ_Y, θ_Z) *satisfying*

$$\theta_Y \leq \eta_{XY}(R_Y), \qquad \theta_Z \leq \eta_{XY}(R_Y) + \eta_{YZ}(R_Z).$$

Testing under Expected Rate Constraints

Point-to-Point Testing under Expected Rate Constraints



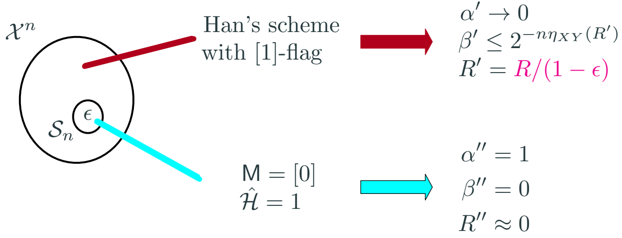
- Message M is a variable-length bit-string
- Rate constraint $\mathbb{E}[\mathrm{len}(\mathsf{M})] \leq nR$

Theorem (Salehkalaibar and Wigger'2020)

The largest ϵ-achievable exponent is

$$\theta^*_{\mathsf{VL},\epsilon}(R) = \eta_{XY}(R/(1-\epsilon)) \qquad \text{(depends on } \epsilon\text{)}$$

Optimal Point-to-Point Variable-Length Scheme



- Set $\mathcal{S}_n$ has probability ϵ
- Average performances

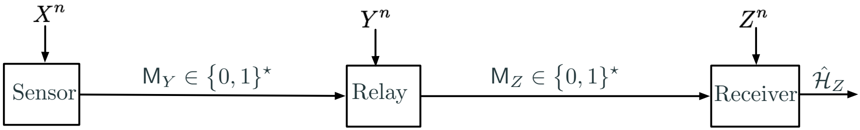
$$\alpha_n \leq \epsilon$$

$$\beta_n \leq (1-\epsilon)2^{-n \cdot \eta_{XY}(R/(1-\epsilon))}$$

and total rate is $(1-\epsilon)R' = R$

$\rightarrow$ Achievability of $\theta = \eta_{XY}(R/(1-\epsilon))$

Two-Hop Testing under Expected Rate Constraints

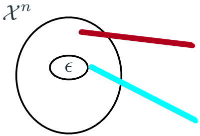


- Messages M_Y and M_Z are variable-length bit-strings
- Rate constraints $\mathbb{E}[\mathrm{len}(\mathsf{M}_Y)] \leq nR_Y$ and $\mathbb{E}[\mathrm{len}(\mathsf{M}_Z)] \leq nR_Z$

Definition

Exponents region $\mathcal{E}_{\mathsf{VL},\epsilon_Y,\epsilon_Z}(R_Y, R_Z)$ is the set of all (ϵ_Y, ϵ_Z)-achievable exponent pairs (θ_Y, θ_Z) under expected rate constraints

Achievability with Expected Rate Constraints for $\epsilon_Y = \epsilon_Z \triangleq \epsilon$



Two-Hop Scheme with [1]-flag

$$\alpha'_Y, \alpha'_Z \to 0$$
$$\beta'_Y \leq 2^{-n\eta_{XY}(R'_Y)}$$
$$\beta'_Z \leq 2^{-n(\eta_{XY}(R'_Y)+\eta_{YZ}(R'_Z))}$$
$$R'_Y = R_Y/(1-\epsilon)$$
$$R'_Z = R_Z/(1-\epsilon)$$

$\mathsf{M}_Y = \mathsf{M}_Z = [0]$
$\hat{\mathcal{H}}_Y = \hat{\mathcal{H}}_Z = 1$

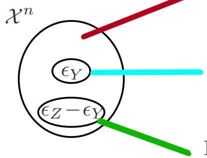
$$\alpha''_Y = \alpha''_Z = 1$$
$$\beta''_Y = \beta''_Z = 0$$
$$R''_Y = R''_Z = 0$$

Theorem (Hamad, Sarkiss, W'2021)

$\mathcal{E}_{\epsilon,\epsilon}(R_Y, R_Z)$ is the set of all pairs (θ_Y, θ_Z) satisfying

$$\theta_Y \leq \eta_{XY}(R_Y/(1-\epsilon)), \qquad \theta_Z \leq \eta_{XY}(R_Y/(1-\epsilon)) + \eta_{YZ}(R_Z/(1-\epsilon)).$$

Scheme for Expected Rate Constraints with $\epsilon_Y < \epsilon_Z$



Two-Hop Scheme with [1]-flag → $\alpha'_Y, \alpha'_Z \to 0$; $\beta'_Y \leq 2^{-n\eta_{XY}(R'_Y)}$; $\beta'_Z \leq 2^{-n(\eta_{XY}(R'_Y)+\eta_{YZ}(R'_Z))}$; $R'_Y, R'_Z = R_Z/(1-\epsilon_Z)$

$\mathsf{M}_Y = \mathsf{M}_Z = [0]$, $\hat{\mathcal{H}}_Y = \hat{\mathcal{H}}_Z = 1$ → $\alpha''_Y = \alpha''_Z = 1$; $\beta''_Y = \beta''_Z = 0$; $R''_Y = R''_Z = 0$

Han scheme on first link, $\mathsf{M}_Z = [0]$, $\hat{\mathcal{H}}_Z = 1$ → $\alpha'''_Y \to 0$; $\beta'''_Y \leq 2^{-n\eta_{XY}(R'''_Y)}$; $\alpha'''_Z = 1$, $\beta'''_Z = 0$; $R'''_Y, R'''_Z = 0$

- Average type-I error prob. $\alpha_Y \to \epsilon_Y$ and $\alpha_Z \to \epsilon_Z$ as $n \to \infty$
- Average type-II error prob. at Relay:
  $\beta_Y \leq (1-\epsilon_Z)2^{-n\eta_{XY}(R'_Y)} + (\epsilon_Z - \epsilon_Y)2^{-n\eta_{XY}(R'''_Y)})$
- Average rate of first link $(1-\epsilon_Z)R'_Y + (\epsilon_Z - \epsilon_Y)R'''_Y$

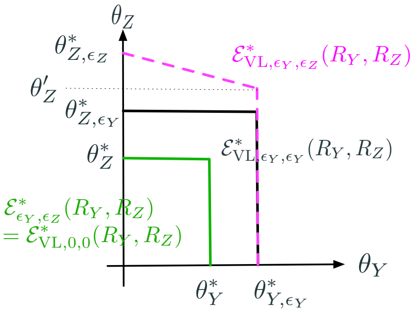
Optimal Exponents Region for $\epsilon_Y < \epsilon_Z$

Theorem (Hamad, W', Sarkiss '2021)

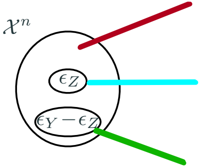
$\mathcal{E}^*_{\text{VL},\epsilon_Y,\epsilon_Z}(R_Y, R_Z)$ set of (θ_Y, θ_Z) pairs satisfying

$$\theta_Y \leq \min\{\eta_{XY}(R'_Y), \eta_{XY}(R'''_Y)\}$$
$$\theta_Z \leq \eta_{XY}(R'_Y) + \eta_{YZ}(R_Z/(1-\epsilon_Z)),$$

for some $R_{Y'}, R'''_Y > 0$ satisfying $R_Y \geq (1-\epsilon_Z)R'_Y + (\epsilon_Z - \epsilon_Y)R'''_Y$.



Scheme for Expected Rate Constraints with $\epsilon_Y > \epsilon_Z$



Two-Hop Scheme with [1,0]-flag → $\alpha'_Y, \alpha'_Z \to 0$, $\beta'_Y \leq 2^{-n\eta_{XY}(R'_Y)}$, $\beta'_Z \leq 2^{-n(\eta_{XY}(R'_Y)+\eta_{YZ}(R'_Z))}$, R'_Y, R'_Z

$\mathsf{M}_Y = \mathsf{M}_Z = [0]$, $\hat{\mathcal{H}}_Y = \hat{\mathcal{H}}_Z = 1$ → $\alpha''_Y = \alpha''_Z = 1$, $\beta''_Y = \beta''_Z = 0$, $R''_Y = R''_Z = 0$

Two-Hop Scheme with [1,1]-flag except $\hat{\mathcal{H}}_Y = 1$ → $\alpha'''_Y = 1,\ \beta'''_Y = 0$, $\alpha'''_Z \to 0$, $\beta'''_Z \leq 2^{-n(\eta_{XY}(R'''_Y)+\eta_{YZ}(R'''_Z))}$, R'''_Y, R'''_Z

- Average type-I error prob. $\alpha_Y \to \epsilon_Y$ and $\alpha_Z \to \epsilon_Z$ as $n \to \infty$
- Average type-II error prob. at Receiver:
  $\beta_Y \leq (1-\epsilon_Y)2^{-n(\eta_{XY}(R'_Y)+\eta_{YZ}(R'_Z))} + (\epsilon_Y - \epsilon_Z)2^{-n(\eta_{XY}(R'''_Y)+\eta_{YZ}(R'''_Z))}$
- Average rate of first link $(1-\epsilon_Y)R'_Y + (\epsilon_Y - \epsilon_Z)R'''_Y$

Optimal Exponents Region for $\epsilon_Y > \epsilon_Z$

Theorem (Hamad, W', Sarkiss '2021)

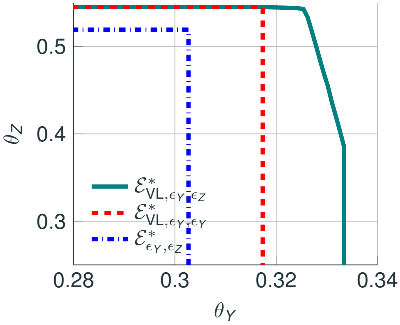
$\mathcal{E}^*_{\mathrm{VL},\epsilon_Y,\epsilon_Z}(R_Y, R_Z)$ *set of* (θ_Y, θ_Z) *pairs satisfying*

$$\theta_Y \leq \eta_{XY}(R'_Y)$$
$$\theta_Z \leq \min\{\eta_{XY}(R'_Y) + \eta_{YZ}(R'_Z), \eta_{XY}(R'''_Y) + \eta_{YZ}(R'''_Z)\}$$

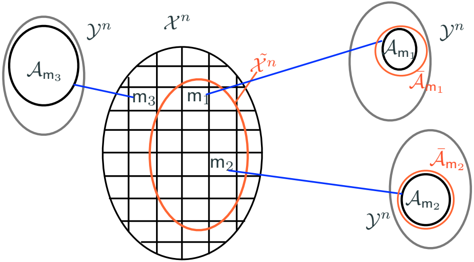
for some $R_{Y'}, R'''_Y, R'_Z, R''''_Z > 0$ *satisfying*
$R_Y \geq (1-\epsilon_Z)R'_Y + (\epsilon_Z - \epsilon_Y)R'''_Y$ *and* $R_Z \geq (1-\epsilon_Z)R'_Z + (\epsilon_Z - \epsilon_Y)R'''_Z$

Two-Hop Setup with $\epsilon_Y > \epsilon_Z$: Numerical Results

- Binary example
  - $X \sim$ Bernoulli(0.5)
  - $Y = X \oplus$ Bernoulli(0.9)
  - $Z = Y \oplus$ Bernoulli(0.8)
- Type-I error probabilities $\epsilon_Y = 0.1$ and $\epsilon_Z = 0.05$
- Rates $R_Y = R_Z = 0.5$



Idea of Converse Proof for Single-Sensor System



Original acceptance region:
$\mathcal{A} = \bigcup_m \{m\} \times \mathcal{A}_m$

New problem:
$\tilde{X}^n$ lives on $\tilde{\mathcal{X}}^n$

New acceptance region:
$\bar{\mathcal{A}} = \bigcup_m \{m\} \times \bar{\mathcal{A}}_m$

- Restrict to subset $\tilde{\mathcal{X}}^n \rightarrow$ change of measure
- Slightly-enlarge the acceptance region $\rightarrow$ blowing-up lemma

New problem has $\tilde{\alpha}_n \approx 0$ and $\mathbb{E}[\mathrm{len}(\tilde{M})] \leq \frac{\mathbb{E}[\mathrm{len}(M)]}{1-\epsilon}$ and $-\frac{1}{n}\log\tilde{\beta}_n \approx -\frac{1}{n}\log\beta_n$

Converse Proof: Bound on Rate

- Pick set $\mathcal{D}_n := \left\{ x^n \in \mathcal{T}_\epsilon^{(n)}(P_X) \colon \mathbb{P}[\hat{\mathcal{H}} = 0 | \mathcal{H} = 0, X^n = x^n] \geq \eta \right\}$ for small parameters $\eta, \epsilon > 0$ that will tend to 0
- Since $\mathbb{P}[\hat{\mathcal{H}} = 0 | \mathcal{H} = 0, X^n = x^n] \geq 1 - \epsilon \quad \rightarrow \quad \mathbb{P}[\mathcal{D}_n] \geq \frac{1-\epsilon-\eta}{1-\eta}$
- Change of measure:

$$(\tilde{X}^n, \tilde{Y}^n, \tilde{\mathsf{M}}) \sim \prod_{i=1}^{n} P_X(x_i) \frac{\mathbb{1}\{x^n \in \mathcal{D}_n\}}{\mathbb{P}[\mathcal{D}_n]} \prod_{i=1}^{n} P_{Y|X}(y_i|x_i) \mathbb{1}\{m = \text{enc}(x^n)\}$$

- $R \geq \frac{1}{n}\mathbb{E}[\mathrm{len}(\mathsf{M})] \geq \frac{1}{n}\mathbb{P}[\mathcal{D}_n] \cdot \mathbb{E}[\mathrm{len}(\tilde{\mathsf{M}})]$

$$\mathbb{E}[\mathrm{len}(\tilde{\mathsf{M}})] \geq H(\tilde{\mathsf{M}} \big| \mathrm{len}(\tilde{\mathsf{M}})) = (H(\tilde{\mathsf{M}}) - H(\mathrm{len}(\tilde{\mathsf{M}}))) \approx H(\tilde{\mathsf{M}})$$

$$\geq I(\tilde{\mathsf{M}}; \tilde{X}^n) = \sum_{i=1}^{n} I(\tilde{\mathsf{M}}; \tilde{X}_i | \tilde{X}^{i-1})$$

$$= \sum_{i=1}^{n} I(\tilde{\mathsf{M}}, \tilde{X}^{i-1}; \tilde{X}_i) + \log(\mathbb{P}[\mathcal{D}_n]) = \sum_{i=1}^{n} I(U_i; \tilde{X}_i) + \log(\mathbb{P}[\mathcal{D}_n])$$

Converse Proof: Marton's Blowing Up Lemma

Lemma (Marton's blowing up lemma)

Let $S_1, S_2, \ldots$ be i.i.d. $\sim P_S$ and $\{\epsilon_n\} \downarrow 0$.
There exist sequences $\{\ell_n\}$ and $\{\zeta_n\}$ both $\downarrow 0$ s.t. for any set $\mathcal{B}_n$:

If $P_S^{\otimes n}(\mathcal{B}_n) \geq exp(-n\epsilon_n)$, then $P_S^{\otimes n}(\bar{\mathcal{B}}_n^{(\ell_n)}) \geq 1 - \zeta_n$.

- Blow-up acceptance regions
  $\bar{\mathcal{A}}_{\mathrm{m}} := \left\{ y^n \colon \exists \tilde{y}^n \ \text{ s.t. } \ d_{\mathrm{H}}(y^n, \tilde{y}^n) \leq \ell_n, \ \tilde{y}^n \in \mathcal{A}_{\mathrm{m}} \right\}$
- By the blowing-up lemma and the definition of $\mathcal{D}_n$

$$\mathbb{P}[(\tilde{\mathrm{M}}, \tilde{Y}^n) \in \bar{\mathcal{A}} \big| \mathcal{H} = 0] \geq 1 - \zeta_n$$

- Since the blowup is very small

$$\begin{aligned} -\frac{1}{n} \log \mathbb{P}[(\tilde{\mathrm{M}}, \tilde{Y}^n) \in \bar{\mathcal{A}} \big| \mathcal{H} = 1] \quad &\approx \quad -\frac{1}{n} \log \mathbb{P}[(\tilde{\mathrm{M}}, \tilde{Y}^n) \in \mathcal{A} \big| \mathcal{H} = 1] \\ &\approx \quad -\frac{1}{n} \log \mathbb{P}[(\mathrm{M}, Y^n) \in \mathcal{A} \big| \mathcal{H} = 1] \end{aligned}$$

Converse Proof: Bound on Exponent

$$-\frac{1}{n}\log \mathbb{P}[(\tilde{\mathrm{M}},\tilde{Y}^n)\in\bar{\mathcal{A}}|\mathcal{H}=1]\approx\frac{1}{1-\zeta_n}D\big(P_{(\tilde{\mathrm{M}},\tilde{Y}^n)|\mathcal{H}=0}\|P_{(\tilde{\mathrm{M}},\tilde{Y}^n)|\mathcal{H}=1}\big)$$

$$\approx\quad\frac{1}{1-\zeta_n}I(\tilde{\mathrm{M}};\tilde{Y}^n)=\frac{1}{1-\zeta_n}\sum_{i=1}^n I(\tilde{\mathrm{M}};\tilde{Y}_i|\tilde{Y}^{i-1})$$

$$\approx\quad\frac{1}{1-\zeta_n}\sum_{i=1}^n I(\tilde{\mathrm{M}},\tilde{Y}^{i-1};\tilde{Y}_i)$$

$$\geq\quad\frac{1}{1-\zeta_n}\sum_{i=1}^n I(\tilde{\mathrm{M}},|\tilde{X}^{i-1};\tilde{Y}_i)=\frac{1}{1-\zeta_n}\sum_{i=1}^n I(U_i;\tilde{Y}_i)$$

- We related the exponent to the rate via the same variables U_i
- Remaining steps by introducing a uniform random variable T for the time index and taking $n\to\infty,\epsilon,\eta\downarrow 0$

Summary

- Distributed hypothesis testing under zero-rate constraints $\rightarrow$ Local type-based decisions and unanimous decision-forwarding
- Distributed hypothesis testing under maximum rate-constraints $\rightarrow$ Quantization and unanimous-decision forwarding
- Distributed hypothesis testing under *expected* rate-constraints:
    - Combine different degenerate versions of optimal fixed-length schemes
    - Rate-boost on all rates
    - Tradeoff between different decisions