Max-Entropy Results under Markov Conditions and Applications to Capacity Problems

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Optimality of Gaussians for Some Capacity Expressions

- Capacity of many Gaussian two-user setups with partially coop. txs:

$$\mathcal{C} = \bigcup_{\substack{X_1 - U - X_2: \\ \mathsf{E}\left[X_\nu^2\right] \leq P_\nu}} \mathcal{R}(X_1, U, X_2)$$

Main Results

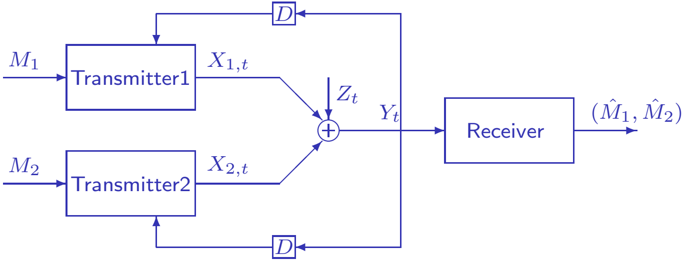
- Technique showing optimality of Gaussian $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ for certain $\mathcal{R}(X_1, U, X_2)$
- Capacity results

Two Known Optimality-Proofs of Gaussians

1. Memoryless Gaussian Multi-Access Channel (MAC) with feedback

2. Memoryless Gaussian MAC without feedback

Memoryless Gaussian MAC with Feedback



- $Y_t = X_{1,t} + X_{2,t} + Z_t \in \mathbb{R}$, $\{Z_t\}$ IID $\sim \mathcal{N}(0,1)$
- Power constraint $\frac{1}{n}\mathsf{E}\left[X_{\nu,t}(M_\nu, Y^{t-1})^2\right] \leq P_\nu$
- Independent Messages; $M_\nu \sim \mathcal{U}\{1, \ldots, \lfloor 2^{nR_\nu} \rfloor\}$
- Capacity region: $\left\{(R_1, R_2) \text{ s.t. } \lim_{n\to\infty} \Pr\left[(M_1, M_2) \neq (\hat{M}_1, \hat{M}_2)\right] = 0\right\}$

Capacity Bounds for Gaussian MAC with Feedback

Ozarow'85:

$$\bigcup_{\substack{(X_1^{\mathcal{G}},X_2^{\mathcal{G}}) \text{ Gaussian:} \\ \mathsf{E}\left[(X_\nu^{\mathcal{G}})^2\right]\leq P_\nu}} \mathcal{R}_{\mathsf{MacFb}}(X_1^{\mathcal{G}}, X_2^{\mathcal{G}}) \subseteq \mathcal{C}_{\mathsf{MacFb}} \subseteq \bigcup_{\substack{(X_1,X_2): \\ \mathsf{E}\left[X_\nu^2\right]\leq P_\nu}} \mathcal{R}_{\mathsf{MacFb}}(X_1, X_2)$$

where

$$\mathcal{R}_{\mathsf{MacFb}}(X_1, X_2) \triangleq \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \leq I(X_1; Y|X_2) \\ R_2 & \leq I(X_2; Y|X_1) \\ R_1 + R_2 & \leq I(X_1 X_2; Y) \end{array} \right\}$$

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where

$$\mathcal{R}_{\mathsf{MacFb}}(X_1, X_2) \triangleq \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \leq & h(X_1 + Z | X_2) - h(Z) \\ R_2 & \leq & h(X_2 + Z | X_1) - h(Z) \\ R_1 + R_2 & \leq & h(X_1 + X_2 + Z) - h(Z) \end{array} \right\}$$

- Capacity result requires optimality of Gaussians for outer bound!

Tool: Conditional Maximum-Entropy Theorem

Theorem (Thomas'91)

$\mathbf{X}$ and $\mathbf{Y}$ of covariance matrix $\mathsf{K}_{\mathbf{XY}}$, and $\mathbf{X}^{\mathcal{G}}, \mathbf{Y}^{\mathcal{G}} \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_{\mathbf{XY}})$. Then,

$$h(\mathsf{A}\mathbf{X}|\mathbf{Y}) \leq h(\mathsf{A}\mathbf{X}^{\mathcal{G}}|\mathbf{Y}^{\mathcal{G}}), \qquad \forall \mathsf{A} \in \mathbb{R}^{\eta \times n_X}$$

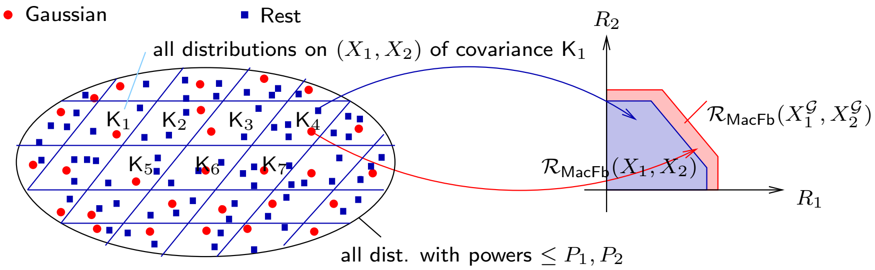
Proof when $\det(\mathsf{A}) \neq 0$:

$$\begin{aligned}
0 &\leq \mathsf{E}_{\mathbf{Y}}\left[D\left(P_{\mathsf{A}\mathbf{X}|\mathbf{Y}=\mathbf{y}} || P_{\mathsf{A}\mathbf{X}^{\mathcal{G}}|\mathbf{Y}^{\mathcal{G}}=\mathbf{y}}\right)\right] \\
&= \mathsf{E}_{\mathbf{Y}}\left[\int \log\left(\frac{\mathsf{d}P_{\mathsf{A}\mathbf{X}|\mathbf{Y}=\mathbf{y}}(\mathsf{A}\mathbf{x})}{\mathsf{d}P_{\mathsf{A}\mathbf{X}^{\mathcal{G}}|\mathbf{Y}^{\mathcal{G}}=\mathbf{y}}(\mathsf{A}\mathbf{x})}\right) \mathsf{d}P_{\mathsf{A}\mathbf{X}|\mathbf{Y}=\mathbf{y}}(\mathbf{x})\right] \\
&= -h(\mathsf{A}\mathbf{X}|\mathbf{Y}) \\
&\quad + \mathsf{E}_{\mathbf{Y}}\left[\int \left(\frac{1}{2}\log\left((2\pi)^{\eta}\frac{|\mathsf{K}_{(\mathsf{A}\mathbf{X})\mathbf{Y}}|}{|\mathsf{K}_{\mathbf{Y}}|}\right) + \frac{1}{2}(\mathsf{A}\mathbf{X})^{\mathsf{T}}\left(\frac{\mathsf{K}_{(\mathsf{A}\mathbf{X})\mathbf{Y}}}{\mathsf{K}_{\mathbf{Y}}}\right)^{-1}\mathsf{A}\mathbf{X}\right) \mathsf{d}P_{\mathsf{A}\mathbf{X}|\mathbf{Y}=\mathbf{y}}(\mathbf{x})\right] \\
&= -h(\mathsf{A}\mathbf{X}|\mathbf{Y}) + h(\mathsf{A}\mathbf{X}^{\mathcal{G}}|\mathbf{Y}^{\mathcal{G}})
\end{aligned}$$

Optimality of Gaussians for Ozarow's Outer Bound

- Conditional Max-Entropy Theorem, $(X_1^{\mathcal{G}}, X_2^{\mathcal{G}}) \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_{\mathsf{X}_1\mathsf{X}_2})$

$$h(X_1 + Z|X_2) - h(Z) \leq h(X_1^{\mathcal{G}} + Z|X_2^{\mathcal{G}}) - h(Z)$$
$$h(X_2 + Z|X_1) - h(Z) \leq h(X_2^{\mathcal{G}} + Z|X_1^{\mathcal{G}}) - h(Z)$$
$$h(X_1 + X_2 + Z) - h(Z) \leq h(X_1^{\mathcal{G}} + X_2^{\mathcal{G}} + Z) - h(Z)$$

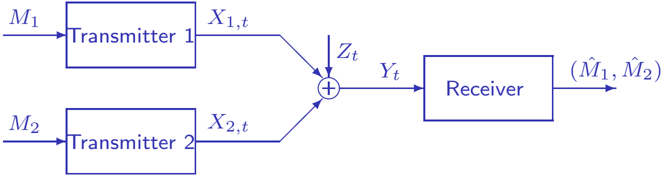


Capacity Region of Gaussian MAC with Feedback

$$C_{\mathsf{MacFb}} = \bigcup_{\substack{(X_1^{\mathcal{G}}, X_2^{\mathcal{G}})\,\text{Gaussian:} \\ \mathsf{E}\left[(X_\nu^{\mathcal{G}})^2\right] \le P_\nu}} \mathcal{R}_{\mathsf{MacFb}}(X_1^{\mathcal{G}}, X_2^{\mathcal{G}})$$

$$= \bigcup_{\rho \in [0,1]} \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \le & \frac{1}{2}\log\left(1 + P_1(1-\rho^2)\right) \\ R_2 & \le & \frac{1}{2}\log\left(1 + P_2(1-\rho^2)\right) \\ R_1 + R_2 & \le & \frac{1}{2}\log\left(1 + P_1 + P_2 + 2\rho\sqrt{P_1 P_2}\right) \end{array} \right\}$$

Memoryless Gaussian MAC without Feedback



- $X_{1,t} = f_{1,t}^{(n)}(M_1)$ and $X_{2,t} = f_{2,t}^{(n)}(M_2)$

Capacity of Gaussian MAC without Feedback

Cover'73, Wyner'74:

$$\mathcal{C}_{\mathsf{Mac}} = \bigcup_{\substack{X_1 - U - X_2: \\ \mathsf{E}\left[X_\nu^2\right] \le P_\nu}} \mathcal{R}_{\mathsf{Mac}}(X_1, U, X_2)$$

where

$$\mathcal{R}_{\mathsf{Mac}}(X_1, U, X_2) = \left\{ (R_1, R_2) : \begin{array}{rl} R_1 & \le I(X_1; Y | X_2 U) \\ R_2 & \le I(X_2; Y | X_1 U) \\ R_1 + R_2 & \le I(X_1 X_2; Y | U) \end{array} \right\}$$

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Suffices to consider Gaussian $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

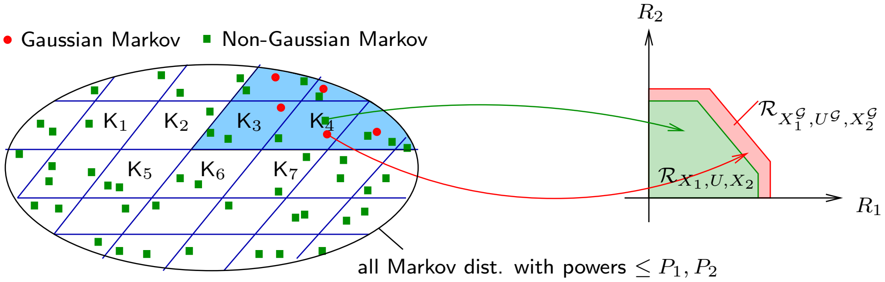
where

$$\mathcal{R}_{\mathsf{Mac}}(X_1, U, X_2) = \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \le & h(X_1 + Z | X_2, U) - h(Z) \\ R_2 & \le & h(X_2 + Z | X_2, U) - h(Z) \\ R_1 + R_2 & \le & h(X_1 + X_2 + Z | U) - h(Z) \end{array} \right\}$$

- Max-entropy not sufficient to show optimality of Gaussians, need Jensen

Max-Entropy Theorem not Sufficient

- For every K there is a Markov tuple $X_1 - U - X_2$
- Only for few K there is a Gaussian Markov tuple $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$



- Approach cannot prove that Gaussian distributions are optimal!

Max-Entropy and Jensen do the Trick!

- Fix $X_1 - U - X_2$ and let $\forall u$: $(X^{\mathcal{G}}_{1,u}, X^{\mathcal{G}}_{2,u}) \sim \mathcal{N}\big(0, \mathsf{K}_{X_1X_2|U=u}\big)$
- Max-Entropy & Jensen:

$$
\begin{aligned}
& h(X_1 + X_2 + Z|U) - h(Z) \\
& \quad \leq \int h(X^{\mathcal{G}}_{1,U} + X^{\mathcal{G}}_{2,U} + Z|U = u)\, \mathsf{d}P_U(u) - h(Z) \quad \text{(Max-Entropy)} \\
& \quad = \int \frac{1}{2}\log(1 + \mathsf{Var}(X_{1,U}|U = u) + \mathsf{Var}(X_{2,U}|U = u))\, \mathsf{d}P_U(u) \\
& \quad \leq \frac{1}{2}\log\big(1 + \mathsf{E}[\mathsf{Var}(X_{1,U}|U)] + \mathsf{E}[\mathsf{Var}(X_{2,U}|U)]\big) \qquad \text{(Jensen)} \\
& \quad \leq \frac{1}{2}\log\big(1 + \mathsf{E}\big[X_1^2\big] + \mathsf{E}\big[X_2^2\big]\big) \\
& \quad \leq \frac{1}{2}\log(1 + P_1 + P_2)
\end{aligned}
$$

- Similarly: $h(X_\nu + Z|U) - h(Z) \leq \frac{1}{2}\log(1 + P_\nu)$.

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$$
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&\quad \leq \frac{1}{2}\log\big(1 + \mathsf{E}[\mathsf{Var}(X_{1,U}|U)] + \mathsf{E}[\mathsf{Var}(X_{2,U}|U)]\big) \qquad \text{(Jensen)} \\
&\quad \leq \frac{1}{2}\log\big(1 + \mathsf{E}\big[X_1^2\big] + \mathsf{E}\big[X_2^2\big]\big) \\
&\quad \leq \frac{1}{2}\log(1 + P_1 + P_2)
\end{aligned}
$$

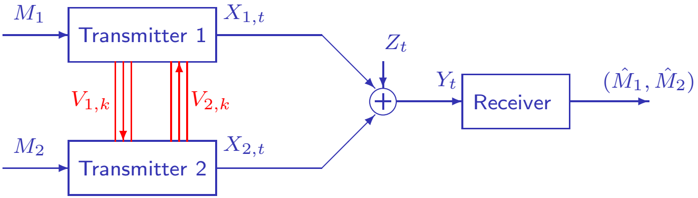
- Similarly: $h(X_\nu + Z|U) - h(Z) \leq \frac{1}{2}\log(1 + P_\nu)$.

Capacity of Gaussian MAC without Feedback

$$C_{\mathsf{Mac}} = \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \leq & \frac{1}{2}\log\left(1 + P_1\right) \\ R_2 & \leq & \frac{1}{2}\log\left(1 + P_2\right) \\ R_1 + R_2 & \leq & \frac{1}{2}\log\left(1 + P_1 + P_2\right) \end{array} \right\}$$

attained by independent X_1, X_2, U with $X_\nu \sim \mathcal{N}(0, P_\nu)$

Gaussian MAC with Conferencing Encoders

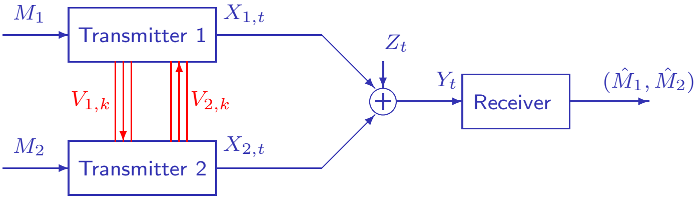


1. phase: Conference (κ sequential uses of noise-free pipes) [Willems'83]

- $V_{1,k} = \varphi_{1,k}\left(M_1, V_{2,1}^{k-1}\right); \qquad V_{2,k} = \varphi_{2,k}\left(M_2, V_{1,1}^{k-1}\right)$

- Rate-limitations: $\sum_{k=1}^{\kappa} \log|\mathcal{V}_{1,k}| \leq nC_{12}$ and $\sum_{k=1}^{\kappa} \log|\mathcal{V}_{2,k}| \leq nC_{21}$

Gaussian MAC with Conferencing Encoders

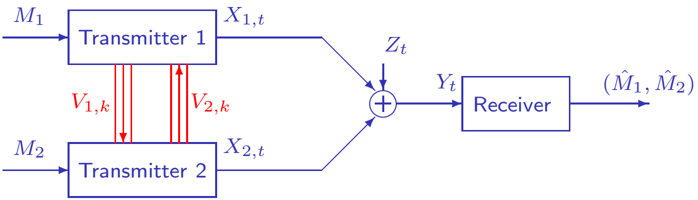


2. phase: Transmission over channel

- $X_{1,t} = f_{1,t}\left(M_1, V_{2,1}^{\kappa}\right);$ $\qquad X_{2,t} = f_{2,t}\left(M_2, V_{1,1}^{\kappa}\right)$

- Power constraints: $\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[X_{\nu,t}^2\right] \leq P_\nu$

Gaussian MAC with Conferencing Encoders



- Discrete Memoryless case solved by Willems'83
- Special Cases:
  - $C_{12} = C_{21} = \infty$: full cooperation (both txs know (M_1, M_2))
  - $C_{12} = 0$, $C_{21} = \infty$: Tx 1 knows (M_1, M_2), Tx 2 only M_2
  - $C_{12} = C_{21} = 0$: no conferencing

Capacity of Gaussian MAC with Conferencing Encoders

Theorem

$$\mathcal{C}_{\mathsf{Conf}} = \bigcup_{\substack{X_1 - U - X_2: \\ \mathsf{E}\left[X_\nu^2\right] \leq P_\nu}} \mathcal{R}_{X_1,U,X_2}$$

where

$$\mathcal{R}_{X_1,U,X_2} = \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \leq & I(X_1; Y | X_2 U) + C_{12} \\ R_2 & \leq & I(X_2; Y | X_1 U) + C_{21} \\ R_1 + R_2 & \leq & I(X_1 X_2; Y | U) + C_{12} + C_{21} \\ R_1 + R_2 & \leq & I(X_1 X_2; Y) \end{array} \right\}$$

Proof similar to Willems'83

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Suffices to consider Gaussian $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

Capacity of Gaussian MAC with Conferencing Encoders

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Suffices to consider Gaussian $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

- Previous proof techniques:

  1. Pure max-entropy approach fails because of Markov-condition!

  2. Max-entropy approach & Jensen fails due to unconditional entropy!

Capacity of Gaussian MAC with Conferencing Encoders

Theorem

$$\mathrm{C}_{\mathsf{Conf}} = \bigcup_{\substack{X_1 - U - X_2: \\ \mathsf{E}[X_\nu^2] \le P_\nu}} \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \le & h(X_1 + Z | X_2, U) - h(Z) + C_{12} \\ R_2 & \le & h(X_2 + Z | X_1 U) - h(Z) + C_{21} \\ R_1 + R_2 & \le & h(X_1 + X_2 + Z | U) - h(Z) + C_{12} + C_{21} \\ R_1 + R_2 & \le & h(X_1 + X_2 + Z) - h(Z). \end{array} \right\}$$

Suffices to consider Gaussian $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

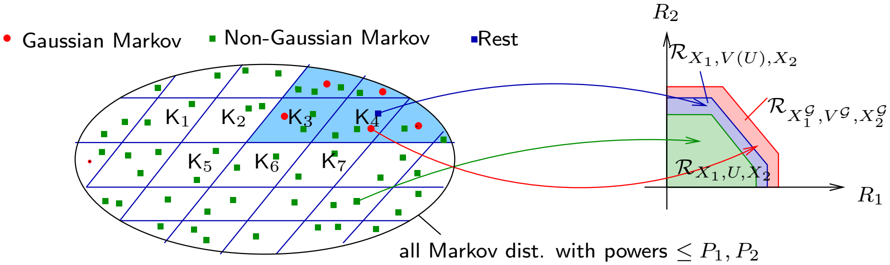
- Proofs:

  1. Max-entropy approach & substitution-trick

  2. Max-entropy approach & Jensen & max-correlation inequality

Substitution-Trick & Max-Entropy Approach

- Fix $X_1 - U - X_2$ and $V(U) \triangleq \mathsf{E}[X_1|U]$
- Shall show:
  1. $\mathcal{R}_{X_1,V,X_2} \supseteq \mathcal{R}_{X_1,U,X_2}$
  2. $(X_1^{\mathcal{G}}, V^{\mathcal{G}}, X_2^{\mathcal{G}}) \sim \mathcal{N}(\mathbf{0}, \mathsf{K}_{X_1VX_2})$ satisfies $X_1^{\mathcal{G}} - V^{\mathcal{G}} - X_2^{\mathcal{G}}$



Proof of Property 1: $\mathcal{R}_{X_1,V,X_2} \supseteq \mathcal{R}_{X_1,U,X_2}$

- Because $V \triangleq \mathsf{E}[X_1|U] = f(U)$:

$$h(X_1 + Z|X_2, U) = h(X_1 + Z|X_2, U, V) \leq h(X_1 + Z|X_2, V)$$

- Similarly:

$$h(X_2 + Z|X_1, U) \leq h(X_1 + Z|X_2, V)$$
$$h(X_1 + X_2 + Z|U) \leq h(X_1 + X_2 + Z|V)$$

- Since all other terms do not depend on U or V:

$$\mathcal{R}_{X_1,U,X_2} \subseteq \mathcal{R}_{X_1,V,X_2}$$

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- Since all other terms do not depend on U or V:

$$\mathcal{R}_{X_1,U,X_2} \subseteq \mathcal{R}_{X_1,V,X_2}$$

- Note: holds also if $\mathbf{X}_1, \mathbf{U}, \mathbf{X}_2$ are vectors and $\mathbf{V} \triangleq \mathsf{E}[\mathbf{X}_1|\mathbf{U}]$

Proof of Property 2: Markov Chain $X_1^{\mathcal{G}} - V^{\mathcal{G}} - X_2^{\mathcal{G}}$

- By Gaussianity, Markov chain equivalent to

$$(X_1^{\mathcal{G}} - aV^{\mathcal{G}}) \perp (V^{\mathcal{G}}, X_2^{\mathcal{G}}) \qquad \text{for some } a \in \mathbb{R}$$

- Proof (recall $V = \mathsf{E}[X_1|U]$)

$$\begin{aligned}
\mathsf{E}\left[(X_1^{\mathcal{G}} - V^{\mathcal{G}})\begin{pmatrix} V^{\mathcal{G}} \\ X_2^{\mathcal{G}} \end{pmatrix}\right] &= \mathsf{E}\left[(X_1 - V)\begin{pmatrix} V \\ X_2 \end{pmatrix}\right] \\
&= \mathsf{E}_U\left[\mathsf{E}\left[(X_1 - V)\begin{pmatrix} V \\ X_2 \end{pmatrix}\middle| U\right]\right] \\
&= \mathsf{E}_U\left[\underbrace{\mathsf{E}[(X_1 - V)|U]}_{=0}\,\mathsf{E}\left[\begin{pmatrix} V \\ X_2 \end{pmatrix}\middle| U\right]\right] \\
&= 0.
\end{aligned}$$

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&= \mathsf{E}_U\left[\mathsf{E}\left[(X_1 - V)\begin{pmatrix} V \\ X_2 \end{pmatrix}\middle| U\right]\right] \\
&= \mathsf{E}_U\left[\underbrace{\mathsf{E}[(X_1 - V)|U]}_{=0}\,\mathsf{E}\left[\begin{pmatrix} V \\ X_2 \end{pmatrix}\middle| U\right]\right] \\
&= 0.
\end{aligned}$$

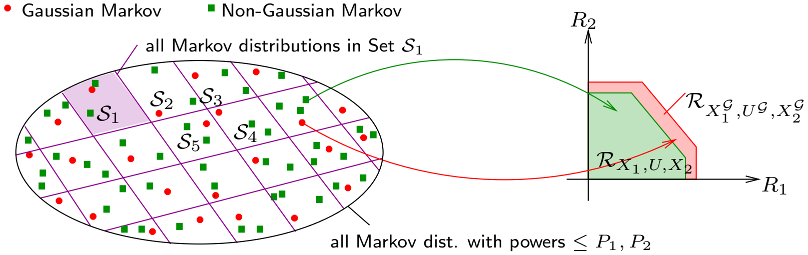
- Holds also if $\mathbf{X}_1, \mathbf{U}, \mathbf{X}_2$ are vectors and $\mathbf{V} \triangleq \mathsf{E}[\mathbf{X}_1|\mathbf{U}]$

Max-Entropy & Jensen & Max-Correlation Inequality

- Different Partition!

  $\mathcal{S}_i$: $X_1 - U - X_2$ with same $\mathsf{E}\left[X_\nu^2\right]$ and $\mathsf{E}[\mathsf{Var}(X_\nu|U)]$ for $\nu \in \{1,2\}$

- Shall show:
  1. In every $\mathcal{S}_i$ there is a Gaussian Markov tuple
  2. Within every $\mathcal{S}_i$ Gaussian Markov triples have the largest region



Proof of Property 2: Gaussians optimal in every $\mathcal{S}_i$

- With Max-Entropy & Jensen:

$$
\begin{aligned}
I(X_1;Y|X_2U) &\leq \frac{1}{2}\log\left(1+\mathsf{E}[\mathsf{Var}(X_1|U)]\right)\\
I(X_2;Y|X_1U) &\leq \frac{1}{2}\log\left(1+\mathsf{E}[\mathsf{Var}(X_2|U)]\right)\\
I(X_1X_2;Y|U) &\leq \frac{1}{2}\log\left(1+\mathsf{E}[\mathsf{Var}(X_1|U)]+\mathsf{E}[\mathsf{Var}(X_2|U)]\right)\\
I(X_1X_2;Y) &\leq \frac{1}{2}\log\left(1+\mathsf{E}\left[X_1^2\right]+\mathsf{E}\left[X_2^2\right]+2\mathsf{E}[X_1X_2]\right)
\end{aligned}
$$

- $\forall\ X_1 - U - X_2$:

$$
\begin{aligned}
\mathsf{E}[X_1X_2] = \mathsf{E}_U[\mathsf{E}[X_1|U]\,\mathsf{E}[X_2|U]] &\leq \sqrt{\mathsf{E}[(\mathsf{E}[X_1|U])^2]}\sqrt{\mathsf{E}[(\mathsf{E}[X_2|U])^2]}\\
&= \sqrt{\mathsf{E}[X_1^2]-\mathsf{E}[\mathsf{Var}(X_1|U)]}\sqrt{\mathsf{E}[X_2^2]-\mathsf{E}[\mathsf{Var}(X_2|U)]}
\end{aligned}
$$

All inequalities hold with equality for Gaussian Markov triples.

Proof of Property 2: Gaussians optimal in every $\mathcal{S}_i$

- With Max-Entropy & Jensen:

$$
\begin{aligned}
I(X_1;Y|X_2U) &\leq \frac{1}{2}\log\left(1+\mathsf{E}[\mathsf{Var}(X_1|U)]\right) \\
I(X_2;Y|X_1U) &\leq \frac{1}{2}\log\left(1+\mathsf{E}[\mathsf{Var}(X_2|U)]\right) \\
I(X_1X_2;Y|U) &\leq \frac{1}{2}\log\left(1+\mathsf{E}[\mathsf{Var}(X_1|U)]+\mathsf{E}[\mathsf{Var}(X_2|U)]\right) \\
I(X_1X_2;Y) &\leq \frac{1}{2}\log\left(1+\mathsf{E}\left[X_1^2\right]+\mathsf{E}\left[X_2^2\right]+2\mathsf{E}[X_1X_2]\right)
\end{aligned}
$$

- $\forall\ X_1 - U - X_2$:

$$
\begin{aligned}
\mathsf{E}[X_1X_2] = \mathsf{E}_U[\mathsf{E}[X_1|U]\,\mathsf{E}[X_2|U]] &\leq \sqrt{\mathsf{E}[(\mathsf{E}[X_1|U])^2]}\sqrt{\mathsf{E}[(\mathsf{E}[X_2|U])^2]} \\
&= \sqrt{\mathsf{E}[X_1^2]-\mathsf{E}[\mathsf{Var}(X_1|U)]}\sqrt{\mathsf{E}[X_2^2]-\mathsf{E}[\mathsf{Var}(X_2|U)]}
\end{aligned}
$$

All inequalities hold with equality for Gaussian Markov triples.

- Note: Extension to vectors seems difficult

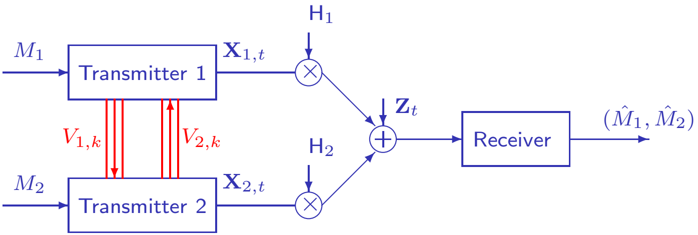
Capacity of Gaussian MAC with Conferencing Encoders

Theorem

$$\mathrm{C}_{\mathsf{Conf}} = \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}: \\ \mathsf{E}\left[(X_\nu^{\mathcal{G}})^2\right] \le P_\nu}} \mathcal{R}_{X_1,U,X_2}$$

$$= \bigcup_{\rho_1,\rho_2 \in [0,1]} \left\{ (R_1,R_2) : \begin{array}{rcl} R_1 & \le & \frac{1}{2}\log(1 + P_1(1-\rho_1^2)) + C_{12} \\ R_2 & \le & \frac{1}{2}\log(1 + P_2(1-\rho_2^2)) + C_{21} \\ R_1 + R_2 & \le & \frac{1}{2}\log(1 + P_1(1-\rho_1^2) + P_2(1-\rho_2^2)) \\ & & +C_{12} + C_{21} \\ R_1 + R_2 & \le & \frac{1}{2}\log(1 + P_1 + P_2 + 2\rho_1\rho_2\sqrt{P_1P_2}) \end{array} \right\}$$

Technique applies to MIMO MAC with Conferencing Enc.

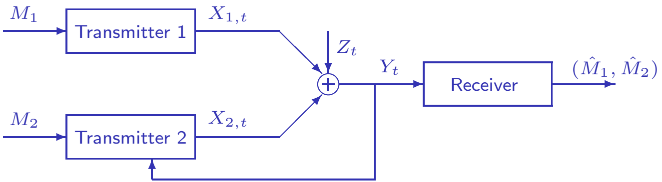


Capacity Region of Gaussian MIMO MAC with Conferencing Encoders

$$\mathrm{C}_{\text{MIMO,Conf}} = \bigcup_{\substack{\mathbf{X}_1 - \mathbf{U} - \mathbf{X}_2 \\ \mathrm{tr}(\mathsf{K}_{X_\nu}) \leq P_\nu}} \left\{ \begin{array}{rcl} R_1 & \leq & I(\mathbf{X}_1; \mathbf{Y} | \mathbf{X}_2 \mathbf{U}) + C_{12}, \\ R_2 & \leq & I(\mathbf{X}_2; \mathbf{Y} | \mathbf{X}_1 \mathbf{U}) + C_{21}, \\ R_1 + R_2 & \leq & I(\mathbf{X}_1 \mathbf{X}_2; \mathbf{Y} | \mathbf{U}) + C_{12} + C_{21}, \\ R_1 + R_2 & \leq & I(\mathbf{X}_1 \mathbf{X}_2; \mathbf{Y}) \end{array} \right\}$$

Gaussian triples $\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}}$ with $\dim(\mathbf{U}^{\mathcal{G}}) = \min_\nu \{\dim(\mathbf{X}_\nu)\}$ suffice

Technique applies to Cover-Leung Region



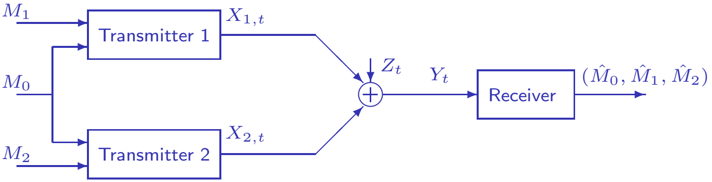
Achievable Region for AWGN MAC with One-sided Feedback [Willems'83]

$$\mathcal{R}_{\mathsf{CL}} \subseteq \mathrm{C}_{\mathsf{OneSidedFB}}$$

$$\mathcal{R}_{\mathsf{CL}} \triangleq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}\left[X_1^2\right] \le P_1, \\ \mathsf{E}\left[X_2^2\right] \le P_2}} \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \le & I(X_1; Y | X_2 U) \\ R_2 & \le & I(X_2; Y | X_1 U) \\ R_1 + R_2 & \le & I(X_1 X_2; Y) \end{array} \right\}$$

Suffices to consider Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$!

Technique applies to MAC with Common/Private Msgs.

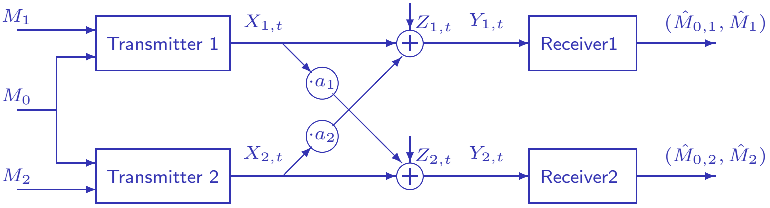


Capacity of Gaussian MAC with Common/Private Msgs. [Slepian/Wolf'73]

$$\mathrm{C}_{\mathsf{ComMsgs}} \triangleq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \leq P_1, \\ \mathsf{E}[X_2^2] \leq P_2}} \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \leq & I(X_1; Y | X_2 U) \\ R_2 & \leq & I(X_2; Y | X_1 U) \\ R_1 + R_2 & \leq & I(X_1 X_2; Y | U) \\ R_0 + R_1 + R_2 & \leq & I(X_1 X_2; Y) \end{array} \right\}$$

Suffices to consider Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$!

Technique applies to IC with Common/Private Msgs.



Capacity region under strong interference [Maric/Yates/Kramer'07]

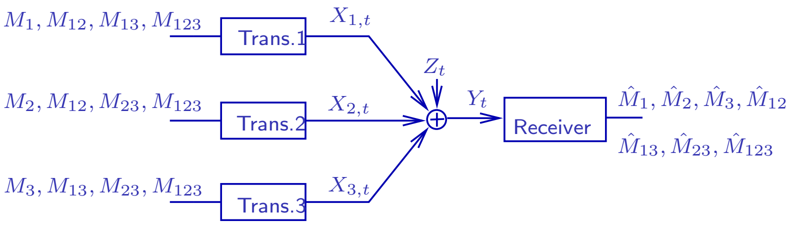
$$\mathrm{C}_{\mathsf{ICPC}} \triangleq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1, \\ \mathsf{E}[X_2^2] \le P_2}} \left\{ \begin{array}{ccc} (R_0, R_1, R_2) : & & \\ R_1 & \le & I(X_1; Y_1 | X_2 U) \\ R_2 & \le & I(X_2; Y_2 | X_1 U) \\ R_1 + R_2 & \le & \min \left\{ \begin{array}{c} I(X_1 X_2; Y_1 | U), \\ I(X_1 X_2; Y_2 | U) \end{array} \right\} \\ R_0 + R_1 + R_2 & \le & \min \left\{ \begin{array}{c} I(X_1 X_2; Y_1), \\ I(X_1 X_2; Y_2) \end{array} \right\} \end{array} \right\}$$

Suffices to consider Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$!

More General Markov Structures?

- Does technique also apply to more involved Markov chains?
- 3-user MAC with unicast-conferencing
- 3-user MAC with common and private messages

3-User AWGN MAC with Common and Private Messages



- Channel: $Y_t = X_{1,t} + X_{2,t} + X_{3,t} + Z_t$
- Common/private messages
- Discrete memoryless case solved by Slepian/Wolf'73 & Han

Capacity of 3 User AWGN MAC with Priv./Common Msgs

Theorem

$$\bigcup_{\substack{U_0,U_{12},U_{13},U_{23} \text{ independent} \\ X_1-(U_0,U_{12},U_{13})-(X_2,X_3,U_{23}) \\ X_2-(U_0,U_{12},U_{23})-(X_1,X_3,U_{13}) \\ X_3-(U_0,U_{13},U_{23})-(X_1,X_2,U_{12}) \\ \mathsf{E}\left[\|X_\nu\|^2\right]\leq P_\nu,\ \nu\in\{1,2,3\}}} \mathcal{R}_{\mathbf{U}X_1X_2X_3} \triangleq \Big\{(R_1,R_2,R_3,R_{12},R_{13},R_{23},R_{123}):$$

$$\forall \text{ distinct } i,j,k \in \{1,2,3\}:$$

$$R_i \leq I(X_i;Y|X_j,X_k,U_0,U_{ij},U_{ik})$$

$$R_i + R_j \leq I(X_i,X_j;Y|X_k,U_0,U_{ij},U_{ik},U_{jk})$$

$$R_1+R_2+R_3 \leq I(X_1,X_2,X_3;Y|U_0,U_{12},U_{13},U_{23})$$

$$R_{ij}+R_i+R_j \leq I(X_i,X_j;Y|X_k,U_0,U_{ik},U_{jk})$$

$$R_{ij}+R_1+R_2+R_3 \leq I(X_1,X_2,X_3;Y|U_0,U_{ik},U_{jk})$$

$$R_{ij}+R_{ik}+R_1+R_2+R_3 \leq I(X_1,X_2,X_3;Y|U_0,U_{jk})$$

$$R_{12}+R_{13}+R_{23}+R_1+R_2+R_3 \leq I(X_1,X_2,X_3;Y|U_0)$$

$$R_{123}+R_{12}+R_{13}+R_{23}+R_1+R_2+R_3 \leq I(X_1,X_2,X_3;Y)\Big\}$$

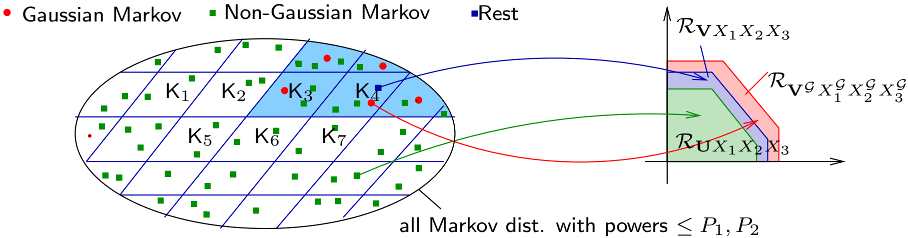
Gaussian $X_1^{\mathcal{G}}, X_2^{\mathcal{G}}, X_3^{\mathcal{G}}, U_0^{\mathcal{G}}, U_{12}^{\mathcal{G}}, U_{13}^{\mathcal{G}}, U_{23}^{\mathcal{G}}$ satisfying Markov chains suffice!

Gaussian-Optimality Proof: Substitution & Max-Entropy

- Choose

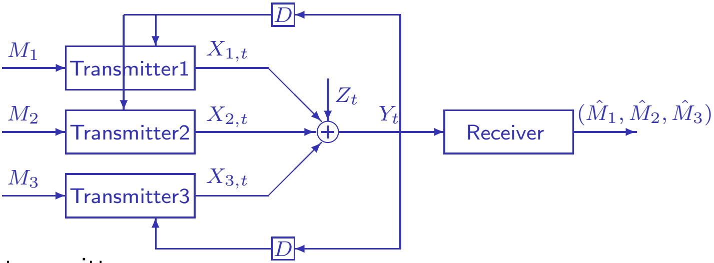
$$V_0(U_0) \triangleq \begin{pmatrix} \mathsf{E}[X_1|U_0] \\ \mathsf{E}[X_2|U_0] \\ \mathsf{E}[X_3|U_0] \end{pmatrix} \text{ and } V_{ik}(U_{ik}, U_0) \triangleq \begin{pmatrix} \mathsf{E}[X_i|U_{ik}, U_0] - \mathsf{E}[X_i|U_0] \\ \mathsf{E}[X_k|U_{ik}, U_0] - \mathsf{E}[X_k|U_0] \end{pmatrix}$$

- Shall show:
  1. $\mathcal{R}_{\mathbf{V}X_1X_2X_3} \supseteq \mathcal{R}_{\mathbf{U}X_1X_2X_3}$
  2. $(V_0^{\mathcal{G}}, V_{12}^{\mathcal{G}}, V_{13}^{\mathcal{G}}, V_{23}^{\mathcal{G}}, X_1^{\mathcal{G}}, X_2^{\mathcal{G}}, X_3^{\mathcal{G}})$ satisfy independence & Markov cond.



The Symmetric K-User Gaussian MAC - Linear-Feedback Sum-Capacity and General Feedback Sum-Capacity

K-User Memoryless Gaussian MAC with Feedback



- $K \geq 2$ transmitters
- Channel: $Y_t = \sum_{\nu=1}^{K} X_{\nu,t} + Z_t$
- Independent messages $M_1, \ldots, M_K$ where $M_\nu \sim \mathcal{U}\{1, \ldots, \lfloor 2^{nR_\nu} \rfloor\}$
- Inputs $X_{\nu,t} = f_{\nu,t}^{(n)}(M_\nu, Y^{t-1})$
- Symmetric Power Constraints: $\frac{1}{n}\mathsf{E}\left[X_{\nu,t}^2\right] \leq P$

Capacity of K-User Gaussian MAC with Feedback

- For $K = 2$, capacity region known (Ozarow'85)
- For $K \geq 3$
  - Best achievable region: Fourier-MEC scheme (Kramer'02) reduces to Ozarow's scheme when $K = 2$
  - Best outer bound: Dependence-Balance bound (Kramer/Gastpar'06)
  - Bounds tight when $P \geq \Pi^\star(K)$ (in this case DB bound = cut-set bound)

Linear-Feedback Codes

- General feedback codes:

$$X_{\nu,t} = f_{\nu,t}(M_\nu, Y^{t-1})$$

- Linear-feedback codes:

$$X_{\nu,t} = L_{\nu,t}(\boldsymbol{\theta}_\nu(M_\nu), Y^{t-1})$$

  where $\boldsymbol{\theta}_\nu$ are arbitrary (vector-valued) and $L_{\nu,t}$ are linear

- Linear-feedback codes include all non-feedback codes and Kramer's code

- Linear-feedback sum-capacity: maximum sum-rate achieved by linear-feedback codes

Symmetric Linear-Feedback Sum-Capacity

Theorem (Ardestanizadeh/Javidi/Kim/Wigger)

The symmetric linear-feedback sum-capacity is

$$C_{\mathsf{L}} = \frac{1}{2}\log(1+\phi^\star KP)$$

where $\phi^\star$ *unique solution in* $[1, K]$ *to*

$$(1+KP\phi)^{K-1} = (1+P\phi(K-\phi))^K$$

Achieved by Kramer's scheme.

When $P \geq \Pi^\star(K)$ *then*

$$C_{\mathsf{L}} = C$$

- Proof in 5 steps ...

Proof-Step 1: Optimality of Gaussians

$$C_{\mathsf{L}} \leq \limsup_{n\to\infty} C_{\mathsf{L}}^{(n)}$$

where

$$C_{\mathsf{L}}^{(n)} \triangleq \max_{\{X_{\nu,t}\}} \frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1})$$

and

1. $X_{\nu,t} = L_{\nu,t}(\mathbf{V}_\nu, Y^{t-1})$ for independent $\{\mathbf{V}_\nu\}$
2. $\frac{1}{n}\mathsf{E}\left[\sum_{t=1}^{n} X_{\nu,t}^2\right] \leq P$

Proof: Fano's Inequality

Proof-Step 1: Optimality of Gaussians

$$C_{\mathsf{L}} \leq \limsup_{n\to\infty} C_{\mathsf{L}}^{(n)}$$

where

$$C_{\mathsf{L}}^{(n)} \triangleq \max_{\{X_{\nu,t}\}} \frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1})$$

and

1. $X_{\nu,t} = L_{\nu,t}(\mathbf{V}_\nu, Y^{t-1})$ for independent $\{\mathbf{V}_\nu\}$
2. $\frac{1}{n}\mathsf{E}\left[\sum_{t=1}^{n} X_{\nu,t}^2\right] \leq P$

Suffices to consider Gaussian $\mathbf{V}_\nu$ (i.e., Gaussian $X_1^n, \ldots, X_K^n$)

Proof: Fano's Inequality & conditional max-entropy theorem

Proof-Step 2: Dependence-Balance Condition

Dependence-balance condition replaces functional relationship:

$$C_{\mathsf{L}}^{(n)} = \max_{\{X_{\nu,t}\}} \frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1})$$

where Gaussian $X_1^n, \ldots, X_K^n$ satisfy

1. $X_{\nu,t} = L_{\nu,t}(\mathbf{V}_\nu, Y^{t-1})$ for independent Gaussian $\{\mathbf{V}_\nu\}$
2. $\frac{1}{n}\mathsf{E}\left[\sum_{t=1}^{n} X_{\nu,t}^2\right] \leq P$

Proof-Step 2: Dependence-Balance Condition

Dependence-balance condition replaces functional relationship:

$$C_{\mathsf{L}}^{(n)} = \max_{\{X_{\nu,t}\}} \frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1})$$

where Gaussian $X_1^n, \ldots, X_K^n$ satisfy

1. $$\frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1}) \leq \frac{1}{n} \frac{1}{K-1} \sum_{t=1}^{n} \sum_{\nu=1}^{K} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1}, X_{\nu,t})$$

2. $\frac{1}{n} \mathsf{E}\left[\sum_{t=1}^{n} X_{\nu,t}^2\right] \leq P$

- Non-convex optimization problem over $\{\mathsf{K}_{X_{1,t},\ldots,X_{K,t}}\}$

Proof-Step 3: Lagrange Duality

For all $\{\lambda_\nu\}, \gamma \geq 0$:

$$C_{\mathsf{L}}^{(n)} \leq \max_{\{X_{\nu,t} \text{ j. G.}\}} \frac{1}{n}\sum_{t=1}^{n} I(X_{1,t},\ldots,X_{K,t};Y_t|Y^{t-1})$$
$$-\gamma\frac{1}{n}\sum_{t=1}^{n} I(X_{1,t},\ldots,X_{K,t};Y_t|Y^{t-1})$$
$$+\frac{1}{n}\frac{\gamma}{K-1}\sum_{t=1}^{n}\sum_{\nu=1}^{K} I(X_{1,t},\ldots,X_{K,t};Y_t|Y^{t-1},X_{\nu,t})$$
$$+\sum_{\nu=1}^{K}\lambda_\nu\left(P-\frac{1}{n}\sum_{t=1}^{n}\mathsf{E}\left[X_{\nu,t}^2\right]\right)$$

- Non-convex optimization problem when $\gamma > 1$
- For $\gamma \in [0,1] \rightarrow$ cutset bound (not sufficient)

Proof-Step 3: Lagrange Duality

For all $\{\lambda_\nu\}, \gamma \geq 0$:

$$
\begin{aligned}
C_{\mathsf{L}}^{(n)} \leq \max_{\{X_{\nu,t} \text{ j. G.}\}} \; & \frac{1}{n}\sum_{t=1}^{n} I(X_{1,t},\ldots,X_{K,t};Y_t|Y^{t-1}) \\
& -\gamma\frac{1}{n}\sum_{t=1}^{n} I(X_{1,t},\ldots,X_{K,t};Y_t|Y^{t-1}) \\
& +\frac{1}{n}\frac{\gamma}{K-1}\sum_{t=1}^{n}\sum_{\nu=1}^{K} I(X_{1,t},\ldots,X_{K,t};Y_t|Y^{t-1},X_{\nu,t}) \\
& +\sum_{\nu=1}^{K}\lambda_\nu\left(P-\frac{1}{n}\sum_{t=1}^{n}\mathsf{E}\left[X_{\nu,t}^2\right]\right)
\end{aligned}
$$

- By symmetry: $\lambda_1 = \ldots \lambda_K = \lambda$ and $\mathsf{K}_{X_{1,t},\ldots,X_{K,t}|Y^{t-1}} = \pi \begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}$

Proof-Step 4: Symmetric & stationary $\{X_{\nu,t}\}$ optimal!

For every $\lambda, \gamma \geq 0$:

$$C_{\mathsf{L}}^{(n)} \leq \max_{\pi \geq 0, \phi \in [0,K]} \frac{1}{2}\log(1+K\pi\phi) \\ -\gamma\left(\frac{1}{2}\log(1+K\pi\phi)+\frac{K}{2(K-1)}\log(1+\pi\phi(K-\phi))\right) \\ +\lambda(P-\pi)$$

- Objective function not concave/convex in (ϕ, π) when $\gamma \geq 1$

Proof-Step 5: Solve Non-Convex Optimization Problem

For every $\lambda, \gamma \geq 0$:

$$C_{\mathsf{L}}^{(n)} \leq \max_{\pi \geq 0, \phi \in [0,K]} \quad \frac{1-\gamma}{2} \log(1 + K\pi\phi) + \gamma \frac{K}{2(K-1)} \log(1 + \pi\phi(K-\phi)) + \lambda(P - \pi)$$

- For fixed γ, π, λ, concave in ϕ $\rightarrow$ choose stationary point $\phi_{\mathsf{s}}(\gamma, \pi) \geq 0$

Proof-Step 5: Solve Non-Convex Optimization Problem

For every $\gamma \geq 0$:

$$C_{\mathsf{L}}^{(n)} \leq \min_{\lambda \geq 0} \max_{\pi \geq 0} \quad \frac{1-\gamma}{2} \log(1 + K\pi\phi_{\mathsf{s}}) + \gamma \frac{K}{2(K-1)} \log(1 + \pi\phi_{\mathsf{s}}(K - \phi_{\mathsf{s}})) + \lambda(P - \pi)$$

- For fixed γ, π, λ, concave in $\phi \rightarrow$ choose stationary point $\phi_{\mathsf{s}}(\gamma, \pi) \geq 0$
- Sum of log-terms concave $\rightarrow$ Slater's condition $\rightarrow$ strong duality

Proof-Step 5: Solve Non-Convex Optimization Problem

For every $\gamma \geq 0$:

$$C_{\mathsf{L}}^{(n)} \leq \max_{\pi \in [0,P]} \; \frac{1-\gamma}{2} \log(1 + K\pi\phi_{\mathsf{s}}) + \gamma \frac{K}{2(K-1)} \log(1 + \pi\phi_{\mathsf{s}}(K - \phi_{\mathsf{s}}))$$

- For fixed γ, π, λ, concave in $\phi \rightarrow$ choose stationary point $\phi_{\mathsf{s}}(\gamma, \pi) \geq 0$
- Sum of log-terms concave $\rightarrow$ Slater's condition $\rightarrow$ strong duality
- Sum of log-terms non-decreasing $\rightarrow \pi^* = P$

Proof-Step 5: Solve Non-Convex Optimization Problem

$$
\begin{aligned}
C_{\mathsf{L}}^{(n)} &\leq \frac{1-\gamma}{2}\log(1+KP\phi_{\mathsf{s}}) + \frac{\gamma K}{2(K-1)}\log(1+P\phi_{\mathsf{s}}(K-\phi_{\mathsf{s}})) \\
&= \frac{1}{2}\log(1+\phi^{\star}KP)
\end{aligned}
$$

- For fixed γ, π, λ, concave in $\phi \rightarrow$ choose stationary point $\phi_{\mathsf{s}}(\gamma, \pi) \geq 0$
- Sum of log-terms concave $\rightarrow$ Slater's condition $\rightarrow$ strong duality
- Sum of log-terms non-decreasing $\rightarrow \pi^{*} = P$
- Choose appropriate γ such that $\phi_{\mathsf{s}}(\gamma, P) = \phi^{\star}$

recall: $(1+KP\phi^{\star})^{K-1} = (1+P\phi^{\star}(K-\phi^{\star}))^{K}$

Proof-Step 1 fails for General Feedback Sum-Capacity C

$$C \leq \limsup_{n\to\infty} C_{\mathsf{L}}^{(n)}$$

where

$$C^{(n)} \triangleq \max_{\{X_{\nu,t}\}} \frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1})$$

and

1. $X_{\nu,t} = f_{\nu,t}(\mathbf{V}_\nu, Y^{t-1})$ for independent $\{\mathbf{V}_\nu\}$
2. $\frac{1}{n}\mathsf{E}\left[\sum_{t=1}^{n} X_{\nu,t}^2\right] \leq P$

~~Suffices to consider Gaussian $\mathbf{V}_\nu$(i.e., Gaussian $X_1^n, \ldots, X_K^n$)~~

Proof: Fano's Inequality & conditional max-entropy theorem

Proof-Step 1 fails for General Feedback Sum-Capacity C

$$C \leq \limsup_{n\to\infty} C_{\mathsf{L}}^{(n)}$$

where

$$C^{(n)} \triangleq \max_{\{X_{\nu,t}\}} \frac{1}{n} \sum_{t=1}^{n} I(X_{1,t}, \ldots, X_{K,t}; Y_t | Y^{t-1})$$

and

1. $X_{\nu,t} - (Y^{t-1}, \mathbf{V}_\nu) - (Y^{t-1}, \mathbf{V}_{\nu'}) - X_{\nu',t}$
2. $\frac{1}{n}\mathsf{E}\left[\sum_{t=1}^{n} X_{\nu,t}^2\right] \leq P$

~~Suffices to consider Gaussian $\mathbf{V}_\nu$(i.e., Gaussian $X_1^n, \ldots, X_K^n$)~~

Proof: Fano's Inequality & conditional max-entropy theorem

Summary

- New technique to prove optimality of Gaussian distributions under Markov-conditions for differential entropy expressions
- Several capacity results for cooperative multi-user networks
- Technique extends to vector case and multiple Markov chains

- Symmetric linear-feedback sum-capacity of K-user Gaussian MAC
- Kramer's scheme optimal for this class, for all P, K
- Problem for general feedback-capacity: optimality-proof of Gaussians