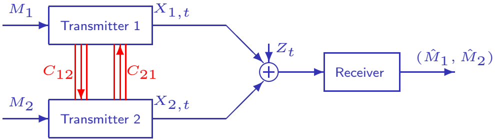
Gaussian Multiple Access Channels with Cooperating Transmitters



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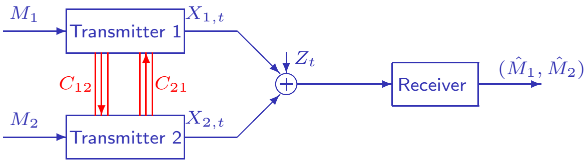
joint work with Shraga Bross, Gerhard Kramer, Amos Lapidoth

Stanford, November, 2009

Part I: Markov Chains and Gaussian Inputs

Multiple-access channel with conferencing encoders

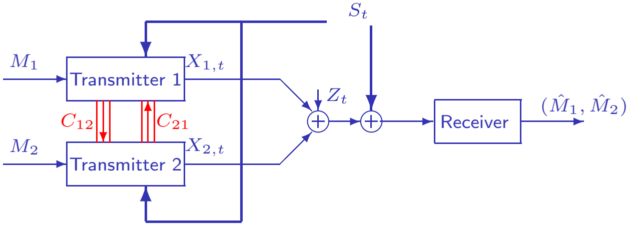


$$\mathcal{C}_{\mathsf{Conf}} \triangleq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}\left[X_1^2\right] \leq P_1,\ \mathsf{E}\left[X_2^2\right] \leq P_2}} \left\{ (R_1, R_2) : \begin{array}{rcll} R_1 & \leq & I(X_1; Y | X_2 U) & + C_{12}, \\ R_2 & \leq & I(X_2; Y | X_1 U) & + C_{21}, \\ R_1 + R_2 & \leq & I(X_1 X_2; Y | U) & + C_{12} + C_{21}, \\ R_1 + R_2 & \leq & I(X_1 X_2; Y) & \end{array} \right\}$$

Capacity region maximized by Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$.

- Proof requires new technique, Max-Entropy theorem not sufficient
- Technique extends to multiple antennas and multiple Markov chains

Part II: Dirty-Paper Coding for MACs with Conferencing



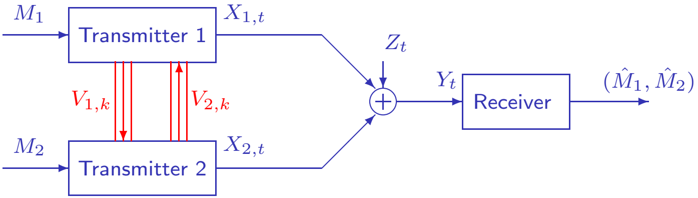
- Costa-type result: Transmitters can cancel non-causally known interference

- Also: Extension of Costa's result to single-user channels with Gaussian interference and (not necessarily Gaussian) dependent noise

Part I:

Markov Chains and Gaussian Inputs

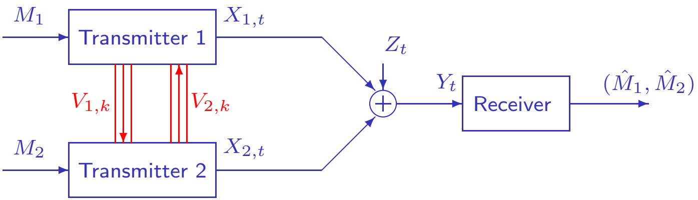
Gaussian MAC with Conferencing Encoders



1. phase: Conference (κ sequential uses of noise-free pipes)

- Messages M_1 and M_2 independent; M_ν uniform over $\{1, \ldots, 2^{nR_\nu}\}$
- $V_{1,k} = \varphi_{1,k}\left(M_1, V_{2,1}^{k-1}\right); \qquad V_{2,k} = \varphi_{2,k}\left(M_2, V_{1,1}^{k-1}\right)$
- Rate-limitations: $\displaystyle\sum_{k=1}^{\kappa} \log|\mathcal{V}_{1,k}| \leq nC_{12}$ and $\displaystyle\sum_{k=1}^{\kappa} \log|\mathcal{V}_{2,k}| \leq nC_{21}$

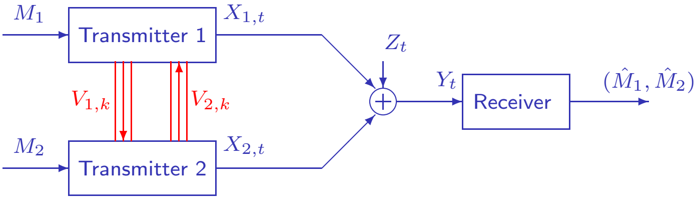
Gaussian MAC with Conferencing Encoders



2. phase: Transmission over channel

- $Y_t = X_{1,t} + X_{2,t} + Z_t;$ $\quad\{Z_t\}$ IID $\sim \mathcal{N}(0, N)$
- $X_{1,t} = f_{1,t}\left(M_1, V_{2,1}^{\kappa}\right);$ $\quad X_{2,t} = f_{2,t}\left(M_2, V_{1,1}^{\kappa}\right)$
- Power constraints: $\quad \frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[X_{\nu,t}^2\right] \le P_\nu, \qquad \nu \in \{1,2\}$
- Capacity region $\mathcal{C}_{\text{Conf}}$:
  Set of all rate-pairs (R_1, R_2) such that $p(\text{error}) \to 0$ as $n \to \infty$

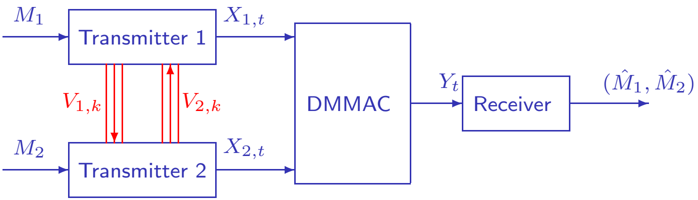
Gaussian MAC with Conferencing Encoders



Special Cases:

- $C_{12} = C_{21} = \infty$: full cooperation (both txs know (M_1, M_2))
- $C_{12} = 0$, $C_{21} = \infty$: Tx 1 knows (M_1, M_2), Tx 2 only M_2
- $C_{12} = C_{21} = 0$: no conferencing

Discrete Memoryless MAC with Conferencing Encoders



Theorem (Willems'83)

$$\mathsf{C}_{\mathsf{DMConf}} = \bigcup_{X_1 - U - X_2} \left\{ (R_1, R_2) : \begin{array}{ccc} R_1 & \leq & I(X_1; Y | X_2 U) + C_{12} \\ R_2 & \leq & I(X_2; Y | X_1 U) + C_{21} \\ R_1 + R_2 & \leq & I(X_1 X_2; Y | U) + C_{12} + C_{21} \\ R_1 + R_2 & \leq & I(X_1 X_2; Y) \end{array} \right\}$$

where $|\mathcal{U}| \leq \min\{|\mathcal{X}_1||\mathcal{X}_2| + 2, |\mathcal{Y}| + 3\}$

Capacity of AWGN MAC with Conferencing Encoders

Theorem

$$C_{\mathsf{Conf}} =$$

$$\bigcup_{\substack{\rho_1,\rho_2 \\ \in[0,1]}} \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \leq & \frac{1}{2}\log\left(1+\frac{P_1\left(1-\rho_1^2\right)}{N}\right)+C_{12} \\ R_2 & \leq & \frac{1}{2}\log\left(1+\frac{P_2\left(1-\rho_2^2\right)}{N}\right)+C_{21} \\ R_1+R_2 & \leq & \frac{1}{2}\log\left(1+\frac{P_1\left(1-\rho_1^2\right)+P_2\left(1-\rho_2^2\right)}{N}\right)+C_{12}+C_{21} \\ R_1+R_2 & \leq & \frac{1}{2}\log\left(1+\frac{P_1+P_2+2\rho_1\rho_2\sqrt{P_1P_2}}{N}\right) \end{array} \right\}$$

Capacity of AWGN MAC with Conferencing Encoders

Theorem

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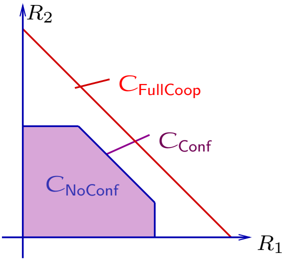
$C_{12} = C_{21} = 0$



Capacity of AWGN MAC with Conferencing Encoders

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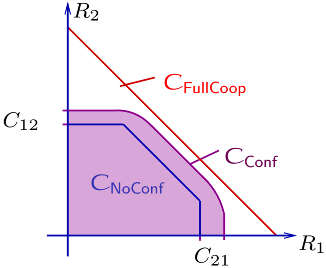
$C_{12}, C_{21} \neq 0$, but "small"



If $C_{12} > 0$ or $C_{21} > 0$

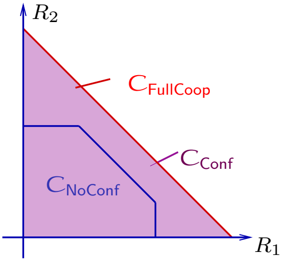
$$C_{\mathsf{NoConf}} \subsetneq C_{\mathsf{Conf}}$$

Capacity of AWGN MAC with Conferencing Encoders

Theorem

$$C_{\mathsf{Conf}} = \bigcup_{\substack{\rho_1,\rho_2 \\ \in[0,1]}} \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \le & \frac{1}{2}\log\left(1 + \frac{P_1\left(1-\rho_1^2\right)}{N}\right) + C_{12} \\ R_2 & \le & \frac{1}{2}\log\left(1 + \frac{P_2\left(1-\rho_2^2\right)}{N}\right) + C_{21} \\ R_1 + R_2 & \le & \frac{1}{2}\log\left(1 + \frac{P_1\left(1-\rho_1^2\right)+P_2\left(1-\rho_2^2\right)}{N}\right) + C_{12} + C_{21} \\ R_1 + R_2 & \le & \frac{1}{2}\log\left(1 + \frac{P_1+P_2+2\rho_1\rho_2\sqrt{P_1P_2}}{N}\right) \end{array} \right\}$$

$$C_{12}, C_{21} \ge \frac{1}{2}\log\left(1 + \frac{P_1+P_2+2\sqrt{P_1P_2}}{N}\right)$$



Superposition-Scheme Achieves Capacity (insp. by Willems'83)

- Conference: Create common message M_0, private messages $M_{1,p}, M_{2,p}$
  - Transmitters split messages: $M_1 = (M_{1,c}, M_{1,p})$ and $M_2 = (M_{2,c}, M_{2,p})$
  - Conference $M_{1,c}$ and $M_{2,c} \Rightarrow$ Common Message $M_0 \triangleq (M_{1,c}, M_{2,c})$
  - Rate of $M_{1,c} < C_{12}$ and rate of $M_{2,c} < C_{21}$

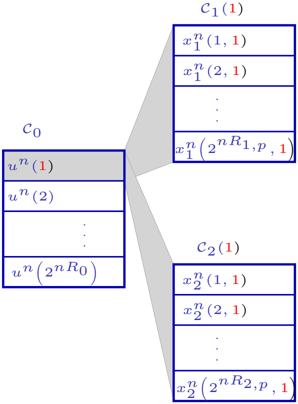
Superposition-Scheme Achieves Capacity (insp. by Willems'83)

- Conference: Create common message M_0, private messages $M_{1,p}, M_{2,p}$

- Superposition:

  - For each $u^n(m_0)$ generate $\mathcal{C}_1(m_0)$ and $\mathcal{C}_2(m_0)$

  - Encode $M_{1,p}$ and $M_{2,p}$ with codebooks $\mathcal{C}_1(M_0)$ and $\mathcal{C}_2(M_0)$



Superposition-Scheme Achieves Capacity (insp. by Willems'83)

- Conference: Create common message M_0, private messages $M_{1,p}, M_{2,p}$
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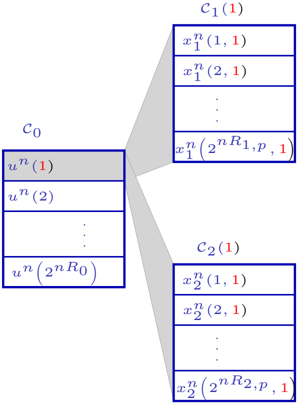
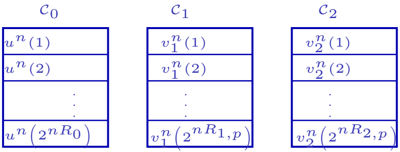


- Joint decoding: Find $(\hat{m}_0, \hat{m}_{1,p}, \hat{m}_{2,p})$ such that $(u^n(\hat{m}_0), x_1^n(\hat{m}_{1,p}, \hat{m}_0), x_2^n(\hat{m}_{2,p}, \hat{m}_0))$ jointly typical

Also Independent Enc./Successive Dec. Achieves Capacity

- Conference as before: common M_0 and private $M_{1,p}, M_{2,p}$

  - Independent codes $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$
  - Addition of codewords:
    $x_\nu^n = \lambda_\nu u^n(M_0) + v_\nu^n(M_{\nu,p})$
    where $\lambda_1 = (1 - \lambda_2) \in [0, 1]$



$\rightarrow$ channel coherently combines $u^n(M_0)$:

- Successive decoding and stripping off

- Capacity region: time-share different decoding orders

Also Independent Enc./Successive Dec. Achieves Capacity

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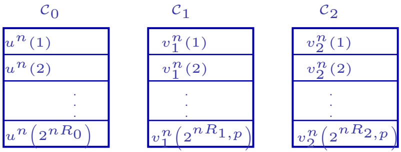

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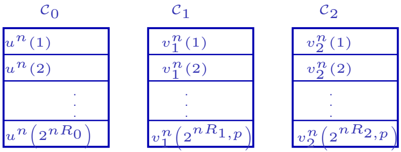
$\rightarrow$ channel coherently combines $u^n(M_0)$:

- Successive decoding and stripping off
  - Decode M_0 from $Y^n = u^n(M_0) + v_1^n(M_{1,p}) + v_2^n(M_{2,p}) + Z^n$;
    subtract $u^n(M_0)$
- Capacity region: time-share different decoding orders

Also Independent Enc./Successive Dec. Achieves Capacity

- Conference as before: common M_0 and private $M_{1,p}, M_{2,p}$



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$\rightarrow$ channel coherently combines $u^n(M_0)$:

- Successive decoding and stripping off
  - Decode $M_{1,p}$ from $Y_0^n = v_1^n(M_{1,p}) + v_2^n(M_{2,p}) + Z^n$;
    subtract $v_1^n(M_{1,p})$

- Capacity region: time-share different decoding orders

Also Independent Enc./Successive Dec. Achieves Capacity

- Conference as before: common M_0 and private $M_{1,p}, M_{2,p}$

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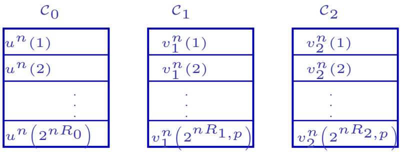


$\rightarrow$ channel coherently combines $u^n(M_0)$:

- Successive decoding and stripping off
  - Decode $M_{2,p}$ from $Y_1^n = v_2^n(M_{2,p}) + Z^n$

- Capacity region: time-share different decoding orders

Converse

- Step 1: Willems's outer bound *with power constraints*:

$$C_{\mathsf{Conf}} \subseteq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \leq P_1,\ \mathsf{E}[X_2^2] \leq P_2}} \mathcal{R}_{X_1,U,X_2}, \tag{1}$$

  where

$$\mathcal{R}_{X_1,U,X_2} \triangleq \left\{ (R_1, R_2) : \begin{array}{rcll} R_1 & \leq & I(X_1; Y|X_2U) & +C_{12}, \\ R_2 & \leq & I(X_2; Y|X_1U) & +C_{21}, \\ R_1 + R_2 & \leq & I(X_1X_2; Y|U) & +C_{12} + C_{21}, \\ R_1 + R_2 & \leq & I(X_1X_2; Y) & \end{array} \right\}$$

- Step 2: In (1) suffices to take *Gaussian* Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

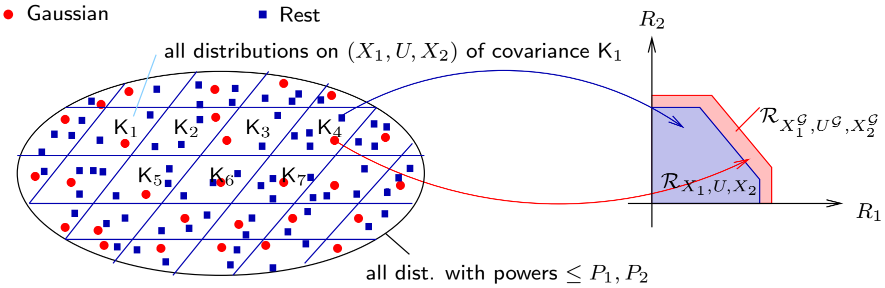
- Step 3: Evaluate $\mathcal{R}_{X_1,U,X_2}$ $\forall$ Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

Step 2: Substitution Approach

Gaussians $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ optimal for $\mathcal{R}_{X_1,U,X_2}$

$$\bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2]\le P_1,\ \mathsf{E}[X_2^2]\le P_2}} \mathcal{R}_{X_1,U,X_2} \stackrel{?}{=} \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}} \\ \mathsf{E}[(X_1^{\mathcal{G}})^2]\le P_1,\ \mathsf{E}[(X_2^{\mathcal{G}})^2]\le P_2}} \mathcal{R}_{X_1^{\mathcal{G}},U^{\mathcal{G}},X_2^{\mathcal{G}}}$$

If there were no Markov condition:



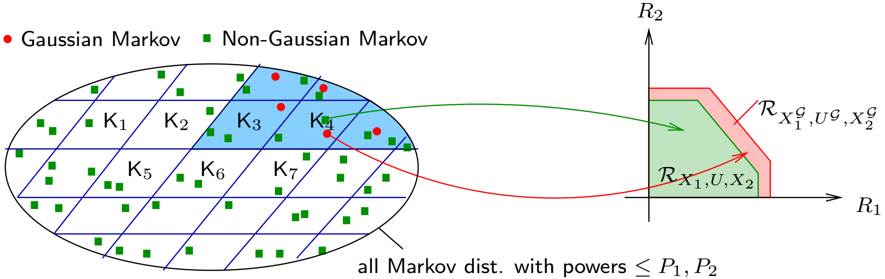
Direct application of Conditional Max-Entropy Theorem [Thomas'81]

Step 2: Substitution Approach

Gaussians $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ optimal for $\mathcal{R}_{X_1,U,X_2}$

$$\bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1,\, \mathsf{E}[X_2^2] \le P_2}} \mathcal{R}_{X_1,U,X_2} \overset{?}{=} \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}} \\ \mathsf{E}[(X_1^{\mathcal{G}})^2] \le P_1,\, \mathsf{E}[(X_2^{\mathcal{G}})^2] \le P_2}} \mathcal{R}_{X_1^{\mathcal{G}},U^{\mathcal{G}},X_2^{\mathcal{G}}}$$

First try with Markov condition $\longrightarrow$ same as without Markov condition



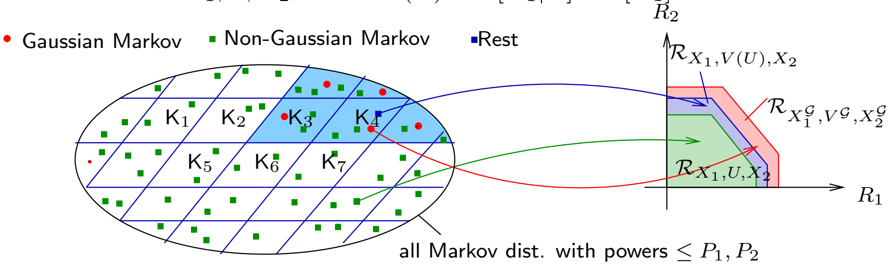
Problem: $\forall$K $\succeq 0$ there is a Markov triple but not necess. a Gaussian Markov!

Step 2: Substitution Approach

Gaussians $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ optimal for $\mathcal{R}_{X_1,U,X_2}$

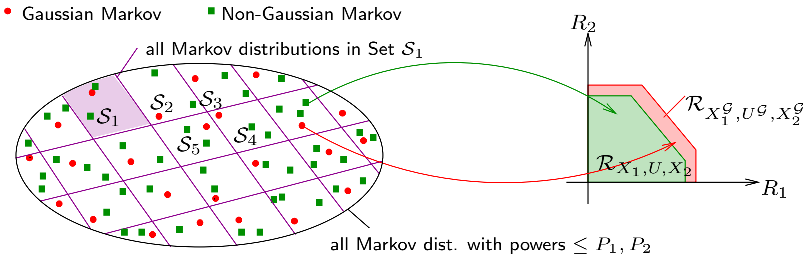
$$\bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1,\ \mathsf{E}[X_2^2] \le P_2}} \mathcal{R}_{X_1,U,X_2} \stackrel{?}{=} \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}} \\ \mathsf{E}[(X_1^{\mathcal{G}})^2] \le P_1,\ \mathsf{E}[(X_2^{\mathcal{G}})^2] \le P_2}} \mathcal{R}_{X_1^{\mathcal{G}},U^{\mathcal{G}},X_2^{\mathcal{G}}}$$

Trick: Consider X_1, V, X_2, where $V(U) = \mathsf{E}[X_1|U] - \mathsf{E}[X_1]$



1. $\mathcal{R}_{X_1,V,X_2}$ larger than $\mathcal{R}_{X_1,U,X_2}$ because $V = f(U)$
2. For X_1, V, X_2 there is a Gaussian Markov of same covariance $\square$

Step 2: Max-Correlation Approach



$\mathcal{S}_i$: $X_1 - U - X_2$ with same $\mathsf{E}\left[X_1^2\right], \mathsf{E}\left[X_2^2\right], \mathsf{E}[\mathsf{Var}(X_1|U)], \mathsf{E}[\mathsf{Var}(X_2|U)]$

1. In every $\mathcal{S}_i$ there is a Gaussian Markov!
2. Within every $\mathcal{S}_i$ Gaussian Markov triples have the largest region

Main tool in proof of 2. (with equality in *Gaussian* case): $\forall\ X_1 - U - X_2$:

$$\mathsf{Cov}[X_1, X_2] \leq \sqrt{\mathsf{E}[X_1^2] - \mathsf{E}[\mathsf{Var}(X_1|U)]}\sqrt{\mathsf{E}[X_2^2] - \mathsf{E}[\mathsf{Var}(X_2|U)]}.$$

Details to Max-Correlation Approach

- With Max-Entropy & Jensen:

$$I(X_1;Y|X_2U) \le \frac{1}{2}\log\left(1+\frac{\mathsf{E}[\mathsf{Var}(X_1|U)]}{N}\right)$$
$$I(X_2;Y|X_1U) \le \frac{1}{2}\log\left(1+\frac{\mathsf{E}[\mathsf{Var}(X_2|U)]}{N}\right)$$
$$I(X_1X_2;Y|U) \le \frac{1}{2}\log\left(1+\frac{\mathsf{E}[\mathsf{Var}(X_1|U)]+\mathsf{E}[\mathsf{Var}(X_2|U)]}{N}\right)$$

- With Max-Entropy & Max-correlation inequality:

$$\begin{aligned}
&I(X_1X_2;Y)\\
&\le \frac{1}{2}\log\left(1+\frac{\mathsf{E}\left[X_1^2\right]+\mathsf{E}\left[X_2^2\right]+2\mathsf{E}[X_1X_2]}{N}\right)\\
&\le \frac{1}{2}\log\left(1+\frac{\mathsf{E}\left[X_1^2\right]+\mathsf{E}\left[X_2^2\right]+2\sqrt{\mathsf{E}[X_1]^2-\mathsf{E}[\mathsf{Var}(X_1|U)]}\sqrt{\mathsf{E}\left[X_2^2\right]-\mathsf{E}[\mathsf{Var}(X_2|U)]}}{N}\right)
\end{aligned}$$

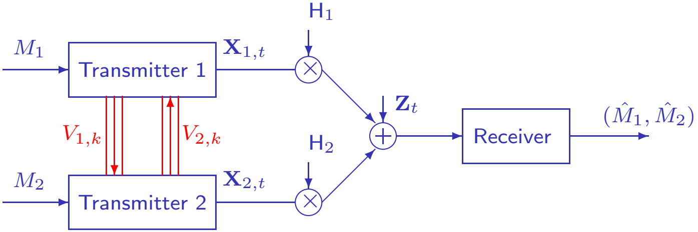
All inequalities hold with equality for Gaussian Markov triples.

Technique Applies for More Two-User Settings

- Capacity of two-user MAC with a common and two private messages (Slepian&Wolf'73)
- Capacity of interference channels with partial transmitter cooperation (Maric/Yates/Kramer'07)
- Capacity of compound MAC with conferencing encoders (Maric/Yates/Kramer'08)
- Achievable region for MAC noise-free (one-sided) feedback (Cover-Leung'81)
- Outer bound for MAC with user-cooperation/noisy cribbing (Tandon/Ulukus'08)

It suffices to consider Gaussian Markov triples!

Multi-Antenna AWGN MAC with Conferencing Encoders



- 2. phase: Transmission over channel
  - $\mathbf{Y}_t = \mathsf{H}_1 \mathbf{X}_{1,t} + \mathsf{H}_2 \mathbf{X}_{2,t} + \mathbf{Z}_t \qquad \{\mathbf{Z}_t\}$ IID $\sim \mathcal{N}(0, N\mathsf{I}_{d_r})$
  - $\mathbf{X}_{1,t} \in \mathbb{R}^{d_{t1}}$, $\mathbf{X}_{2,t} \in \mathbb{R}^{d_{t2}}$ and $\mathbf{Y}_t \in \mathbb{R}^{d_r}$
  - $\mathsf{H}_1, \mathsf{H}_2$ fixed and known to everyone
  - Power constraints: $\qquad \frac{1}{n} \sum_{t=1}^{n} \mathsf{E}\left[\|X_{\nu,t}\|^2\right] \leq P_\nu, \qquad \nu \in \{1, 2\}$

Capacity of Multi-Antenna AWGN MAC with Conferencing Encoders

Theorem

$$C_{\mathsf{MIMO,Conf}} = \bigcup_{\substack{\mathsf{A}_1,\mathsf{A}_2,\mathsf{B}_1,\mathsf{B}_2: \\ \mathrm{tr}(\mathsf{A}_1\mathsf{A}_1^\mathsf{T}+\mathsf{B}_1\mathsf{B}_1^\mathsf{T})\le P_1 \\ \mathrm{tr}(\mathsf{A}_2\mathsf{A}_2^\mathsf{T}+\mathsf{B}_2\mathsf{B}_2^\mathsf{T})\le P_2}} \left\{ \begin{array}{lcl} (R_1,R_2): & & \\ \quad R_1 & \le & \frac{1}{2}\log\left(\det\left(\mathsf{I}+\mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}\right)\right)+C_{12} \\ \quad R_2 & \le & \frac{1}{2}\log\left(\det\left(\mathsf{I}+\mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}\right)\right)+C_{21} \\ \quad R_1+R_2 & \le & \frac{1}{2}\log\left(\det\left(\mathsf{I}+\mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}+\mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}\right)\right) \\ & & +C_{12}+C_{21} \\ \quad R_1+R_2 & \le & \frac{1}{2}\log\Big(\det\Big(\mathsf{I}+\mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}+\mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} \\ & & \quad +\mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}+\mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} \\ & & \quad +\mathsf{H}_1\mathsf{B}_1\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}+\mathsf{H}_2\mathsf{B}_2\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}\Big)\Big) \end{array} \right\}$$

Superposition-Scheme Achieves Capacity (insp. by Willems'83)

As before:

- Conference as before: common $M_0 = (M_{1,p}, M_{2,p})$, private $M_{1,p}, M_{2,p}$
- Superposition $M_{1,p}$ or $M_{2,p}$ on top of M_0
- Jointly decode Messages $M_0, M_{1,p}, M_{2,p}$

MIMO: conferenced bits describe common beamforming direction

Multi-Antenna Converse: 3 Steps

- Step 1: Extend Willems's outer bound *with power constraints* to MIMO:

$$\mathsf{C}_{\mathsf{MIMO,Conf}} \subseteq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1,\ \mathsf{E}[X_2^2] \le P_2}} \mathcal{R}_{\mathbf{X}_1,\mathbf{U},\mathbf{X}_2}, \tag{2}$$

  where

$$\mathcal{R}_{\mathbf{X}_1,\mathbf{U},\mathbf{X}_2} \triangleq \left\{ (R_1, R_2) : \begin{array}{rcll} R_1 & \le & I(\mathbf{X}_1;\mathbf{Y}|\mathbf{X}_2\mathbf{U}) & +C_{12}, \\ R_2 & \le & I(\mathbf{X}_2;\mathbf{Y}|\mathbf{X}_1\mathbf{U}) & +C_{21}, \\ R_1 + R_2 & \le & I(\mathbf{X}_1\mathbf{X}_2;\mathbf{Y}|\mathbf{U}) & +C_{12} + C_{21}, \\ R_1 + R_2 & \le & I(\mathbf{X}_1\mathbf{X}_2;\mathbf{Y}) & \end{array} \right\}$$

- Step 2: In (2) suffices to take *Gaussian* Markov triples $\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}}$ where $\mathbf{U}^{\mathcal{G}} \in \mathbb{R}^{\min\{d_{t1}, d_{t2}\}}$

- Step 3: Evaluate $\mathcal{R}_{X_1,U,X_2}$ $\forall$ Gaussian Markov triples $\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}}$ with $\mathbf{U}^{\mathcal{G}} \in \mathbb{R}^{\min\{d_{t1}, d_{t2}\}}$

Main "Step 2": Max-Correlation Approach?

Gaussians $\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}}$ optimal also under Markov condition

$$\bigcup_{\substack{\mathbf{X}_1 - \mathbf{U} - \mathbf{X}_2 \\ \mathsf{E}\left[\|\mathbf{X}_\nu\|^2\right] \leq P_\nu}} \mathcal{R}_{\mathbf{X}_1, \mathbf{U}, \mathbf{X}_2} \stackrel{?}{=} \bigcup_{\substack{\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}} \\ \mathsf{E}\left[\|\mathbf{X}_\nu\|^2\right] \leq P_\nu \\ \dim(\mathbf{U}^{\mathcal{G}}) = \min\{d_{t1}, d_{t2}\}}} \mathcal{R}_{\mathbf{X}_1^{\mathcal{G}}, \mathbf{U}^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}}$$

- Max-correlation inequality

$$\mathsf{Cov}[X_1, X_2] \leq \sqrt{\mathsf{E}[X_1^2] - \mathsf{E}[\mathsf{Var}(X_1|U)]}\sqrt{\mathsf{E}[X_2^2] - \mathsf{E}[\mathsf{Var}(X_2|U)]}$$

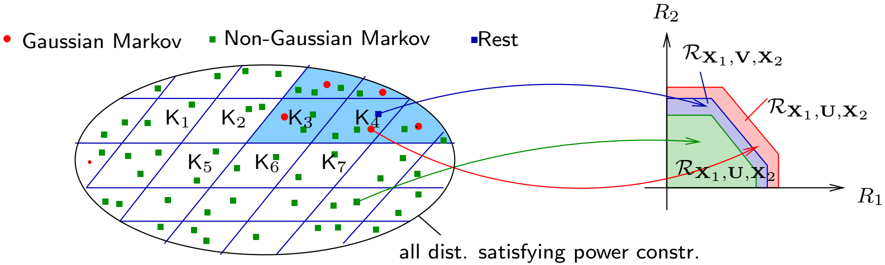
seems difficult to extend to vector case!

Main "Step 2": Substitution Approach

Gaussians $\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}}$ optimal also under Markov condition

$$\bigcup_{\substack{\mathbf{X}_1 - \mathbf{U} - \mathbf{X}_2 \\ \mathsf{E}\left[\|\mathbf{X}_\nu\|^2\right] \le P_\nu}} \mathcal{R}_{\mathbf{X}_1,\mathbf{U},\mathbf{X}_2} \stackrel{?}{=} \bigcup_{\substack{\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}} \\ \mathsf{E}\left[\|\mathbf{X}_\nu\|^2\right] \le P_\nu \\ \dim(\mathbf{U}^{\mathcal{G}}) = \min\{d_{t1}, d_{t2}\}}} \mathcal{R}_{\mathbf{X}_1^{\mathcal{G}},\mathbf{U}^{\mathcal{G}},\mathbf{X}_2^{\mathcal{G}}}$$

Choose $\mathbf{V}(\mathbf{U}) = \mathsf{E}[\mathbf{X}_1|\mathbf{U}] - \mathsf{E}[\mathbf{X}_1]$

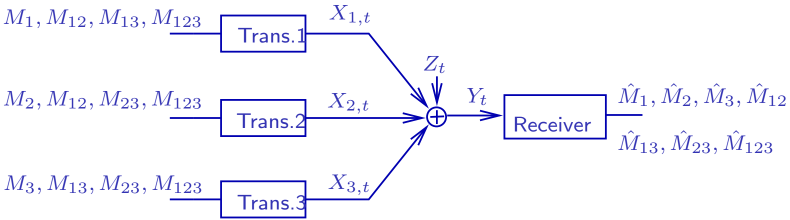


$\longrightarrow$ Bound on dimensionality of $\mathbf{V}$ for free!

More General Markov Structures?

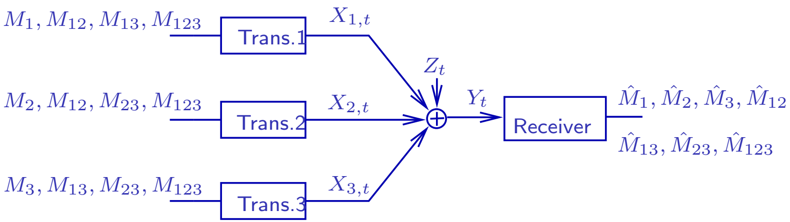
- Does technique also apply for more involved Markov chains?
- 3-user MAC with unicast-conferencing
- 3-user MAC with common and private messages

3-User AWGN MAC with Common and Private Messages



- Channel: $Y_t = X_{1,t} + X_{2,t} + X_{3,t} + Z_t$
- Common/private messages :
  - 7 independent messages $M_1, M_2, M_3, M_{12}, M_{13}, M_{23}, M_{123}$
  - M_ν known to Transmitter ν only
  - $M_{\nu,\nu'}$ known to Transmitter ν & Transmitter ν'
  - M_{123} known to all transmitters

3-User AWGN MAC with Common and Private Messages



- 7 independent messages $M_1, M_2, M_3, M_{12}, M_{13}, M_{23}, M_{123}$
- of rates $R_1, R_2, R_3, R_{12}, R_{13}, R_{23}, R_{123}$
- Capacity region: set of all rate seven-tuples s.t. $p(\text{error}) \to 0$
- Discrete memoryless setup: Slepian/Wolf'73 & Han

Capacity of 3 User AWGN MAC with Priv./Common Msgs

Theorem

$$\bigcup_{\substack{U_0,U_{12},U_{13},U_{23} \text{ independent} \\ X_1-(U_0,U_{12},U_{13})-(X_2,X_3,U_{23}) \\ X_2-(U_0,U_{12},U_{23})-(X_1,X_3,U_{13}) \\ X_3-(U_0,U_{13},U_{23})-(X_1,X_2,U_{12}) \\ \mathsf{E}\left[\|X_\nu\|^2\right]\leq P_\nu,\ \nu\in\{1,2,3\}}} \mathcal{R}_{\mathbf{U},X_1,X_2,X_3} \triangleq \Bigg\{(R_1,R_2,R_3,R_{12},R_{13},R_{23},R_{123}):$$

$$\begin{aligned}
R_1 &\leq I(X_1;Y|X_2,X_3,U_0,U_{12},U_{13}) \\
R_2 &\leq I(X_2;Y|X_1,X_3,U_0,U_{12},U_{23}) \\
R_3 &\leq I(X_3;Y|X_1,X_2,U_0,U_{13},U_{23}) \\
R_1+R_2 &\leq I(X_1,X_2;Y|X_3,U_0,U_{12},U_{13},U_{23}) \\
R_1+R_3 &\leq I(X_1,X_3;Y|X_2,U_0,U_{12},U_{13},U_{23}) \\
R_2+R_3 &\leq I(X_2,X_3;Y|X_1,U_0,U_{12},U_{13},U_{23}) \\
&\cdots \quad \cdots \quad \cdots \\
R_{13}+R_{23}+R_1+R_2+R_3 &\leq I(X_1,X_2,X_3;Y|U_0,U_{12}) \\
R_{12}+R_{13}+R_{23}+R_1+R_2+R_3 &\leq I(X_1,X_2,X_3;Y|U_0) \\
R_0+R_{12}+R_{13}+R_{23}+R_1+R_2+R_3 &\leq I(X_1,X_2,X_3;Y)
\end{aligned}\Bigg\}$$

Gaussian $X_1^{\mathcal{G}}, X_2^{\mathcal{G}}, X_3^{\mathcal{G}}, U_0^{\mathcal{G}}, U_{12}^{\mathcal{G}}, U_{13}^{\mathcal{G}}, U_{23}^{\mathcal{G}}$ satisfying Markov chains suffice!

Converse: Gaussians Optimal under Mult. Markov Chains

Main Step "2": Gaussians Optimal for $\mathcal{R}_{\mathbf{U},X_1,X_2,X_3}$

$$\bigcup_{\substack{U_0,U_{12},U_{13},U_{23} \text{ independent} \\ X_1-(U_0,U_{12},U_{13})-(X_2,X_3,U_{23}) \\ X_2-(U_0,U_{12},U_{23})-(X_1,X_3,U_{13}) \\ X_3-(U_0,U_{13},U_{23})-(X_1,X_2,U_{12}) \\ \mathsf{E}\left[\|X_\nu\|^2\right]\le P_\nu,\ \nu\in\{1,2,3\}}} \mathcal{R}_{\mathbf{U},X_1,X_2,X_3} \overset{?}{=} \bigcup_{\substack{U_0^{\mathcal{G}},U_{12}^{\mathcal{G}},U_{13}^{\mathcal{G}},U_{23}^{\mathcal{G}} \text{ independent} \\ X_1^{\mathcal{G}}-(U_0^{\mathcal{G}},U_{12}^{\mathcal{G}},U_{13}^{\mathcal{G}})-(X_2^{\mathcal{G}},X_3^{\mathcal{G}},U_{23}^{\mathcal{G}}) \\ X_2^{\mathcal{G}}-(U_0^{\mathcal{G}},U_{12}^{\mathcal{G}},U_{23}^{\mathcal{G}})-(X_1^{\mathcal{G}},X_3^{\mathcal{G}},U_{13}^{\mathcal{G}}) \\ X_3^{\mathcal{G}}-(U_0^{\mathcal{G}},U_{13}^{\mathcal{G}},U_{23}^{\mathcal{G}})-(X_1^{\mathcal{G}},X_2^{\mathcal{G}},U_{12}^{\mathcal{G}}) \\ \mathsf{E}\left[\|X_\nu^{\mathcal{G}}\|^2\right]\le P_\nu,\ \nu\in\{1,2,3\}}} \mathcal{R}_{\mathbf{U}^{\mathcal{G}},X_1^{\mathcal{G}},X_2^{\mathcal{G}},X_3^{\mathcal{G}}}$$

- Max-correlation approach: OK, but more involved correlation-ineq.
- Substitution approach: OK with following auxiliary r.v.

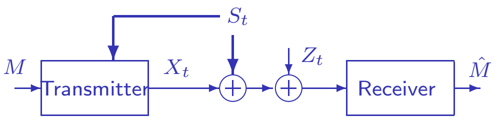
$$V_0 \triangleq \begin{pmatrix} \mathsf{E}[X_1|U_0]-\mathsf{E}[X_1] \\ \mathsf{E}[X_2|U_0]-\mathsf{E}[X_2] \\ \mathsf{E}[X_3|U_0]-\mathsf{E}[X_3] \end{pmatrix},$$

$$V_{\nu\nu'} \triangleq \begin{pmatrix} \mathsf{E}[X_\nu|U_{\nu\nu'},U_0]-\mathsf{E}[X_\nu|U_0] \\ \mathsf{E}[X_{\nu'}|U_{\nu\nu'},U_0]-\mathsf{E}[X_{\nu'}|U_0] \end{pmatrix}, \quad (\nu,\nu')\in\{(1,2),(1,3),(2,3)\}$$

Part II:

Dirty-Paper MAC with Conferencing Encoders

Single-User Writing on Dirty Paper (Costa'82)



- $\{S_t\} \sim$ IID $\mathcal{N}(0, Q)$ $\{Z_t\} \sim$ IID $\mathcal{N}(0, N)$
- Transmitter knows interference $S^n \triangleq (S_1, \ldots, S_n)$ non-causally
- Input power constraint: $\frac{1}{n}\mathsf{E}\left[\|x^n(M, S^n)\|^2\right] \leq P$

Theorem (Costa'82)

Capacity is $\frac{1}{2}\log\left(1 + \frac{P}{N}\right)$ $\rightarrow$ *Interference can be perfectly canceled!*

- Extension to ergodic noise (Cohen/Lapidoth'02) or arbitrary interference (Erez/Zamir'05)

Dirty-Paper Extension to Dependent Noise & Interfer.

- $\{S_t\} \sim$ IID $\mathcal{N}(0, Q)$
- Noise $\{Z_t\}$ can depend on interference if
  - it has empirical noise variance approximately N
  - it is approximately orthogonal to interference $\{S_t\}$

Noise is such that $\forall$ suff. small $\delta > 0$ $\exists \epsilon^*(\delta) \in (0, N)$ s.t.

1. $\frac{P+\alpha Q}{\sqrt{P+\alpha^2 Q}\sqrt{P+N+Q+(3-2\alpha)\epsilon^*(\delta)}} \geq \sqrt{1 - \frac{N}{P+N} 2^4 \delta} \frac{P}{P+\alpha^2 Q}$
2. $\lim_{n\to\infty} \Pr\left[\left|\frac{1}{n}\|Z^n\|^2 - N\right| \leq \epsilon^*(\delta)\right] = 1$
3. $\forall \epsilon > 0$: $\lim_{n\to\infty} \Pr\left[-\epsilon \leq \frac{1}{n}\langle Z^n, S^n\rangle \leq \epsilon^*(\delta)\right] = 1.$

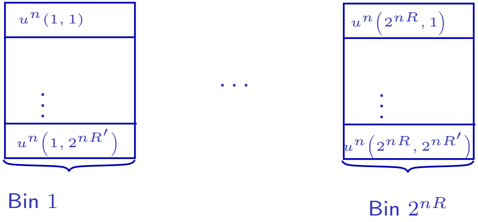
Theorem

Rate $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$ *achievable*
$\rightarrow$ *same performance as over Gaussian channel without interference*

Proof: Refined analysis of scheme by Cohen/Lapidoth'02

Dirty-Paper Scheme (Cohen&Lapidoth'02)

For simplicity: S^n uniform over n-sphere of radius nQ

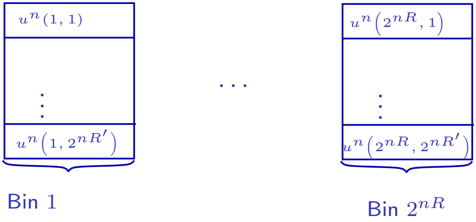


- 2^{nR} bins each with $2^{nR'}$ codewords
- $\{u^n(m,k)\}$ IID uniform over n-sphere of radius $r = \sqrt{n(P + \alpha^2 Q)}$
- $\alpha = \frac{P}{P+N}$

- Encoding: In Bin $M = m$ choose $u^n(m, k^*(s^n))$ closest to s^n & send $x^n = u^n(m, k^*(s^n)) - \alpha s^n$
- Decoding: Find $u^n(\hat{m}, \hat{k})$ closest to y^n and declare $\hat{m}$

Dirty-Paper Scheme (Cohen&Lapidoth'02)

For simplicity: S^n uniform over n-sphere of radius nQ



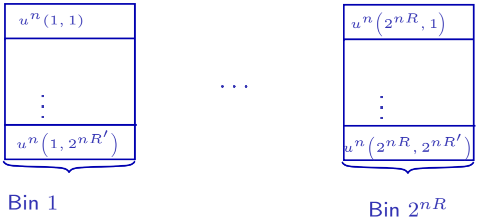
- 2^{nR} bins each with $2^{nR'}$ codewords
- $\{u^n(m,k)\}$ IID uniform over n-sphere of radius $r = \sqrt{n(P+\alpha^2 Q)}$
- $\alpha = \frac{P}{P+N}$

- Encoding: In Bin $M = m$ choose $u^n(m, k^*(s^n))$ closest to s^n & send $x^n = u^n(m, k^*(s^n)) - \alpha s^n$
- Decoding: Find $u^n(\hat{m}, \hat{k})$ closest to y^n and declare $\hat{m}$

For proper choice of R', r, α scheme achieves $R = \frac{1}{2}\log\left(1 + \frac{P}{N}\right)$
$\rightarrow$ can cancel interference perfectly

Dirty-Paper Scheme (Cohen&Lapidoth'02)

For simplicity: S^n uniform over n-sphere of radius nQ

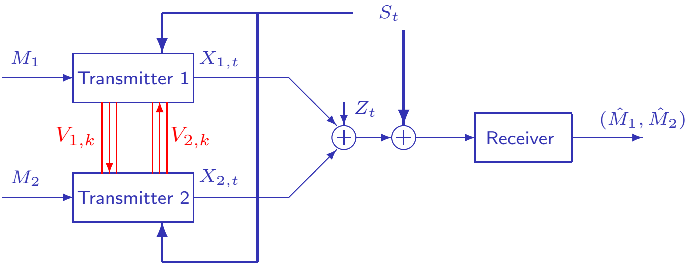


- 2^{nR} bins each with $2^{nR'}$ codewords
- $\{u^n(m,k)\}$ IID uniform over n-sphere of radius $r = \sqrt{n(P + \alpha^2 Q)}$
- $\alpha = \frac{P}{P+N}$

- Encoding: In Bin $M = m$ choose $u^n(m, k^*(s^n))$ closest to s^n & send $x^n = u^n(m, k^*(s^n)) - \alpha s^n$
- Decoding: Find $u^n(\hat{m}, \hat{k})$ closest to y^n and declare $\hat{m}$

- X^n and $u^n(M, K^*(S^n))$ depend on S^n!
- Receiver can only recover $u^n(M, K^*(S^n))$ but not X^n !

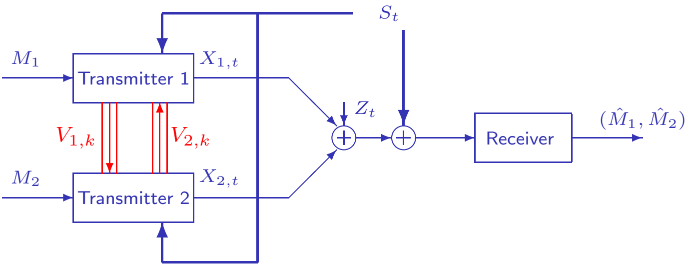
Dirty-Paper MAC with Conferencing Encoders



- $\{S_t\} \sim$ IID $\mathcal{N}(0, Q)$
- Transmitters know interference $S^n \triangleq (S_1, \ldots, S_n)$ non-causally
- Inputs $X_\nu^n \triangleq (X_{\nu,1}, \ldots, X_{\nu,n})$ at Transmitter ν:

$$X_\nu^n = f_{\nu,t}\left(M_\nu, V_{\bar{\nu}}^\kappa, S^n\right), \qquad \nu, \bar{\nu} \in \{1,2\},\ \nu \neq \bar{\nu}$$

Dirty-Paper MAC with Conferencing Encoders



2 Settings:

- Transmitters learn S^n before the conference
  - $V_{1,k} = \varphi_{1,k}(M_1, V_2^{k-1}, S^n)$ and $V_{2,k} = \varphi_{1,k}(M_2, V_1^{k-1}, S^n)$
- Transmitters learn S^n after the conference

Related Results

- Dirty-paper MAC (Gel'fand/Pinsker'84 & Kim et al. '04)
- MAC with degraded message set and intereference known to only one transmitter (Kotagiri et al.'06 & Somekh-Baruch et al.'06)
- Doubly-dirty MAC (Philosof et al.'07)

Interference Non-Causally Known at Txs can be Canceled

Theorem

In two-user Gaussian MAC with conferencing encoders transmitters can cancel non-causally known interference:

$$C_{\text{Int,before}} = C_{\text{Int,after}} = C_{\text{Conf}}$$

Special cases:

- MAC (Gel'fand/Pinsker, and Kim et al.) when $C_{1,2} = C_{2,1} = 0$
- MAC with degraded message sets when $C_{1,2} = \infty$ and $C_{2,1} = 0$

Remark: Achievability holds for all ergodic noises!

Converse

- Reveal interference also to receiver
- Receiver can subtract interference off
- After subtraction: interference is independent of channel $\rightarrow$ does not influence capacity

Achievability: Independent Encoding/Successive Decoding

- Conference as before: common $M_0 = (M_{1,c}, M_{2,c})$, private $M_{1,p}, M_{2,p}$
- Independent dirty-paper coding of $M_0, M_{1,p}, M_{2,p}$
- Transmitters add up coded sequences
- Successive decoding and stripping off

- Independent encoding & successive decoding achieve capacity when there is no interference
- With dirty-paper coding we achieve same performance as without interference

Encoding: Independent Dirty-Paper Coding

- With independent dirty-paper codes (of parameters determined later)
  - Encode M_0: $\tilde{X}_0^n = (u_\nu^n(M_0, K_0^*(S^n)) - \alpha_0 S^n)$
  - Encode $M_{\nu,p}$: $\tilde{X}_\nu^n = (u_\nu^n(M_{\nu,p}, K_\nu^*(S)) - \alpha_\nu S^n)$

- Transmitters scale and add up codewords.

$$X_\nu^n = \tilde{X}_\nu^n + \lambda_\nu \tilde{X}_0^n, \qquad \lambda_1 = (1 - \lambda_2) \in [0, 1]$$

- Channel coherently adds up transmissions of M_0:

$$\mathbf{Y} = \tilde{X}_0^n + \tilde{X}_1^n + \tilde{X}_2^n + S^n + Z^n$$

Successive Decoding & Stripping Off

$$Y^n = \underbrace{u_0^n(M_0, K_0^*(S^n)) - \alpha_0 S^n}_{\tilde{X}_0^n} + \tilde{X}_1^n + \tilde{X}_2^n + S^n + Z^n$$

1. Decode M_0 treating $\tilde{X}_1^n, \tilde{X}_2^n$ as noise & strip off $u_0^n(M_0, K_0(S^n)) \to Y_0^n$

- Steps 1. & 2.: Noise and interference dependent but $\approx$ orthogonal
- Our costa-extension: With proper parameters dirty-paper coding achieves same performance as without interference

Successive Decoding & Stripping Off

$$Y_0^n = \qquad - \alpha_0 S^n + \tilde{X}_1^n + \tilde{X}_2^n + S^n + Z^n$$

1. Decode M_0 treating $\tilde{X}_1^n, \tilde{X}_2^n$ as noise & strip off $u_0^n(M_0, K_0(S^n)) \to Y_0^n$

- Steps 1. & 2.: Noise and interference dependent but $\approx$ orthogonal
- Our costa-extension: With proper parameters dirty-paper coding achieves same performance as without interference

Successive Decoding & Stripping Off

$$Y_0^n = \tilde{X}_1^n + \tilde{X}_2^n + (1-\alpha_0)S^n + Z^n$$

1. Decode M_0 treating $\tilde{X}_1^n, \tilde{X}_2^n$ as noise & strip off $u_0^n(M_0, K_0(S^n)) \to Y_0^n$
2. Decode $M_{1,p}$ treating $\tilde{X}_2^n$ as noise & strip off $u_1^n(M_{1,p}, K_1(S^n)) \to Y_1^n$

- Steps 1. & 2.: Noise and interference dependent but $\approx$ orthogonal
- Our costa-extension: With proper parameters dirty-paper coding achieves same performance as without interference

Successive Decoding & Stripping Off

$$Y_0^n = \underbrace{u_1^n(M_{1,p}, K_1^*(S)) - \alpha_1 S^n}_{\tilde{X}_1^n} + \tilde{X}_2^n + (1-\alpha_0)S^n + Z^n$$

1. Decode M_0 treating $\tilde{X}_1^n, \tilde{X}_2^n$ as noise & strip off $u_0^n(M_0, K_0(S^n)) \rightarrow Y_0^n$
2. Decode $M_{1,p}$ treating $\tilde{X}_2^n$ as noise & strip off $u_1^n(M_{1,p}, K_1(S^n)) \rightarrow Y_1^n$

- Steps 1. & 2.: Noise and interference dependent but $\approx$ orthogonal
- Our costa-extension: With proper parameters dirty-paper coding achieves same performance as without interference

Successive Decoding & Stripping Off

$$Y_1^n = \underbrace{u_2^n(M_{2,p}, K_2^*(S)) - \alpha_2 S^n}_{\tilde{X}_2^n} + (1 - \alpha_0 - \alpha_1) S^n + Z^n$$

1. Decode M_0 treating $\tilde{X}_1^n, \tilde{X}_2^n$ as noise & strip off $u_0^n(M_0, K_0(S^n)) \to Y_0^n$
2. Decode $M_{1,p}$ treating $\tilde{X}_2^n$ as noise & strip off $u_1^n(M_{1,p}, K_1(S^n)) \to Y_1^n$
3. Decode $M_{2,p}$

- Steps 1. & 2.: Noise and interference dependent but $\approx$ orthogonal
- Our costa-extension: With proper parameters dirty-paper coding achieves same performance as without interference

Summary

- Capacity region of two-user Gaussian MAC with conferencing encoders
- New technique for proving Gaussians maximize mutual informations under Markov conditions
- Technique useful also to optimize Slepian-Wolf region (with many users) and multi-antenna regions
- Costa-type result for two-user Gaussian MAC with conferencing encoders
- Extension of Costa's result to Gaussian interference and dependent (not necessarily Gaussian) noise