Testing Against Independence with Multiple Decision Centers

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joint work with Roy Timo

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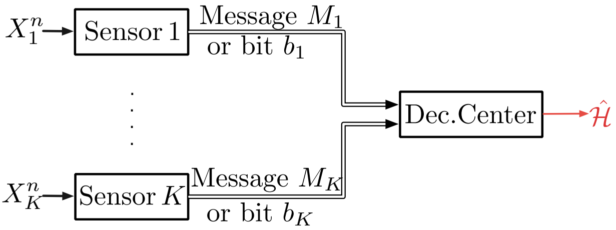
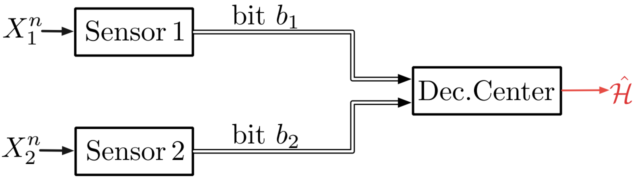
Previous setups: hypothesis testing with 1 decision center

- $(X_1^n, X_2^n, \ldots, X_K^n)$ i.i.d. $\sim P_{X_1 X_2 \cdots X_K}$ if $\mathcal{H} = 0$ and
  $\sim Q_{X_1 X_2 \cdots X_K}$ if $\mathcal{H} = 1$



- Type-I error probability $\Pr[\hat{H} = 1 | H = 0] \leq \epsilon$
- Type-II error probabiilty $\Pr[\hat{H} = 0 | H = 1] \leq e^{-\theta n}$ largest θ?

Previous results: single-bit or zero-rate communication



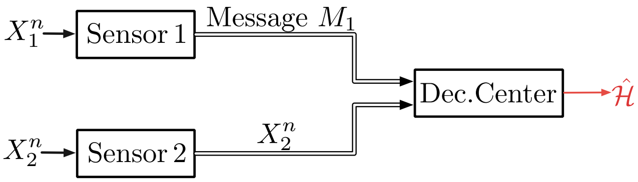
- Han['87]: Optimal error exponent:

$$\theta^* = \min_{\substack{P_{\tilde{X}_1\tilde{X}_2}: \\ P_{\tilde{X}_1}=P_{X_1} \\ P_{\tilde{X}_2}=P_{X_2}}} D\big(P_{\tilde{X}_1\tilde{X}_2}||Q_{X_1X_2}\big)$$

- Shalaby&Papamarcou['92]: Same with 0-rate M_1/M_2 (instead of b_1/b_2)

$$\lim_{n\to\infty} \frac{\log|\mathcal{M}_k|}{n} = 0, \qquad k = 1, 2.$$

Previous results: positive-rate communication



Csiszar&Ahlswede['86], Shimokawa/Han/Amari['94], Rahman&Wagner['12]:

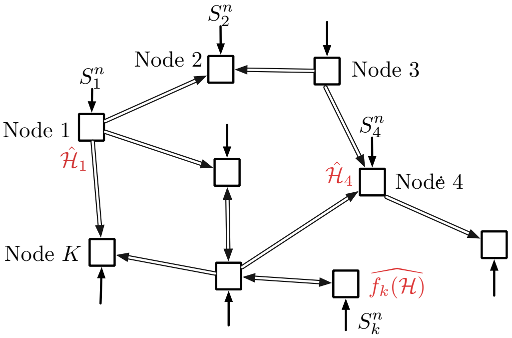
- Test for independence $Q_{X_1X_2} = P_{X_1}P_{X_2}$:

$$\theta^* = \max_{U \to X_1 \to X_2} I(U;X_2), \qquad \text{s.t. } R \geq I(U;X_1)$$

- Extension to conditional independence test
- General achievability; converse does not match

- Interaction and secrecy: Xiang&Kim['12/'13], Zhao&Lai['14/'15], Katz/Piantanida/Couillet/Debbah['15/'16], Liao/Sankar/Tan['16]

Untouched problems in information theory



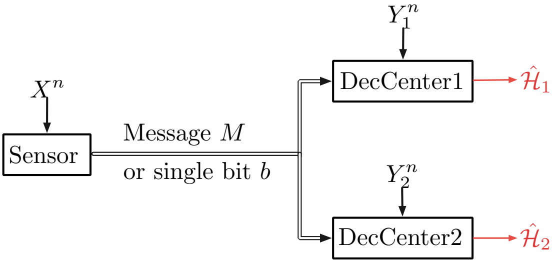
S_k^n: Node k's measured signal

Certain nodes guess $\mathcal{H}$ or a function thereof

- Multiple decision centers
- Multi-hop communication

1 Sensor and 2 decision centers: the problem setup

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H} = 0$ and $\sim Q_{XY_1Y_2}$ if $\mathcal{H} = 1$

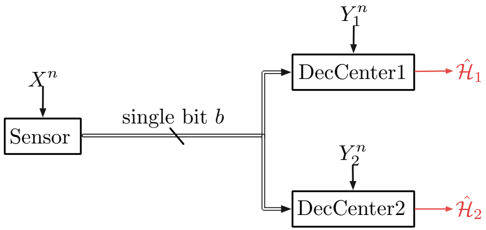


- Type-I error probabilities $\alpha_{k,n} := \Pr[\hat{H}_k = 1 | H = 0] \to 0$ as $n \to \infty$
- Type-II error probabiilties $\beta_{k,n} := \Pr[\hat{H}_k = 0 | H = 1] \leq e^{-n\theta_k}$

Optimal exponents region (θ_1, θ_2) ?

Single-bit communication: the coding scheme

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H}=0$ and $\sim Q_{XY_1Y_2}$ if $\mathcal{H}=1$

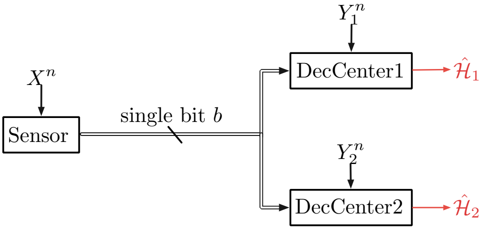


- Sensor: Sends $b=1$ if X^n in typical set $\mathcal{T}_\mu(P_X)$, and $b=0$ otherwise.

- Dec.Center k: If $(b=1)$ & $(Y_k^n \in \mathcal{T}_\mu(P_Y))$, set $\hat{H}_k=0$;
  otherwise, set $\hat{H}_k=1$

Single-bit communication: optimal error-exponents

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H}=0$ and $\sim Q_{XY_1Y_2}$ if $\mathcal{H}=1$



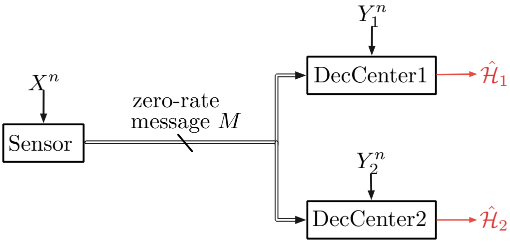
Optimal error-exponents:

$$\theta_1^\star = \min_{\substack{\tilde{X},\tilde{Y}_1:\\ P_{\tilde{X}}=P_X\\ P_{\tilde{Y}_1}=P_{Y_1}}} D\big(Q_{\tilde{X}\tilde{Y}_1}||P_{XY_1}\big); \qquad \theta_2^\star = \min_{\substack{\tilde{X},\tilde{Y}_2:\\ P_{\tilde{X}}=P_X\\ P_{\tilde{Y}_2}=P_{Y_2}}} D\big(Q_{\tilde{X}\tilde{Y}_2}||P_{XY_2}\big)$$

- No tradeoff because of multiple decision centers

Zero-rate communication

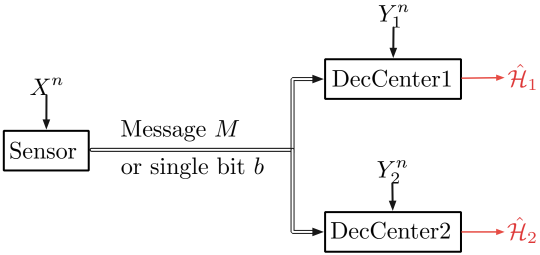
$$\lim_{n\to\infty} \frac{\log|\mathcal{M}|}{n} = 0$$



- Same exponents as with one communication bit
- No tradeoff between multiple decision centers

Positive communication rate: independence test

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H} = 0$ and $\sim P_X \cdot P_{Y_1Y_2}$ if $\mathcal{H} = 1$

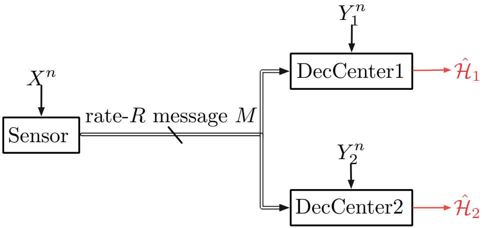


$$M \in \mathcal{M} = \{1, \ldots, 2^{nR}\}$$

Rate-exponents region (R, θ_1, θ_2) ?

Test for Independence: coding scheme

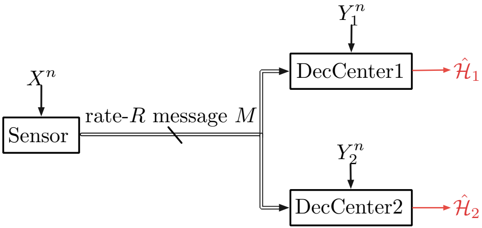
- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H}=0$ and $\sim P_X \cdot P_{Y_1Y_2}$ if $\mathcal{H}=1$



- Sensor: Quantize X^n into $U^n(m) \rightarrow$ send $M=m$
- Dec. Center k: If $(U^n(m), Y_k^n) \in \mathcal{T}_\mu(P_{UY})$, set $\hat{\mathcal{H}}_k = 0$;
  otherwise, set $\hat{\mathcal{H}}_k = 1$

Test for Independence: coding scheme

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H} = 0$ and $\sim P_X \cdot P_{Y_1Y_2}$ if $\mathcal{H} = 1$



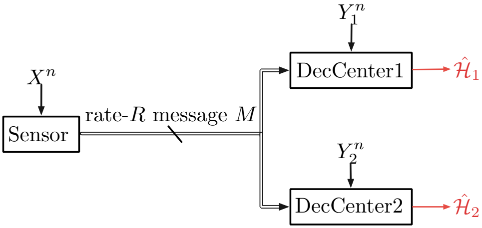
- Sensor: Quantize X^n into $U^n(m) \rightarrow$ send $M = m$

- Dec. Center k: If $(U^n(m), Y_k^n) \in \mathcal{T}_\mu(P_{UY})$, set $\hat{\mathcal{H}}_k = 0$;
  otherwise, set $\hat{\mathcal{H}}_k = 1$

$\rightarrow$ Communication completely ignores receiver side-information!

Test for independence: optimal error exponents

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H} = 0$ and $\sim P_X \cdot P_{Y_1Y_2}$ if $\mathcal{H} = 1$



Rate-exponents region:

$\theta_1 \leq I(U; Y_1) \qquad \theta_2 \leq I(U; Y_2) \qquad R > I(U; X)$

for some $U \to X \to (Y_1, Y_2)$

- No tradeoff between decisions except for common quantisation U

Test for independence: the converse

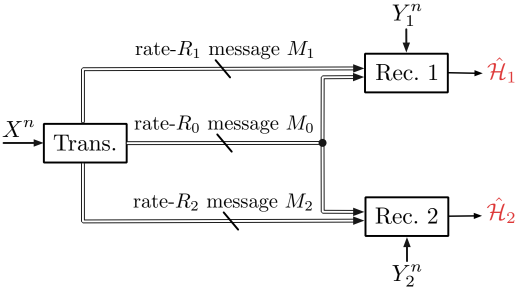
$$
\begin{aligned}
&-\frac{1}{n}\log \beta_{1,n} \\
&\leq (1-\epsilon)^{-1}\frac{1}{n} D\big(P_{MY_1^n|\mathcal{H}=0}\big\|P_{MY_1^n|\mathcal{H}=1}\big) \\
&= (1-\epsilon)^{-1}\frac{1}{n} I(M;Y_1^n) \\
&= (1-\epsilon)^{-1}\frac{1}{n}\sum_{t=1}^{n} I(M,Y_1^{t-1};Y_{1,t}) \\
&\leq (1-\epsilon)^{-1}\sum_{t=1}^{n} I(M,Y_1^{t-1},Y_2^{t-1};Y_{1,t}) \\
&= (1-\epsilon)^{-1} I(U_n;Y_{1,n}).
\end{aligned}
$$

Test for independence: the converse

$$\begin{aligned}
&-\frac{1}{n}\log\beta_{1,n}\\
&\le (1-\epsilon)^{-1}\frac{1}{n}D\big(P_{MY_1^n|\mathcal{H}=0}\big\|P_{MY_1^n|\mathcal{H}=1}\big)\\
&= (1-\epsilon)^{-1}\frac{1}{n}I(M;Y_1^n)\\
&= (1-\epsilon)^{-1}\frac{1}{n}\sum_{t=1}^{n}I(M,Y_1^{t-1};Y_{1,t})\\
&\le (1-\epsilon)^{-1}\sum_{t=1}^{n}I(M,Y_1^{t-1},Y_2^{t-1};Y_{1,t})\\
&= (1-\epsilon)^{-1}I(U_n;Y_{1,n}).
\end{aligned}$$

$$\begin{aligned}
R &\ge \frac{1}{n}H(M)\\
&\ge \frac{1}{n}I(M;X^n,Y_1^n,Y_2^n)\\
&\ge \frac{1}{n}\sum_{t=1}^{n}I(M,Y_1^{t-1},Y_2^{t-1};X_t)\\
&= I(U_n;X_n),
\end{aligned}$$

Test for independence over Gray-Wyner network



Optimal rate-exponents region:

$$\theta_1 \leq I(U_0, U_1; Y_1) \qquad \theta_2 \leq I(U_0, U_2; Y_2)$$

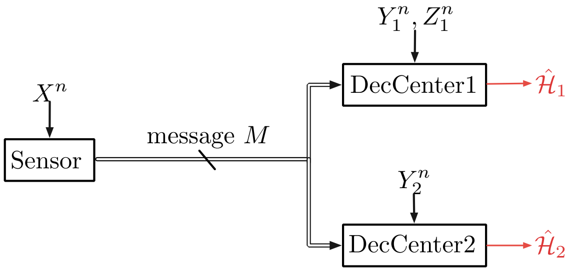
$$R_0 \geq I(U_0; X) \qquad R_1 \geq I(U_1; X|U_0) \qquad R_2 \geq I(U_2; X|U_0)$$

for some $(U_0, U_1, U_2) \to X \to (Y_1, Y_2)$

- No tradeoff between decisions except for common quantisation U_0
- Receiver side-information completely ignored in communication!

An example where things change: conditional indep.

- $(X^n, Y_1^n, Y_2^n, Z_1^n)$ i.i.d. $\sim P_{XY_1Y_2Z_1}$ if $\mathcal{H} = 0$;
  $\sim P_{XZ_1} \cdot P_{Y_1|Z_1} P_{Y_2}$ if $\mathcal{H} = 1$

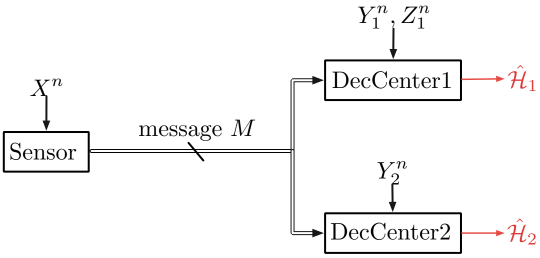


- Sensor: Use Heegard-Berger (binning) with side-informations Z_1^n to lossily communicate X^n to both centers

- Dec. Center 1: Uses Z_1^n to recover quantization; then tests against Y_1^n

Conditional independence: Optimal Performance

- $(X^n, Y_1^n, Y_2^n, Z_1^n)$ i.i.d. $\sim P_{XY_1Y_2Z_1}$ if $\mathcal{H} = 0$;
  $\sim P_{XZ_1} \cdot P_{Y_1|Z_1} P_{Y_2}$ if $\mathcal{H} = 1$



Optimal rate-exponents region: tradeoff between centers!

$$\theta_1 \leq I(U_0, U_1; Y_1 | Z_1) \qquad \theta_2 \leq I(U_0; Y_2)$$

$$R_0 \geq I(U_0; X) + I(U_1; X | U_0, Z_1) \qquad \text{for some } (U_0, U_1) \to X \to (Y_1, Y_2)$$

General hypothesis tests with positive rates

- (X^n, Y_1^n, Y_2^n) i.i.d. $\sim P_{XY_1Y_2}$ if $\mathcal{H} = 0$;
  $\sim Q_{XY_1Y_2}$ if $\mathcal{H} = 1$
- Send different quantizations (U_0^n, U_1^n) and (U_0^n, U_2^n) to decision centers 1 and 2
- Adapt bin sizes to quality of $Q_{XY_1Y_2}$
  $\rightarrow$ The better Y_1 and Y_2 under Q, the less needs to be transmitted!

Communication wants to exploit side-information under $Q_{XY_1Y_2}$
Important tradeoff due to multiple decision centers

Summary

- Information-theoretic hypothesis testing with two decision centers
- Single bit or zero-rate communication:
  · no tradeoff due to multiple decision centers
- Testing for independence with positive rate:
  · some tradeoff due to multiple centers (choice of quantization);
  · optimal communication ignores receiver side-information
- Other tests at positive rates:
  · tradeoff due to multiple centers becomes important;
  · optimal communication accounts for receiver side-information