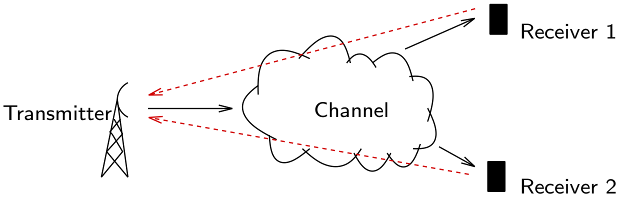
Coding Schemes for Memoryless Broadcast Channels with Rate-Limited Feedback



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joint work with Selma Belhadj Amor, Yossef Steinberg, and Youlong Wu

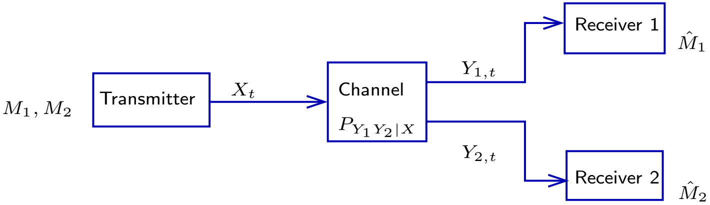
Supelec, 18 February 2014

Part I:

Discrete Memoryless Gaussian BC with Rate-Limited Feedback

joint work with Youlong Wu

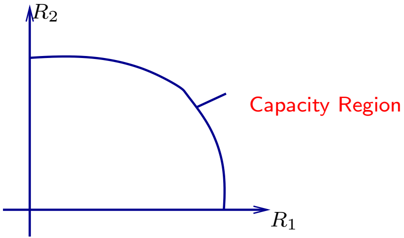
Discrete Memoryless BC without Feedback



- Rx i wants to learn $M_i \in \{1, \ldots, 2^{nR_i}\}$
- Inputs $X_t = f_t(M_1, M_2)$
- Finite input and output alphabets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$
- Memoryless channel $P_{Y_1 Y_2|X}$

Capacity Region

- Rates of communication $R_1, R_2 \geq 0$
- Capacity region: Pairs (R_1, R_2) s.t. $p(\text{error})$ arbitrarily small



Marton's Achievable Region

(R_1, R_2) achievable if:

$$R_1 \le I(U_0, U_1; Y_1)$$

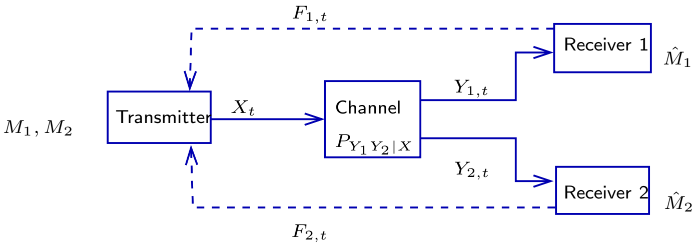
$$R_2 \le I(U_0, U_2; Y_2)$$

$$R_1 + R_2 \le I(U_1; Y_1 | U_0) + I(U_2; Y_2 | U_0) + \min_i I(U_0; Y_i) - I(U_1; U_2 | U_0)$$

for some (U_0, U_1, U_2) s.t. $(U_0, U_1, U_2) - \circ - X - \circ - (Y_1, Y_2)$

- In general capacity region unknown
- For some channels Marton's region equals capacity

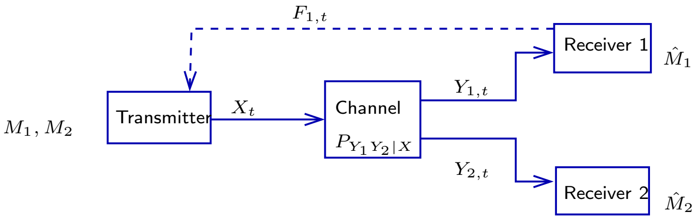
Rate-Limited Feedback Pipes



- $F_{i,t} = \phi_{i,t}(Y_i^t)$ where $|\mathcal{F}_{i,1}| \cdots |\mathcal{F}_{i,n}| \leq nR_{\mathsf{FB},i}, \quad i = 1, 2$

- Two-sided feedback: $X_t = f_t(M_1, M_2, F_1^{t-1}, F_2^{t-1})$
- One-sided feedback, $R_{\mathsf{FB},2} = 0$: $X_t = f_t(M_1, M_2, F_1^{t-1})$
- Perfect feedback, $R_{\mathsf{FB},1} = R_{\mathsf{FB},2} = \infty$: $X_t = f_t(M_1, M_2, Y_1^{t-1}, Y_2^{t-1})$

Rate-Limited Feedback Pipes



- $F_{i,t} = \phi_{i,t}(Y_i^t)$ where $|\mathcal{F}_{i,1}| \cdots |\mathcal{F}_{i,n}| \leq nR_{\mathsf{FB},i}, \quad i = 1, 2$
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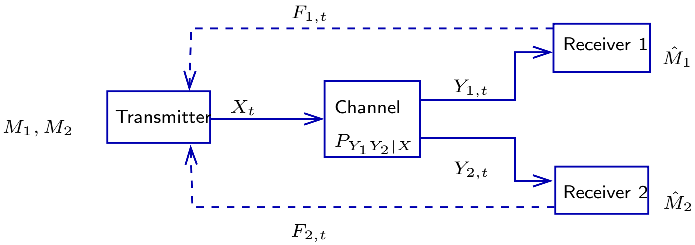

- $F_{i,t} = \phi_{i,t}(Y_i^t)$ where $|\mathcal{F}_{i,1}| \cdots |\mathcal{F}_{i,n}| \leq nR_{\mathsf{FB},i}, \quad i = 1, 2$

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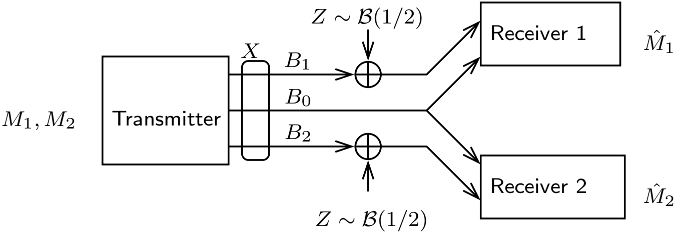
Previous Results for Perfect Feedback

- Capacity region not known in general
- El Gamal'78: No feedback-gain for physically degraded BCs
- Dueck'80, Ozarow'85, Kramer'00, Wang'09, Tassiulas&Georgiadis'10, Shayevitz&W'10, Maddah-Ali&Tse'10: Feedback-gain for some BCs
- Kramer'00, Shayevitz-W'13, Venkataramanan-Pradhan'13: achievable region for general memoryless BCs (difficult to evaluate)
- Outer bound: a genie reveals Y_1 to Receiver 2 or vice versa

For most BCs: don't know whether feedback helps

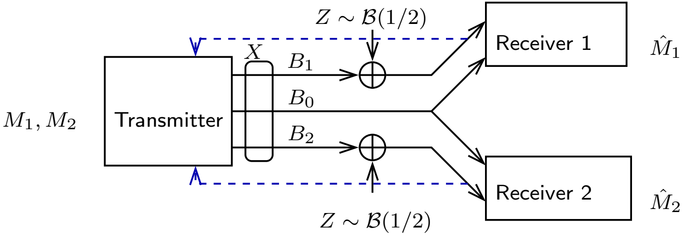
First Intuition how Feedback Helps: Dueck's Example



Without feedback:

- Top and bottom links useless
- Capacity: $0 \leq R_1 + R_2 \leq 1$

First Intuition how Feedback Helps: Dueck's Example



With feedback, $R_{\mathsf{FB},1} = 1$ or $R_{\mathsf{FB},2} = 1$:

- Feedback: $F_{i,t} = Y_{i,t}$
- Transmitter sends $B_{0,t} = Z_{t-1}$
- Capacity: $0 \leq R_1 \leq 1$ and $0 \leq R_2 \leq 1$

$\Rightarrow (X, Y_1, Y_2)$ determine common update info Z useful to both Rxs

Generalized Scheme for $R_{\mathsf{FB},1} = R_{\mathsf{FB},2} = \infty$ (Shayevitz&W'10)

Block 1	Block 2	Block 3	$\dots$	Block B	Block $B+1$
fresh data	fresh data	fresh data		fresh data	
	update info.	update info.		update info.	update info.

- Block-Markov strategy
- Send fresh data $M_{1,b}, M_{2,b}$ & update info using Marton's scheme
- Update infos $J_{0,b-1}, J_{i,b-1}$ for Receiver i:
  - Indices to compress $(U_{0,b-1}, U_{1,b-1}, U_{2,b-1}, Y_{1,b-1}, Y_{2,b-1})$ for SI $Y_{i,b-1}$
- Backward decoding:
  1. Use $J_{0,b}, J_{i,b}, Y_{i,b}$ to reconstruct compression $V_{i,b}$
  2. Decode $M_{i,b}, J_{0,b-1}, J_{i,b-1}$ based on improved outputs $(Y_{i,b}, V_{i,b})$

Generalized Scheme for $R_{\mathsf{FB},1} = R_{\mathsf{FB},2} = \infty$ (Shayevitz&W'10)

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- Block-Markov strategy
- Send fresh data $M_{1,b}, M_{2,b}$ & update info using Marton's scheme
- Update infos $J_{0,b-1}, J_{i,b-1}$ for Receiver i:
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- Backward decoding: separate source-channel coding
  1. Use $J_{0,b}, J_{i,b}, Y_{i,b}$ to reconstruct compression $V_{i,b}$
  2. Decode $M_{i,b}, J_{0,b-1}, J_{i,b-1}$ based on improved outputs $(Y_{i,b}, V_{i,b})$

Shayevitz-W.'10 Region for Perfect Feedback

Achievable Region

(R_1, R_2) achievable, if for some $P_{U_0U_1U_2}, P_{X|U_0U_1U_2}, P_{V_0V_1V_2|U_0U_1U_2}$:

$$\begin{aligned}
R_1 &\leq I(U_0, U_1; Y_1, V_1) - I(U_0, U_1, U_2, Y_2; V_0, V_1 | Y_1) \\
R_2 &\leq I(U_0, U_2; Y_2, V_2) - I(U_0, U_1, U_2, Y_1; V_0, V_2 | Y_2) \\
R_1 + R_2 &\leq I(U_1; Y_1, V_1 | U_0) + I(U_2; Y_2, V_2 | U_0) + \min_{i \in \{1,2\}} I(U_0; Y_i, V_i) \\
&\quad - I(U_1; U_2 | U_0) - \max_{i \in \{1,2\}} I(U_0, U_1, U_2, Y_1, Y_2; V_0 | Y_i) \\
R_1 + R_2 &\leq I(U_1, U_0; Y_1, V_1) + I(U_2, U_0; Y_2, V_2) - I(U_1; U_2 | U_0) \\
&\quad - I(U_0, U_1, U_2, Y_2; V_0, V_1 | Y_1) - I(U_0, U_1, U_2, Y_1; V_0, V_2 | Y_2)
\end{aligned}$$

More Comments on the Shayevitz-W'10 Region

- Achieves capacity of generalized Dueck example
- Improves over nofeedback capacity for a noisy Blackwell DMBC
- Recovers BEC-BC capacity results under public erasure events

- Update info should have common part $J_{0,b}$ useful to both rxs
- Tradeoff: update-info sent at expense of fresh data!

  $\rightarrow$ Identifying good update info/compression is hard in general

- Scheme applies to generalized feedback: replace (Y_1, Y_2) by $\tilde{Y}$

Later Related Works for Channels with Stale State $\tilde{Y} = S$

- Maddah-Ali&Tse'10:

$$Q = \begin{cases} 0 & \text{w.p. } 1/3 \\ 1 & \text{w.p. } 1/3 \\ 2 & \text{w.p. } 1/3 \end{cases}, \quad V_0 = V_i = \begin{cases} \emptyset & \text{if } Q = 0 \\ Y_1 & \text{if } Q = 1 \\ Y_2 & \text{if } Q = 2 \end{cases}, \quad X = \begin{cases} U_0 & \text{if } Q = 0 \\ U_1 & \text{if } Q = 1 \\ U_2 & \text{if } Q = 2 \end{cases}$$

- Yang/Kobayashi/Gesbert/Yi'11: $Q \sim \text{Bern}(2/3)$,

$$V_0 = V_i = \begin{cases} \emptyset & \text{if } Q = 0 \\ (\hat{\eta}_1, \hat{\eta}_2) & \text{if } Q = 1 \end{cases}, \quad X = \begin{cases} U_1 + U_2 & \text{if } Q = 0 \\ U_0 + U_1 + U_2 & \text{if } Q = 0 \end{cases}$$

where $U_1 - U_0 - U_2$ form a Markov chain

- Chen&Elia'13:

$$V_0 = V_1 = V_2 = (\bar{\bar{l}}^{(1)}, \bar{\bar{l}}^{(2)}), \qquad X = U_0 + U_1 + U_2,$$

where $U_1 - U_0 - U_2$ form a Markov chain

Our New Coding Schemes

- Rate-limited and one-sided feedback
- (Even low-rate) Feedback increases capacity of large class of DMBCs
- Ideas:
  - identify "resource hole" in Marton's scheme and occupy it with feedback
  - present a way to construct common info. useful for both receivers

$\rightarrow$ We'll provide an incremental description of our schemes

(Strictly) Less-Noisy DMBCs

Less-Noisy DMBC $Y_2 \succeq Y_1$

For every auxiliary $U \multimap X \multimap (Y_1, Y_2)$:

$$I(U; Y_2) \geq I(U; Y_1)$$

- Binary Symmetric BC
- Binary Erasure BC
- Binary Symmetric/Erasure BC for certain parameters

(Strictly) Less-Noisy DMBCs

Strictly Less-Noisy DMBC $Y_2 \succ Y_1$

For every auxiliary $U \multimap X \multimap (Y_1, Y_2)$ with $I(U;Y_1) > 0$:

$$I(U;Y_2) > I(U;Y_1)$$

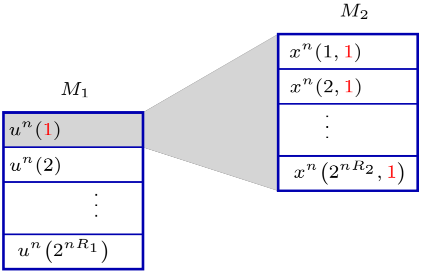
- Asymmetric Binary Symmetric BC
- Asymmetric Binary Erasure BC
- Binary Symmetric/Erasure BC for certain parameters

No Feedback Capacity of Less-Noisy DMBCs, $Y_2 \succeq Y_1$

- Capacity: all rate pairs (R_1, R_2) where for some $U - X - (Y_1, Y_2)$

$$R_1 \le I(U; Y_1)$$
$$R_2 \le I(X; Y_2 | U)$$

- Achieved by superposition coding (degenerate Marton coding)



- Rx 2 can decode M_1 because $I(U; Y_2) \ge I(U; Y_1)$

No Feedback Capacity of Less-Noisy DMBCs, $Y_2 \succeq Y_1$

- Capacity: all rate pairs (R_1, R_2) where for some $U - X - (Y_1, Y_2)$

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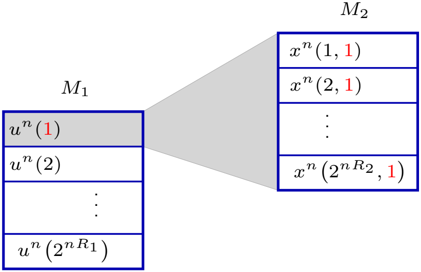


- If $I(U; Y_2) > I(U; Y_1)$, Rx 2 can decode extra message in cloud center

- Problem: Rx 1 cannot decode, unless it knows extra message

Feedback-Scheme for BCs $Y_2 \succ Y_1$ when $R_{\mathsf{FB},1} > 0$

- Superposition coding with cloud center $U_b^N(M_{1,b}, \tilde{M}_{\mathsf{FB},1,b-1})$ and satellite $X_b^N(M_{2,b}|M_{1,b}, \tilde{M}_{\mathsf{FB},1,b-1})$

- $\tilde{M}_{\mathsf{FB},1,b} = F_{1,b}^N$ feedback bits sent by Rx 1 $\quad R_{\mathsf{FB},1} > \tilde{R}_1$
- $\tilde{M}_{\mathsf{FB},1,b}$ quantizes $Y_{1,b}^N$ for given SI $Y_{2,b}^N$ and U_b^N $\quad \tilde{R}_1 > I(Y_1; \hat{Y}_1|UY_2)$

- Rx 1 decodes $M_{1,b}$ $\quad R_1 < I(U; Y_1)$
- Rx 2 decodes $(M_{1,b}, \tilde{M}_{\mathsf{FB},1,b-1})$ $\quad R_1 + \tilde{R}_1 < I(U; Y_2)$
- Rx 2 reconstructs $\hat{Y}_{1,b-1}^N$ and decodes $M_{2,b-1}$ $\quad R_2 < I(X; Y_2, \hat{Y}_1|U)$

A New Achievable Region

New achievable region

(R_1, R_2) achievable, if for some $P_U P_{X|U} P_{Y_1 Y_2|X} P_{\hat{Y}_1|UY_1}$:

$$
\begin{aligned}
R_1 &\leq I(U; Y_1) \\
R_1 &\leq I(U; Y_2) - I(\hat{Y}_1; Y_1|U, Y_2) \\
R_2 &\leq I(X; \hat{Y}_1, Y_2|U)
\end{aligned}
$$

and $\quad I(\hat{Y}_1; Y_1|U, Y_2) \leq R_{\mathsf{FB},1}$.

- Always includes nofeedback capacity for $Y_2 \succeq Y_1$

A Simpler Achievable Region

Corollary I

(R_1, R_2) achievable, if for some $P_U P_{X|U} P_{\hat{Y}_1|UY_1}$:

$$R_1 \leq I(U; Y_1)$$
$$R_2 \leq I(X; \hat{Y}_1 Y_2|U) = I(X; Y_2|U) + I(X; \hat{Y}_1|U, Y_2)$$

and $$I(\hat{Y}_1; Y_1|U, Y_2) \leq \min\{R_{\mathsf{FB},1}, I(U; Y_2) - I(U; Y_1)\}.$$

- Sending $\hat{Y}_1$ is purely beneficial: not bothering Rx 1 and helping Rx 2
- Rate-gain is $I(X; \hat{Y}_1|U, Y_2)$ with $I(\hat{Y}_1; Y_1|U, Y_2)$ feedback bits
- Scheme good when $R_{\mathsf{FB},1} \approx (I(U; Y_2) - I(U; Y_1))$

With Coded Time-Sharing and Backward Decoding

- Superposition all codebooks on an IID sequence Q^n
- Include Q^n in joint typicality checks
- Receiver 2 jointly reconstructs $\hat{Y}_1$ & decodes cloud center and satellite

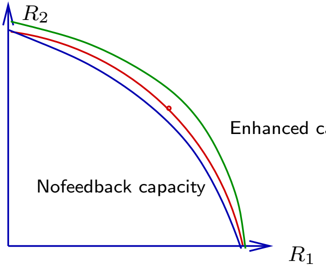
New achievable region

(R_1, R_2) achievable, if for some $P_Q P_{U|Q} P_{X|UQ} P_{Y_1 Y_2|X} P_{\hat{Y}_1|UY_1Q}$:

$$\begin{aligned} R_1 &\leq I(U; Y_1|Q) \\ R_1 + R_2 &\leq I(X; \hat{Y}_1, Y_2|Q) - I(\hat{Y}_1; Y_1 U|Y_2, Q) \\ R_1 + R_2 &\leq I(U; Y_1|Q) + I(X; \hat{Y}_1 Y_2|U, Q) \end{aligned}$$

and $\quad I(\hat{Y}_1; Y_1|U, X, Y_2, Q) \leq R_{\mathsf{FB},1}$.

If $R_{\mathsf{FB},1} > 0$, Feedback Increases Entire Capacity Region

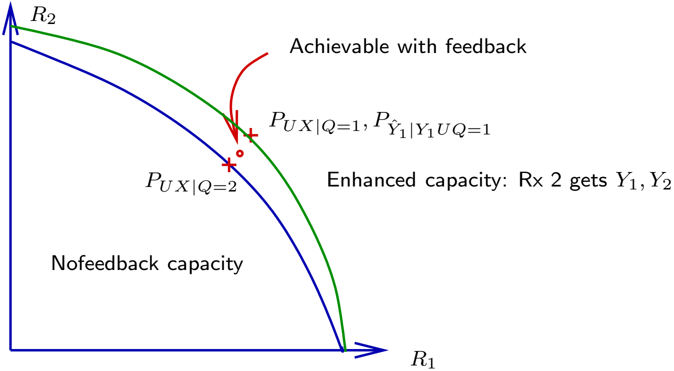


For Any DMBC $Y_2 \succ Y_1$, when $R_{\mathsf{FB},1} > 0$

Feedback improves all $(R_1 > 0, R_2 > 0)$ of the nofeedback capacity, unless (R_1, R_2) lies on boundary of capacity of enhanced channel

- For most strictly less-noisy BCs: all $(R_1 > 0, R_2 > 0)$ increased!
- Ex.: Asymmetric Binary Symmetric, Binary Erasure, Gaussian BC

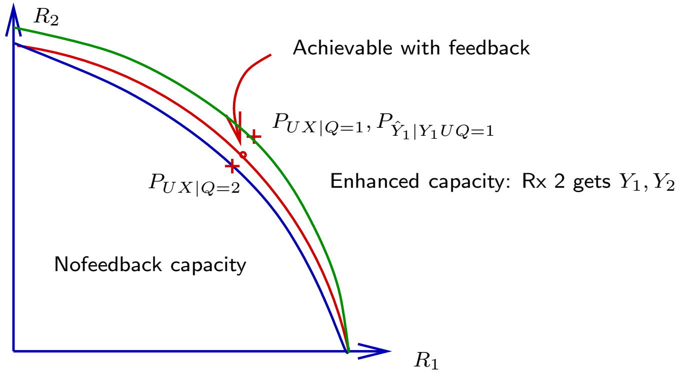
Proof that Feedback Increases Entire Capacity Region



- $Q = 1$: Use $P_{UX|Q=1}$ and $\hat{Y}_1 = Y_1$
- $Q = 2$: Use $P_{UX|Q=2}$ and $\hat{Y}_1$ =const.

  $\rightarrow P_Q(1)$ upper bounded by feedback-rate constraint!

Proof that Feedback Increases Entire Capacity Region



- $Q = 1$: Use $P_{UX|Q=1}$ and $\hat{Y}_1 = Y_1$
- $Q = 2$: Use $P_{UX|Q=2}$ and $\hat{Y}_1$ =const.

  $\rightarrow P_Q(1)$ upper bounded by feedback-rate constraint!

Usefulness of Feedback

- Previous result holds for larger class of *strictly essentially-less noisy BCs*

  $\to$ only need $I(U;Y_2) > I(U;Y_1)$ on boundary of $\mathcal{C}_{\mathsf{FB}}$

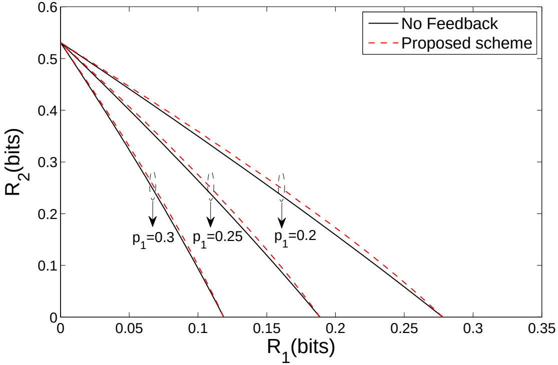
- Improvement over superposition coding region also when $I(U;Y_2) > I(U;Y_1)$ only for some points on its boundary

- Can exchange superposition coding with Marton coding:

  $\to$ feedback helps if $I(U_0;Y_2) > I(U_0;Y_1)$ for some points on boundary

Ex 1: Asymmetric Binary Symmetric BC

- $Y_i = X \oplus Z_i$ with independent Z_i ~Bern(p_i) and $0 < p_2 < p_1 < 1/2$



$p_2 = 0.1$

$R_{\text{FB},1} = 0.8$

- U ~Bern(q) and $X = U \oplus W_1$ with W_1 ~Bern(r)
- $\hat{Y}_1 = U \oplus Y_1 \oplus W_2$ with W_2 ~Bern(s)

Ex 2: BSC/BEC-BC with $1-e > 1-H(p)$ $(Y_2 \succeq Y_1)$

- BSC(p) to Y_1 and independent BEC(e) to Y_2
- Without feedback $\mathcal{C}_{\mathsf{NoFB}} = \bigcup_{s\in[0,1/2]} \left\{ \begin{array}{ccc} R_1 & \leq & 1-H_b(s\star p) \\ R_2 & \leq & (1-e)H_b(s) \\ R_1+R_2 & \leq & 1-e \end{array} \right\}$
- With feedback $R_{\mathsf{FB},1} > 0$:

$$\mathcal{C}_{\mathsf{FB}} \supseteq \bigcup_{s\in[0,1/2]} \left\{ \begin{array}{ccc} R_1 & \leq & 1-H_b(s*p) \\ R_2 & \leq & (1-e)H_b(s)+\gamma e(H_b(s*p)-H_b(p)) \\ R_1+R_2 & \leq & 1-e-\gamma H_b(p) \end{array} \right\}$$

where $\gamma \leq \frac{R_{\mathsf{FB},1}}{(1-e)H_b(p)+e}$

- $\mathcal{C}_{\mathsf{Fb}} \supsetneq \mathcal{C}_{\mathsf{NoFB}}$

Ex 2: BSC/BEC-BC for $1-e<1-H(p)$

- Y_1 more capable or essentially less noisy than Y_2
- Without feedback $\mathcal{C}_{\mathsf{NoFB}} = \bigcup_{\alpha\in[0,1]} \left\{ \begin{array}{lcl} R_1 & \leq & \alpha(1-H_b(p)) \\ R_2 & \leq & (1-\alpha)(1-e) \end{array} \right\}$
- With feedback $R_{\mathsf{FB},2}>0$:

$$\mathcal{C}_{\mathsf{FB}} \supseteq \bigcup_{\alpha\in[0,1]} \left\{ \begin{array}{ccc} R_1 & \leq & \alpha(1-H_b(p)) + \alpha(1-e)\gamma H_b(p) \\ R_2 & \leq & (1-\alpha)(1-e) \\ R_1+R_2 & \leq & 1-H_b(p)-(1-\alpha)\gamma H_b(e). \end{array} \right\}$$

where $\gamma \leq \frac{R_{\mathsf{FB},2}}{(1-e)H_b(p)+H_b(e)}$

- $\mathcal{C}_{\mathsf{Fb}} \supsetneq \mathcal{C}_{\mathsf{NoFB}}$
  $\rightarrow$they have no boundary points $(R_1>0, R_2>0)$ in common!

Extension to Two-Sided Feedback

- Marton-coding with cloud center $u_{0,b}(M_{1,c,b}, M_{2,c,b}, \tilde{M}_{\mathsf{FB},1,b-1}, \tilde{M}_{\mathsf{FB},2,b-1})$
- Satellites $u_{1,b}(M_{1,p,b}, K_{1,b})$ and $u_{2,b}(M_{2,p}, K_{2,b})$
- Feedback messages $\tilde{M}_{\mathsf{FB},1,b}$ and $\tilde{M}_{\mathsf{FB},2,b}$ as before: compress outputs $Y_{1,b}$ or $Y_{2,b}$!

$\tilde{M}_{\mathsf{FB},1,b-1}$ "transparent" for Receiver 1, $\tilde{M}_{\mathsf{FB},2,b-1}$ for Receiver 2

$\rightarrow$ can "double-book" resources in cloud-center

$\rightarrow$ same resource useful for both receivers

Sliding-Window Decoding or Backward Decoding

Sliding-window decoding:

- Rx 1 decodes cloud center $M_{1,c,b}, M_{2,c,b}, \tilde{M}_{\mathsf{FB},2,b-1}$ based on $Y_{1,b}$
- Rx 1 compresses $Y_{1,b}$ for SI $Y_{2,b}$ and $U_{0,b} \rightarrow \tilde{M}_{\mathsf{FB},1,b}$
- Rx 1 jointly reconstructs $\hat{Y}_{2,b-1}$ and decodes $M_{1,p,b-1}$ based on $(Y_{1,b-1}^N, \hat{Y}_{2,b-1})$

Backward decoding:

- Rx 1 compresses $Y_{1,b}$ for SI $Y_{2,b} \rightarrow \tilde{M}_{\mathsf{FB},1,b}$
- Rx 1 reconstructs $\hat{Y}_{2,b}$ and decodes $M_{1,c,b}, M_{2,c,b}, \tilde{M}_{\mathsf{FB},2,b-1}, M_{1,p,b}$ based on $(Y_{1,b}, \hat{Y}_{2,b})$.

Achievable Region for Two-Sided Feedback

Achievable Region Sliding-Window Decoding

(R_1, R_2) achievable, if for some $P_Q P_{U_0 U_1 U_2|Q} P_{\hat{Y}_1|Y_1 U_0 Q} P_{\hat{Y}_2|Y_2 U_0 Q}$ and $f\colon \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Q} \to \mathcal{X}$:

$$\begin{aligned} R_1 &\leq \Gamma + I(U_1; Y_1, \hat{Y}_2 | U_0, Q) \\ R_2 &\leq \Gamma + I(U_2; Y_2, \hat{Y}_1 | U_0, Q) \\ R_1 + R_2 &\leq \Gamma + I(U_1; Y_1, \hat{Y}_2 | U_0, Q) + I(U_2; Y_2, \hat{Y}_1 | U_0, Q) - I(U_1; U_2 | U_0, Q) \end{aligned}$$

and

$$\begin{aligned} I(\hat{Y}_1; Y_1 | U_0, Y_2, Q) &\leq R_{\mathsf{FB},1} \\ I(\hat{Y}_2; Y_2 | U_0, Y_1, Q) &\leq R_{\mathsf{FB},2} \end{aligned}$$

where
$\Gamma := \min\{I(U_0; Y_1|Q) - I(\hat{Y}_2; Y_2|U_0 Y_1 Q), I(U_0; Y_2|Q) - I(\hat{Y}_1; Y_1|U_0 Y_2 Q)\}$

Achievable Region with Backward Decoding

Achievable Region Backward Decoding

(R_1, R_2) achievable, if for some $P_Q P_{U_0U_1U_2|Q} P_{\hat{Y}_1|Y_1Q} P_{\hat{Y}_2|Y_2Q}$ and $f : \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Q} \to \mathcal{X}$:

$$
\begin{aligned}
R_1 &\le I(U_0, U_1; Y_1, \tilde{Y}_2|Q) - I(\tilde{Y}_2; Y_2|Y_1, Q) \\
R_2 &\le I(U_0, U_2; Y_2, \tilde{Y}_1|Q) - I(\tilde{Y}_1; Y_1|Y_2, Q) \\
R_1 + R_2 &\le I(U_0, U_1; Y_1, \tilde{Y}_2|Q) - I(\tilde{Y}_2; Y_2|Y_1, Q) \\
&\quad + I(U_2; Y_2, \tilde{Y}_1|U_0, Q) - I(U_1; U_2|U_0) \\
R_1 + R_2 &\le I(U_0, U_2; Y_2, \tilde{Y}_1|Q) - I(\tilde{Y}_1; Y_1|Y_2, Q) \\
&\quad + I(U_1; Y_1, \tilde{Y}_2|U_0, Q) - I(U_1; U_2|U_0) \\
R_1 + R_2 &\le I(U_0, U_1; Y_1, \tilde{Y}_2|Q) - I(\tilde{Y}_2; Y_2|Y_1, Q) \\
&\quad + I(U_0, U_2; Y_2, \tilde{Y}_1|Q) - I(\tilde{Y}_1; Y_1|Y_2, Q) - I(U_1; U_2|U_0)
\end{aligned}
$$

and

$$
\begin{aligned}
I(\hat{Y}_1; Y_1|Y_2, Q) &\le R_{\mathsf{FB},1} \\
I(\hat{Y}_2; Y_2|Y_1, Q) &\le R_{\mathsf{FB},2}
\end{aligned}
$$

Discussion of our Two-Sided Feedback Schemes

- Can "double-book" resources in cloud center
- Can improve over nofeedback even when $I(U_0; Y_1) = I(U_0; Y_2)$
  (e.g., Blackwell BC with states by Kim/Chia/El Gamal'13).
- Encoder only relays feedback info.
  seems unable to recover Dueck's scheme
- Improvement: Encoder could process feedback info:
  reconstruct outputs $\hat{Y}_{1,b}, \hat{Y}_{2,b}$ and compress $(U_{0,b}, U_{2,b}, U_{2,b}, \hat{Y}_{1,b}, \hat{Y}_{2,b})$

New Scheme: Encoder Processes Feedback Info

- Marton-coding with cloud centers $u_{0,b}(M_{1,c,b}, M_{2,c,b}, \tilde{M}_{V,b-1})$
- Satellites $u_{1,b}(M_{1,p,b}, K_{1,b})$ and $u_{2,b}(M_{2,p}, K_{2,b})$
- Encoder:
  - Obtains $\tilde{M}_{\mathsf{FB},1,b}$ and $\tilde{M}_{\mathsf{FB},2,b}$ and reconstructs $\tilde{Y}_{1,b}, \tilde{Y}_{2,b}$
  - Compress $U_{0,b}, U_{1,b}, U_{2,b}, \tilde{Y}_{1,b}, \tilde{Y}_{2,b}$ into V_b for SI $Y_{1,b}$ or $Y_{2,b}$ $\rightarrow$ $\tilde{M}_{V,b-1}$
- Backward Decoding:
  - Rx 1 compresses $Y_{1,b}$ for SI $U_{0,b}, U_{1,b}, U_{2,b}$ $\rightarrow$ $\tilde{M}_{\mathsf{FB},1,b-1}$
  - Rx 1 jointly reconstructs V_b and decodes $M_{1,c,b}, M_{2,c,b}, \tilde{M}_{V,b-1}, M_{1,p,b}$

Achievable Region when Encoder Processes Fb Info

Achievable Region for Two-Sided Feedback—Encoder Processes Fb

(R_1, R_2) achievable, if for some function $f\colon \mathcal{X} \to \mathcal{U}_0 \times \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{Q}$ and some $P_Q, P_{U_0U_1U_2|Q}, P_{\tilde{Y}_1|Y_1Q}, P_{\tilde{Y}_2|Y_2Q}, P_{V|U_0U_1U_2\tilde{Y}_1\tilde{Y}_2}$

$$
\begin{aligned}
R_1 &\le I(U_0, U_1; Y_1, \tilde{Y}_1, V) - I(V; U_0, U_1, U_2, \tilde{Y}_2|\tilde{Y}_1, Y_1) \\
R_2 &\le I(U_0, U_2; Y_2, \tilde{Y}_2, V) - I(V; U_0, U_1, U_2, \tilde{Y}_1|\tilde{Y}_2, Y_2) \\
R_1 + R_2 &\le I(U_0, U_1; Y_1, \tilde{Y}_1, V) + I(U_2; Y_2, \tilde{Y}_2, V|U_0) \\
&\quad - I(V; U_0, U_1, U_2, \tilde{Y}_2|\tilde{Y}_1, Y_1) - I(U_1; U_2|U_0) \\
R_1 + R_2 &\le I(U_0, U_2; Y_2, \tilde{Y}_2, V) + I(U_1; Y_1, \tilde{Y}_1, V|U_0) \\
&\quad - I(V; U_0, U_1, U_2, \tilde{Y}_1|\tilde{Y}_2, Y_2) - I(U_1; U_2|U_0) \\
R_1 + R_2 &\le I(U_0, U_1; Y_1, \tilde{Y}_1, V) - I(V; U_0, U_1, U_2, \tilde{Y}_2|\tilde{Y}_1, Y_1) \\
&\quad + I(U_0, U_2; Y_2, \tilde{Y}_2, V) - I(V; U_0, U_1, U_2, \tilde{Y}_1|\tilde{Y}_2, Y_2) - I(U_1; U_2|U_0)
\end{aligned}
$$

where

$$
\begin{aligned}
I(Y_1; \tilde{Y}_1|U_0, U_1, U_2, \tilde{Y}_2) &\le R_{\mathsf{FB},1} \\
I(Y_2; \tilde{Y}_2|U_0, U_1, U_2, \tilde{Y}_1) &\le R_{\mathsf{FB},2} \\
I(Y_1, Y_2; \tilde{Y}_1, \tilde{Y}_2|U_0, U_1, U_2) &\le R_{\mathsf{FB},1} + R_{\mathsf{FB},2}.
\end{aligned}
$$

In the Limit as $R_{\mathsf{FB},1}, R_{\mathsf{FB},2} \to \infty$

- Our region improves over the Shayevitz-W'10 region for perfect feedback specialized to $V_0 = V_1 = V_2$

- Joint source-channel coding gain (as in Tuncel'06 or NoisyNetworkCoding):

  $\min\{a-c, b-d\}$ instead of $\min\{a,b\} - \max\{c,d\}$

Noisy Feedback Channel (Instead of a Feedback Pipe)

- Feedback in block b: use a capacity $R_{\mathsf{FB},i}$ achieving code to send $\tilde{M}_{\mathsf{FB},i,b-1}$

- Error probability for feedback code in a block $\rightarrow \epsilon_{\mathsf{fb},i}^{(n)}$

- Probability of feedback error in *any of the* blocks: $(\epsilon_{\mathsf{FB},1}^{(n)} + \epsilon_{\mathsf{FB},2}^{(n)})B$

- Overall probability of error at most $(\epsilon^{(n)} + \epsilon_{\mathsf{FB},1}^{(n)} + \epsilon_{\mathsf{FB},2}^{(n)})B$

Our rates remain achievable for noisy feedback channels of capacities $R_{\mathsf{FB},1}$ and $R_{\mathsf{FB},2}$ if receivers can code over feedback links!

First Summary

Rate-limited feedback can:

- allow to exploit unutilized resources in Marton's cloud center
- allow to "double-book" resources in Marton's cloud center

Any positive feedback rate increases capacity of:

- all strictly essentially less-noisy DMBCs
- some more capable DMBCs (BSC/BEC-BC)

Results hold with noisy or noise-free rate-limited feedback channels, if receivers can code over feedback channels

Part II:

Memoryless Gaussian BC with Perfect Feedback

joint work with Selma Belhadj Amor and Yossef Steinberg

Scalar Gaussian BC with Perfect Feedback

- $X_t, Y_{1,t}, Y_{2,t} \in \mathbb{R}$
- $Y_{i,t} = h_i X_t + Z_{i,t}$,
- $Z_{1,t}$ and $Z_{2,t}$ independent $\mathcal{N}(0,1)$
- Average block power constraint: $\mathsf{E}\left[\frac{1}{n}\sum_{t=1}^{n} X_t^2\right] \leq P$
- $X_t = f_t(M_1, M_2, Y_1^{t-1}, Y_2^{t-1})$

Capacity of Scalar Gaussian BC with Feedback

- Capacity region $\mathcal{C}^{\mathsf{fb}}_{\mathsf{BC}}$ unknown
- Sum-capacity $C^{\mathsf{fb}}_{\mathsf{BC},\Sigma}$ at high SNR: $\lim_{P\to\infty}(C_{\mathsf{BC},\Sigma} - C_{\mathsf{High}}) = 0$ (Gastpar/Lapidoth/Steinberg/W'12)

$$C_{\mathsf{High}} \triangleq \begin{cases} C_{\mathsf{Coop},\Sigma}, & \text{if } |\rho_z| < 1 \\ C_1(P) + C_2(P), & \text{if } |\rho_z| = 1, h_1 \neq h_2\rho_z \\ C_1(P) & \text{if } |\rho_z| = 1, h_1 = h_2\rho_z \end{cases}$$

ρ_z: noise correlation (only here ρ_z can be $\neq 0$)

Achievable Regions with Feedback

- Ozarow&Leung'84: Schalkwijk-Kailath type scheme
- Elia'04, Wu et al.'05, Ardenistazadeh et al.'10: control-based scheme
- Gastpar/Lapidoth/Steinberg/W'12

- All these schemes are linear-feedback schemes
- Optimal linear-feedback schemes unknown!

Linear-Feedback Schemes for BC

- Encoding in two steps:
  - $(M_1, M_2) \mapsto \mathbf{W}^n$ (arbitrary!)
  - $\mathbf{X}^n = \mathbf{W}^n + \mathsf{A}_{1,\mathsf{BC}} \mathbf{Y}_1^n + \mathsf{A}_{2,\mathsf{BC}} \mathbf{Y}_2^n$ (linear!)

    $\mathsf{A}_{1,\mathsf{BC}}$ and $\mathsf{A}_{2,\mathsf{BC}}$ strictly lower-triangular

- Decoding can be arbitrary

- *Linear-feedback capacity region* $\mathcal{C}_{\mathsf{BC}}^{\mathsf{linfb}}$:

  set of all rate pairs (R_1, R_2) achievable with linear-feedback scheme

- Tricky part: identify optimal $\mathsf{A}_{1,\mathsf{BC}}^{\star}$ and $\mathsf{A}_{2,\mathsf{BC}}^{\star}$

Closer Look at Previous Linear-Feedback Schemes for BC

- $\mathbf{W}^n = \mathbf{w}_1\Theta_1(M_1) + \mathbf{w}_2\Theta_2(M_2)$ $\qquad \Theta_1, \Theta_2 \in \mathbb{R}$
- $X_t \propto X_{1,t} + \gamma X_{2,t}$
- Ozarow-Leung: $X_{i,t}$ is LMMSE of Θ_i given $\mathbf{Y}_1^{t-1}$
  (inspired by optimal single-user and MAC schemes)
- Elia, Wu etal, Ardestinazadeh etal: $X_{i,1} = \Theta_i$ and $X_{i,t+1} = b_i X_{i,t} + Y_{i,t}$
  $\rightarrow$ optimal b_i found through LQG control
  $\rightarrow$ achieves MAC sum-capacity under sum-power constraint (Duality!)

Scalar Gaussian MAC with Feedback

- $Y_t = h_1 X_{1,t} + h_2 X_{2,t} + Z_t,$ where $Z_t \sim \mathcal{N}(0,1)$
- Total power constraint: $\frac{1}{n}\sum_{t=1}^{n}(\mathsf{E}\left[X_{1,t}^2\right] + \mathsf{E}\left[X_{2,t}^2\right]) \leq P$
- Linear-feedback encoding:
  - $M_i \mapsto \mathbf{W}_i^n$ (arbitrary!)
  - $\mathbf{X}_i^n = \mathbf{W}_i^n + \mathsf{A}_{i,\mathsf{MAC}}\mathbf{Y}^n$ (linear!)
- (LMMSE-based) Linear-feedback scheme optimal: $\mathcal{C}_{\mathsf{MAC}}^{\mathsf{fb}} = \mathcal{C}_{\mathsf{MAC}}^{\mathsf{linfb}} =$

$$\bigcup_{\substack{P_1,P_2\geq 0:\\ P_1+P_2=P}} \bigcup_{\rho\in[0,1]} \left\{ \begin{array}{ccc} R_1 & \leq & \frac{1}{2}\log\left(1+h_1^2P_1(1-\rho^2)\right), \\ R_2 & \leq & \frac{1}{2}\log\left(1+h_2^2P_2(1-\rho^2)\right), \\ R_1+R_2 & \leq & \frac{1}{2}\log\left(1+h_1^2P_1+h_2^2P_2+2\sqrt{h_1^2h_2^2P_1P_2}\rho\right) \end{array} \right\}$$

Scalar MAC-BC Duality for Linear-Feedback Schemes

Linear-Feedback Capacity of Scalar Gaussian BC

$$\mathcal{C}_{\mathsf{BC}}^{\mathsf{linfb}} = \mathcal{C}_{\mathsf{MAC}}^{\mathsf{linfb}} = \mathcal{C}_{\mathsf{MAC}}^{\mathsf{fb}}$$

- Feedback always increases capacity of scalar Gaussian BC
- $\mathsf{A}_{i,\mathsf{BC}} = \bar{\mathsf{A}}_{i,\mathsf{MAC}}$ give same performance on BC and MAC
  ($\bar{\mathsf{A}}$: mirror image of A along counter-diagonal)
- $\Rightarrow$ Obtain optimal BC parameters: $\mathsf{A}^{\star}_{i,\mathsf{BC}} = \bar{\mathsf{A}}^{\star}_{i,\mathsf{MAC}} = \mathsf{A}^{\star}_{i,\mathsf{MAC}}$ (Toeplitz)
- $\Rightarrow$Message-point schemes are also sum-rate optimal for BC

Proof Idea

- Optimal (multi-letter) coding schemes:
  - Transform η channel uses into new super channel use

    $\rightarrow$ described by $\mathsf{A}_{\mathsf{BC},i}$ and $\mathsf{A}_{\mathsf{MAC},i}$
  - Code over these η-input/η-output MACs/BCs without using feedback
- For $\mathsf{A}_{\mathsf{BC},i} = \bar{\mathsf{A}}_{\mathsf{MAC},i}$ super channels have same nofeedback capacity
  - Proof uses nofeedback BC-MAC duality & operations on superchannels

Extension I: One-Sided Feedback

- Feedback only from BC-Receiver 1 or only to MAC-Transmitter 1

Scalar Gaussian MAC-BC Duality with One-Sided Linear-Feedback Schemes

$$\mathcal{C}_{\text{BC}}^{\text{linfb@Rx1}} = \mathcal{C}_{\text{MAC}}^{\text{linfb@Tx1}}$$

- $\mathcal{C}_{\text{BC}}^{\text{linfb@Rx1}}$ and $\mathcal{C}_{\text{MAC}}^{\text{linfb@Tx1}}$ both unknown
- Lapidoth/W'06 achievable region transfers to BC

  $\rightarrow$ One-sided feedback always increases capacity of scalar Gaussian BC
- Bhaskaran'08 and Lapidoth/Steinberg/W'10 regions transfer to MAC

Extension II: $K \geq 3$ User BC and MAC

K-User Scalar Gaussian BC-MAC Duality with Linear-Feedback Schemes

$$\mathcal{C}^{\text{linfb}}_{\text{BC},K} = \mathcal{C}^{\text{linfb}}_{\text{MAC},K}$$

For equal $h_1 = h_2 = \ldots = h_K$:

- Linear-feedback sum-capacity $\mathcal{C}^{\text{linfb}}_{\text{MAC},K,\Sigma}$ and optimal $\mathsf{A}^{\star}_{i,\text{MAC}}$ known (Kramer'00 and Ardestanizadeh etal. 10)

  $\Rightarrow$ Obtain $\mathcal{C}^{\text{linfb}}_{\text{BC},K,\Sigma}$ and optimal $\mathsf{A}^{\star}_{i,\text{BC}}$

- $\mathcal{C}^{\text{linfb}}_{\text{MAC},K,\Sigma} = \mathcal{C}^{\text{fb}}_{\text{MAC},K,\Sigma}$ when P sufficiently large (Kramer'00)

Multi-Antenna Gaussian BC

- BC: $\mathbf{X}_t \in \mathbb{R}^{\nu}$, $\mathbf{Y}_{i,t} \in \mathbb{R}^{\kappa_i}$ and $\mathbf{Y}_{i,t} = \mathsf{H}_i \mathbf{X}_t + \mathbf{Z}_{i,t}$,
- MAC: $\mathbf{X}_{i,t} \in \mathbb{R}^{\kappa_i}$, $\mathbf{Y}_t \in \mathbb{R}^{\nu}$ and $\mathbf{Y}_t = \mathsf{H}_i^{\mathsf{T}} \mathbf{X}_{1,t} + \mathsf{H}_i^{\mathsf{T}} \mathbf{X}_{2,t} + \mathbf{Z}_t$,
- $\mathbf{Z}_t$, $\mathbf{Z}_{1,t}$, $\mathbf{Z}_{2,t}$ independent $\mathcal{N}(0, \mathsf{I})$
- Total block-power constraints:

BC: $\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[\|\mathbf{X}_t\|^2\right] \leq P$; MAC: $\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[\|\mathbf{X}_{1,t}\|^2 + \|\mathbf{X}_{2,t}\|^2\right] \leq P$

BC-MAC Duality without Feedback (Vishwanath etal.'04, Weingarten etal.'06)

$$\mathcal{C}_{\mathsf{BC,MIMO}}^{\mathsf{NoFB}} = \mathcal{C}_{\mathsf{MAC,MIMO}}^{\mathsf{NoFB}}$$

MIMO BC-MAC Duality with Linear-Feedback Schemes

Multi-Antenna BC-MAC Duality with Linear-Feedback Schemes

$$\mathcal{C}^{\mathsf{linfb}}_{\mathsf{BC,MIMO}} = \mathcal{C}^{\mathsf{linfb}}_{\mathsf{MAC,MIMO}}$$

- $\mathcal{C}^{\mathsf{linfb}}_{\mathsf{MAC,MIMO}} = \mathcal{C}^{\mathsf{fb}}_{\mathsf{MAC,MIMO}}$ and optimal $\mathsf{A}^{\star}_{i,\mathsf{MAC}}$ known when transmitters or receiver single-antenna (Jafar etal'06)

  $\Rightarrow$ Obtain $\mathcal{C}^{\mathsf{linfb}}_{\mathsf{BC,SIMO}}$ and $\mathcal{C}^{\mathsf{linfb}}_{\mathsf{BC,MISO}}$ and corresponding optimal $\mathsf{A}^{\star}_{i,\mathsf{BC}}$

Summary

- New coding schemes for DMBC with rate-limited or noisy feedback
  - Feedback allows to occupy unused resources in Marton's cloud center
  - Feedback allows to "double-book" resources in Marton's cloud center
- Any positive feedback rate increases capacity for large class of DMBCs
- MIMO BC-MAC duality with linear-feedback schemes
  - Can transfer MAC results to BC
  - Tells us how to code over Gaussian BC when using feedback linearly