Tracking a Threshold Crossing Time of a Gaussian Random Walk Through Correlated Observations

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Abstract: Given two dependent stochastic processes X and Y, and a stopping time τ on X, the tracking stopping time problem consists in finding a stopping time η on Y that best tracks τ, e.g., so as to minimize the mean absolute deviation $\mathbb{E}|\eta - \tau|$.

This problem formulation applies in several areas including control, communication, and finance. However, the problem is in general hard to solve analytically as it generalizes the well-known (Bayesian) change-point detection problem for which solutions have been reported only for specific settings.

In this paper we provide an analytical solution to a tracking stopping time problem that cannot be formulated as a change-point problem. For the setting where X and Y are correlated Gaussian random walks, and where τ is the crossing time of some given threshold, we provide upper and lower bounds on $\inf_\eta \mathbb{E}|\eta - \tau|$ whose main asymptotic terms coincide as the threshold tends to infinity. The results immediately extend to the continuous time setting where X and Y are correlated standard Brownian motions with drift.

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1. Background

The tracking stopping time (TST) problem is defined as follows. Let $X = \{X_t\}_{t\geq 0}$ be a discrete-time stochastic process and let τ be a stopping time defined over X. Statistician has access to X only through correlated observations $Y = \{Y_t\}_{t\geq 0}$. Knowing the probability distribution of (X, Y) and the stopping rule τ, Statistician wishes to find a stopping η so as to minimize the mean $\mathbb{E}|\eta - \tau|$. (Recall that a stopping time with respect to a stochastic process $\{X_t\}_{t\geq 0}$ is a random variable τ taking values in the positive integers such

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that $\{\tau = t\} \in \mathcal{F}_t$, for all $t \geq 0$, where $\mathcal{F}_t$ denotes the σ-algebra generated by $X_0, X_1, \ldots, X_t$.)

The TST problem formulation, introduced in [8], naturally generalizes to continuous time and other delay penalty functions such as $\mathbb{E}(\eta - \tau)_+$ for a fixed 'false-alarm' probability level $\mathbb{P}(\eta < \tau)$. Important situations are when the observation process is a noisy version of X, a delayed version of X, or represents partial information with respect to X — at time t, $X_t = (\widetilde{X}_t, \widetilde{Y}_t)$ and Statistician observes only $Y_t = \widetilde{Y}_t$. For specific examples of applications of the TST problem related to monitoring, forecasting, and communication we refer to [8].

In [8], an algorithmic approach is proposed for discrete-time settings where all the X_i's and Y_i's take values in a common finite alphabet (otherwise the X and Y processes are arbitrary), and where τ is bounded by some constant $c \geq 1$. Given the probability distribution of (X, Y) and the stopping rule of τ, the algorithm outputs the minimum reaction delay $\mathbb{E}(\eta - \tau)_+$ together with an optimal stopping rule, for all false-alarm probability levels $\mathbb{P}(\eta < \tau) \leq \alpha$, $\alpha \in [0, 1]$. Under certain conditions on (X, Y) and τ, the computational complexity of this algorithm is polynomial in c.

What motivated an algorithmic approach for the TST problem, is that it generalizes the Bayesian change-point detection problem, a long studied problem with applications to industrial quality control that dates back to the 1940's [1], and for which analytical solutions have been reported only for specific, mostly asymptotic, settings.

In the Bayesian change-point problem, there is a random variable θ, taking values in the positive integers, and two probability distributions P_0 and P_1. Under P_0, the conditional density function of Y_t given $Y_1, Y_2, \ldots, Y_{t-1}$ is $f_0(Y_t|Y_1, Y_2, \ldots, Y_{t-1})$, for every $t \geq 0$. Under P_1, the conditional density function of Y_t given $Y_1, Y_2, \ldots, Y_{t-1}$ is $f_1(Y_t|Y_1, Y_2, \ldots, Y_{t-1})$, for every $t \geq 0$. The observed process is distributed according P^θ which assigns the same conditional density functions as P_0 for all $t < \theta$, and the same conditional density functions as P_1 for all $t \geq \theta$.

The Bayesian change-point problem typically consists in finding a stopping time η, with respect to $\{Y_t\}$, that minimizes some function of the delay $\eta - \tau$. Shiryaev [9, 10], for instance, considered minimizing

$$\mathbb{E}(\eta - \theta)_+ + \lambda \mathbb{P}(\eta < \theta)$$

for some given constant $\lambda \geq 0$. Assuming a geometric prior on the change-point θ, and that before and after θ the observations are independent with common density function f_0, for $t < \theta$, and f_1 for $t \geq \theta$, Shiryaev showed that an optimal η stops as soon as the posterior probability that a change occurred exceeds a certain fixed threshold. Later, Yakir [12] generalized Shiryaev's result by considering finite-state Markov chains. For more general prior distributions on θ, the problem is known to become difficult to handle. However, in the limit of small false-alarm probabilities $\mathbb{P}(\eta < \theta) \to 0$, Lai [3] and, later, Tartakovsky and Veeravalli [11], derived asymptotically optimal detection policies for the Bayesian change-point problem under general assumptions on the distributions of the change-point and observed process. (For the non-Bayesian version of the change-point problem we refer the reader to [5, 7].)

It can be shown that any Bayesian change-point problem can be formulated as a TST problem, and that a TST problem cannot, in general, be formulated as a Bayesian change-point problem [8]. The TST problem therefore generalizes the Bayesian change-point problem, which is analytically tractable only in special cases.

Our main contribution relates to the situation where X and Y are correlated Gaussian random walks given by $X_0 = Y_0 = 0$, $X_t = s \cdot t + \sum_{i=1}^{t} V_i$ and $Y_t = X_t + \varepsilon \sum_{i=1}^{t} W_i$, for $t \geq 1$ and some arbitrary constant $s > 0$ and $\varepsilon > 0$. The V_i's and W_i's are assumed to be independent standard Gaussian (i.e., zero mean unit variance) random variables. The stopping time to be tracked is the threshold crossing moment $\tau_l = \inf\{t \geq 0 : X_t \geq l\}$ for some arbitrary threshold level $l > 0$. For this setting, we provide upper and lower bounds on $\inf_\eta \mathbb{E}|\eta - \tau_l|$ that imply

$$\inf_\eta \mathbb{E}|\eta - \tau_l| = \sqrt{\frac{2l\varepsilon^2}{\pi s^3(1+\varepsilon^2)}}(1+o(1)) \quad (l \to \infty) \tag{1.1}$$

for fixed $s > 0$ and $\varepsilon > 0$. Interestingly, (1.1) is still valid if we let η be an estimator of τ that depends on the entire sequence Y_0^∞; causality doesn't come at the expense of increased delay in the above asymptotic regime.

For the particular case where the random walks have no drift, i.e., $s = 0$, we show that $\mathbb{E}|\eta - \tau_l|^r = \infty$ whenever $r \geq 1/2$, $\varepsilon > 0$, and $l > 0$, for any estimate η of τ_l that potentially may also depend on the entire observation process Y_0^∞.

The above results naturally extends to the continuous time setting where $\sum_{i=1}^{t} V_i$ and $\sum_{i=1}^{t} W_i$ are replaced by two independent standard Brownian motions. In particular, (1.1) remains valid for fixed $s > 0$ and $\varepsilon > 0$.

Section 2 contains the main results and Section 3 is devoted to the proofs.

2. Problem Formulation and Main Results

We consider the discrete-time processes

$$X: \quad X_0 = 0 \qquad X_t = \sum_{i=1}^{t} V_i + st \qquad t \geq 1$$

$$Y: \quad Y_0 = 0 \qquad Y_t = X_t + \varepsilon \sum_{i=1}^{t} W_i \qquad t \geq 1$$

where $V_1, V_2, \ldots$ and $W_1, W_2, \ldots$ are two independent sequences of independent standard (i.e., zero mean unit variance) Gaussian random variables, and where $s > 0$ and $\varepsilon > 0$ are arbitrary constants.

Given the threshold crossing time

$$\tau_l = \inf\{t \geq 0 : X_t \geq l\}$$

for some arbitrary level $l > 0$, we aim at finding a stopping time with respect to observation process Y that best tracks τ_l. Specifically, we consider the optimization problem

$$\inf_{\eta} \mathbb{E}|\eta - \tau_l|\,, \tag{2.1}$$

where the minimization is over all stopping times η defined with respect to the natural filtration induced by the Y process.

To avoid trivial situations, we restrict l and ε to be strictly positive. When $l = 0$ or $\varepsilon = 0$, (2.1) is equal to zero: for $l = 0$, $\eta = 0$ is optimal, and for $\varepsilon = 0$, $\eta = \tau_l$ is optimal.

The reason for restricting our attention to the case where also s is strictly positive is that, when $s = 0$, (2.1) is infinite for all $l > 0$ and $\varepsilon > 0$. In fact, Proposition 2.1, given at the end of this section, provides a stronger statement: for $s = 0$, $\varepsilon > 0$, and $l > 0$, we have $\mathbb{E}|\eta - \tau_l|^r = \infty$ for any $r \geq 1/2$ and any estimator $\eta = \eta(Y_0^\infty)$ of τ_l that may depend on the entire observation process Y_0^∞ (i.e., η need not be a stopping time).

The following theorem provides a non-asymptotic upper bound on (2.1) which is achieved by a threshold crossing stopping time applied to a certain estimate of the X process:

Theorem 2.1 (Upper bound)**.** *Fix $\varepsilon > 0$, $s > 0$, $l > 0$, and define $\hat{X}_t$ as*

$$\hat{X}_0 = 0 \qquad \hat{X}_t = st + \frac{1}{1+\varepsilon^2}(Y_t - st) \quad \text{for } t \geq 1.$$

Then, the stopping time $\eta = \inf\{t \geq 0 : \hat{X}_t \geq l\}$ satisfies

$$\mathbb{E}\,|\eta - \tau_l| \leq \sqrt{\frac{2l\varepsilon^2}{\pi(1+\varepsilon^2)s^3}} + \frac{6}{s}\left(\frac{l}{(2\pi s)^3}\right)^{1/4} + \sqrt{\frac{8(s+2)}{\pi s^3}} + 10 + \frac{20}{s}. \tag{2.2}$$

The next theorem provides a non-asymptotic lower bound on $\mathbb{E}|\eta - \tau_l|$ for any estimate $\eta = \eta(Y_0^\infty)$ of τ_l that has access to the entire sequence Y_0^∞. The function $Q(x)$ is defined as $Q(x) = (2\pi)^{-1/2}\int_x^\infty \exp(-u^2/2)du$.

Theorem 2.2 (Lower bound)**.** *Let $\varepsilon > 0$ and $l/s \geq 2$ with $s > 0$. Then, for any integer n such that $1 \leq n < l/s$, the following lower bound holds:*

$$\begin{aligned}\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau_l| \geq{}& \sqrt{\frac{2n\varepsilon^2}{\pi s^2(1+\varepsilon^2)}}\left(1 - Q\left(\frac{l-sn}{\sqrt{n(1+\varepsilon)}}\right)\right) \\ &- \sqrt{\frac{2}{\pi s^3}}\left(l - sn + \sqrt{\frac{n}{2\pi}}\right)^{1/2} - 2 - \frac{4}{s}.\end{aligned} \tag{2.3}$$

When n approaches l/s and l/s tends to infinity in a suitable way, the upper and lower bounds (2.2) and (2.3) become tight. The following result is an immediate consequence of these bounds:

Theorem 2.3 (Asymptotics). *Let q be a constant such that $1/2 < q < 1$. In the asymptotic regime where $l/s \geq 2$,*

$$s\left(\frac{l}{s}\right)^{q-1/2} \longrightarrow \infty\,,$$

and

$$\left(\frac{l}{s}\right)^{1-q}\frac{\varepsilon^2}{1+\varepsilon^2} \longrightarrow \infty\,,$$

we have

$$\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta-\tau_l| = \inf_{\eta} \mathbb{E}|\eta-\tau_l| = \sqrt{\frac{2l\varepsilon^2}{\pi s^3(1+\varepsilon^2)}}\,[1+o(1)]\,. \tag{2.4}$$

In particular, (2.4) holds in the limit $l\to\infty$ for fixed $s>0$ and $\varepsilon>0$.

To prove Theorem 2.1, we consider $\eta = \inf\{t\geq 0 : \hat{X}_t^{(c)} \geq l\}$, where $\hat{X}_t^{(c)}$ is the estimate of X_t defined as $\hat{X}_t^{(c)} = st + c(Y_t - st)$, then optimize over $c \geq 0$. It should be noted that, in the asymptotic regime (given by Theorem 2.3) where the upper and lower bounds on $\inf_\eta \mathbb{E}|\eta-\tau_l|$ coincide, the optimal c (equal to $1/(1+\varepsilon^2)$) is the value for which the variance of $X_t - \hat{X}_t^{(c)}$ is minimized.

Let us now consider the setting where $\sum_{i=1}^t V_i$ and $\sum_{i=1}^t W_i$ are replaced by standard Brownian motions, i.e., with the X and the Y processes being defined as

$$\begin{aligned}
X:&\quad X_0 = 0 \qquad X_t = B_t + st \quad \text{for } t>0\\
Y:&\quad Y_0 = 0 \qquad Y_t = X_t + \varepsilon N_t \quad \text{for } t>0
\end{aligned}$$

where $\{B_t\}_{t>0}$ and $\{N_t\}_{t>0}$ are two independent standard Brownian motions. The previous results easily extend to the Brownian motion setting. Indeed, the analysis is simpler than for the Gaussian random walk setting as there is no 'excess over threshold' for a Brownian motion — the value of a Brownian motion the first time it crosses a certain level equals this level.

Theorems 2.4, 2.5, and 2.6 are analogous to Theorems 2.1, 2.2, and 2.3, respectively.

Theorem 2.4 (Upper bound: Brownian motion with drift). *Fix $\varepsilon>0$, $s>0$, $l>0$, and define $\hat{X}_t$ as*

$$\hat{X}_0 = 0 \qquad \hat{X}_t = st + \frac{1}{1+\varepsilon^2}(Y_t - st) \quad \textit{for } t>0.$$

Then, the stopping time $\eta = \inf\{t\geq 0 : \hat{X}_t = l\}$ satisfies

$$\mathbb{E}\,|\eta-\tau_l| \leq \sqrt{\frac{2l\varepsilon^2}{\pi(1+\varepsilon^2)s^3}} + \frac{6}{s}\left(\frac{l}{(2\pi s)^3}\right)^{1/4}.$$

Theorem 2.5 (Lower bound: Brownian motion with drift)**.** *Let $\varepsilon > 0$, $s > 0$, and $l > 0$, and let n be such that $1 \leq n < l/s$. Then,*

$$\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau_l| \geq \sqrt{\frac{2n\varepsilon^2}{\pi s^2(1+\varepsilon^2)}} \left(1 - Q\left(\frac{l - sn}{\sqrt{n(1+\varepsilon)}}\right)\right) - \sqrt{\frac{2}{\pi s^3}} \left(l - sn + \sqrt{\frac{n}{2\pi}}\right)^{1/2}.$$

The following Theorem is an immediate consequence of Theorems 2.4 and 2.5.

Theorem 2.6 (Asymptotics : Brownian motion with drift)**.** *Theorem 2.3 is also valid in the Brownian motion setting.*

We end this section with a proposition related to the particular case where $s = 0$, which we referred to earlier. When $s = 0$, $\varepsilon > 0$, and $l > 0$, it is impossible to finitely track τ_l, even having access to the entire observation process Y_0^∞: for any estimate $\eta = \eta(Y_0^\infty)$, $\mathbb{E}(|\eta - \tau_l|^r) = \infty$ for all $r \geq 1/2$. The proposition is valid in both the Gaussian random walk and the Brownian motion settings.

Proposition 2.1. *Let $s = 0$ and let $f(x)$, $x \geq 0$, be a non-negative and non-decreasing function such that*

$$\mathbb{E}f(\tau_h/2) = \infty \tag{2.5}$$

for some constant $h > 0$. Then,

i. *$\mathbb{E}f(|\tau_l - \eta|) = \infty$ for any estimate $\eta = \eta\,(Y_0^\infty)$, whenever $\varepsilon > 0$ and $l > 0$.*
ii. *If $f(x) = x^r$, $r \geq 1/2$, then (2.5) holds for all $h > 0$, whenever $\varepsilon > 0$ and $l > 0$. (Hence, $\mathbb{E}|\tau_l - \eta|^r = \infty$ for any estimate $\eta = \eta\,(Y_0^\infty)$ of τ_l whenever $r \geq 1/2$, $s = 0$, $\varepsilon > 0$, and $l > 0$.)*

3. Proofs of Results

In this section we prove Theorems 2.1 and 2.2 and Proposition 2.1. Theorems 2.4 and 2.5 are proved in the same way as Theorems 2.1 and 2.2, by merely ignoring the boundary crossing overshoot. The proofs of Theorems 2.4 and 2.5 are therefore omitted.

Throughout the paper, V and W denote standard Gaussian random variables.

3.1. Useful results

The following result, given in [6, Theorem 2, equation (7)], provides an upper bound on overshoot that is uniform in the crossing level l.

Theorem 3.1 ([6])**.** *Let $Z_1, Z_2, \ldots$ be i.i.d. random variables such that $\mathbb{E}Z_1 \geq 0$. Define $S_t = Z_1 + Z_2 + \ldots + Z_t$, $\mu_l = \inf\{t \geq 1 : S_t \geq l\}$, and $R_{\mu_l} = S_{\mu_l} - l$. Then,*

$$\sup_{l \geq 0} \mathbb{E}(R_{\mu_l}^p) \leq \frac{2(p+2)}{(p+1)} \frac{\mathbb{E}|Z_1|^{p+2}}{\mathbb{E}(Z_1^2)} \qquad \textit{for all} \quad p > 0.$$

Overshoot has been extensively studied and various other bounds have been exhibited (see, e.g.,[2, 4]). However, to the best of our knowledge, the bound given by Theorem 3.1 is a tightest known bound in the sense that it hasn't been improved for all $s \geq 0$ and $p > 0$. In particular, it is tighter than Lorden's bound [4] for small values of s.

While our non-asymptotic results (Theorems 2.1 and 2.2) can easily be improved with tighter overshoot estimates, our main asymptotic result, Theorem 2.3, doesn't.

Corollary 3.1. *Let $Z_1, Z_2, \dots$ be i.i.d. random variables according to a mean $s \geq 0$ and variance $\sigma^2 \geq 0$ Gaussian distribution, and let S_t, μ_l, and R_{μ_l} be defined as in Theorem 3.1. Then,*

$$\sup_{l \geq 0} \mathbb{E}(R_{\mu_l}) \leq 2s + 4\sigma \,, \tag{3.1}$$

and

$$l \leq s\mathbb{E}\mu_l \leq l + 2s + 4\sigma. \tag{3.2}$$

Proof of Corollary 3.1. Since

$$\mathbb{E}|Z_1|^2 = s^2 + \sigma^2 \quad \text{and} \quad \mathbb{E}|Z_1|^4 = \mathbb{E}(s + \sigma V)^4 = s^4 + 6s^2\sigma^2 + 3\sigma^4 \,,$$

we have

$$\sup_{l \geq 0} \mathbb{E}(R^2_{\mu_l}) \leq \frac{8}{3}\left[s^2 + 5\sigma^2 - \frac{2\sigma^4}{s^2 + \sigma^2}\right],$$

from Theorem 3.1 with $p = 2$. Therefore,

$$\begin{aligned} \sup_{l \geq 0} \mathbb{E}(R_{\mu_l}) &\leq \sqrt{\sup_{l \geq 0} \mathbb{E}(R^2_{\mu_l})} \\ &\leq \sqrt{\frac{8}{3}\left[s^2 + 5\sigma^2 - \frac{2\sigma^4}{s^2 + \sigma^2}\right]} \\ &\leq 2s + 4\sigma \,, \end{aligned}$$

which gives (3.1).

Since

$$l \leq \mathbb{E}S_{\mu_l} \leq l + \sup_{l \geq 0} \mathbb{E}(R_{\mu_l}) \,,$$

and $\mathbb{E}S_{\mu_l} = s\mathbb{E}\mu_l$ by Wald's equation, inequality (3.2) follows from (3.1). ☐

Lemma 3.1. *The following inequalities hold for all $l > 0$ and $s > 0$:*

$$\mathbb{E}(l - s\tau_l)_+ \leq \mathbb{E}(s\tau_l - l)_+ \leq \sqrt{\frac{l}{2\pi s}} + s + 2 \,, \tag{3.3}$$

$$\mathbb{E}\,|s\tau_l - l| \leq \sqrt{\frac{2l}{\pi s}} + 2s + 4 \,, \tag{3.4}$$

$$\mathbb{E}(X_{\tau_l} - s\tau_l)_+ \leq \sqrt{\frac{l}{2\pi s}} + 3s + 6. \tag{3.5}$$

Proof of Lemma 3.1. Throughout the proof we use $\lfloor x \rfloor$ to denote the largest integer not greater than x.

By definition, $X_{\tau_l} \geq l$, hence $l \leq \mathbb{E}X_{\tau_l} = s\mathbb{E}\tau_l$ from Wald's equation. Using the identity $x = x_+ - (-x)_+$, we therefore get

$$0 \leq \mathbb{E}(\tau_l - l/s) = \mathbb{E}(\tau_l - l/s)_+ - \mathbb{E}(l/s - \tau_l)_+ \,,$$

i.e.,

$$\mathbb{E}(l - s\tau_l)_+ \leq \mathbb{E}(s\tau_l - l)_+ \,. \tag{3.6}$$

We upper bound the right-side of (3.6) as

$$\begin{aligned} \mathbb{E}\left(\tau_l - l/s\right)_+ &= \mathbb{E}\left(\tau_l - l/s; \tau_l > l/s\right) \\ &\leq \mathbb{E}\left(\tau_l - l/s; X_{\lfloor l/s \rfloor} \leq l\right) \\ &= \mathbb{E}\left(\tau_{-G}; G \leq 0\right) \end{aligned} \tag{3.7}$$

where G is defined as

$$G = X_{\lfloor l/s \rfloor} - l.$$

Since $G \leq \sum_{i=1}^{\lfloor l/s \rfloor} V_i \stackrel{\mathrm{d}}{=} \sqrt{\lfloor l/s \rfloor} V$,[1] using Corollary 3.1 with $\sigma^2 = 1$ yields

$$\begin{aligned} \mathbb{E}\left(\tau_{-G}; G \leq 0\right) &\leq \mathbb{E}\left[\frac{-G}{s} + 2 + \frac{4}{s}; G \leq 0\right] \\ &\leq \sqrt{\frac{l}{s^3}}\mathbb{E}(V)_+ + 1 + \frac{2}{s} \\ &= \sqrt{\frac{l}{2\pi s^3}} + 1 + \frac{2}{s}. \end{aligned} \tag{3.8}$$

From (3.6),(3.7), and (3.8) we get

$$\mathbb{E}(l - s\tau_l)_+ \leq \mathbb{E}(s\tau_l - l)_+ \leq \sqrt{\frac{l}{2\pi s}} + s + 2\,, \tag{3.9}$$

which gives (3.3).

Inequality (3.4) is an immediate consequence of (3.3).

Since $X_{\tau_l} \geq l$, we have

$$\mathbb{E}\left(X_{\tau_l} - s\tau_l\right)_+ \leq \mathbb{E}\left(X_{\tau_l} - l\right) + \mathbb{E}\left(l - s\tau_l\right)_+ \,.$$

This, together with (3.9) and the inequality

$$\mathbb{E}\left(X_{\tau_l} - l\right) \leq 2s + 4 \tag{3.10}$$

obtained from Corollary 3.1, proves (3.5). ☐

[1] We use '$\stackrel{\mathrm{d}}{=}$' to denote equality in distribution.

Proof of Theorem 2.1. We prove Theorem 2.1 by considering estimates of the form

$$\eta^{(c)} = \inf\{t \geq 1 : \hat{X}_t^{(c)} \geq l\} \,,$$

where $\hat{X}$ is defined as

$$\hat{X}_0^{(c)} = 0 \qquad \hat{X}_t^{(c)} = st + c(Y_t - st) = st + c\left[\sum_{i=1}^{t} V_i + \varepsilon \sum_{i=1}^{t} W_i\right] \quad t \geq 1$$

for some constant $c \geq 0$. To obtain the right-side of (2.2), we first upper bound $\mathbb{E}|\eta^{(c)} - \tau_l|$, $c \geq 0$, then optimize the bound over c.

Note that, for $c = 0$, we have $\eta^{(0)} = l/s$, and (3.4) gives

$$\mathbb{E}\left|\eta^{(0)} - \tau_l\right| \leq \sqrt{\frac{2l}{\pi s^3}} + 2 + \frac{4}{s}. \tag{3.11}$$

We now bound $\mathbb{E}|\eta^{(c)} - \tau_l|$ for arbitrary values of $c \geq 0$. Since

$$|x| = 2x_+ - x \,,$$

we have

$$\mathbb{E}\left|\eta^{(c)} - \tau_l\right| = 2\mathbb{E}\left(\eta^{(c)} - \tau_l\right)_+ - \mathbb{E}\left(\eta^{(c)} - \tau_l\right). \tag{3.12}$$

Applying Corollary 3.1 to τ_l and η yields

$$\mathbb{E}(\eta^{(c)} - \tau_l) \geq -\frac{2s+4}{s} \,,$$

hence from (3.12)

$$\mathbb{E}\left|\eta^{(c)} - \tau_l\right| \leq 2\mathbb{E}\left(\eta^{(c)} - \tau_l\right)_+ + \frac{2s+4}{s}. \tag{3.13}$$

Below, we upper bound $\mathbb{E}(\eta^{(c)} - \tau_l)_+$ then use (3.13) to deduce a bound on $\mathbb{E}\left|\eta^{(c)} - \tau_l\right|$.

For notational convenience, throughout the calculations we often omit the superscript $^{(c)}$ and simply write $\hat{X}_t$ and η in place of $\hat{X}_t^{(c)}$ and $\eta^{(c)}$. Similarly, we often drop the subscript $_l$ and write τ instead of τ_l.

Let us introduce the auxiliary stopping time

$$\nu = \inf\{t \geq \tau : \hat{X}_t \geq l\}.$$

Note that ν is defined with respect to both processes X and Y and that $\nu \geq \max\{\eta, \tau\}$. It follows that

$$\begin{aligned}
\mathbb{E}\left(\eta - \tau\right)_+ &\leq \mathbb{E}\left(\nu - \tau; \eta > \tau\right) \\
&\leq \mathbb{E}\left(\nu - \tau; \hat{X}_\tau \leq l\right) \\
&= \frac{1}{s}\mathbb{E}\left(\hat{X}_\nu - \hat{X}_\tau; \hat{X}_\tau \leq l\right) \qquad (3.14)
\end{aligned}$$

where the second inequality holds since $\{\eta > \tau\} \subseteq \{Y_\tau \le l\}$ and where for the last equality we used Wald's equation .

Since the random walk $\hat{X}$ has incremental steps with mean s and variance $c^2(1+\varepsilon^2)$, from Corollary 3.1 we get

$$\begin{aligned}
\mathbb{E}\left(\hat{X}_\nu - \hat{X}_\tau; \hat{X}_\tau \le l\right) &\le \mathbb{E}\left[l + 2s + 4c\sqrt{1+\varepsilon^2} - \hat{X}_\tau; \hat{X}_\tau \le l\right] \\
&\le \mathbb{E}\left[X_\tau + 2s + 4c\sqrt{1+\varepsilon^2} - \hat{X}_\tau; \hat{X}_\tau \le X_\tau\right] \\
&\le s + 2c\sqrt{1+\varepsilon^2} + \mathbb{E}(X_\tau - \hat{X}_\tau)_+ \,,
\end{aligned}$$

hence from (3.14)

$$\mathbb{E}\left(\eta^{(c)} - \tau\right)_+ \le \frac{1}{s}\mathbb{E}(X^{(c)}_\tau - \hat{X}_\tau)_+ + \frac{s + 2c\sqrt{1+\varepsilon^2}}{s}. \tag{3.15}$$

Before we compute a bound on $\mathbb{E}(\hat{X}^{(c)}_\tau - X_\tau)_+$ for general values of $c \ge 0$, we consider the simpler case $c = 1$.

Case $c = 1$: Here $\hat{X}^{(1)}_t = Y_t$ and $\eta^{(1)} = \inf\{t \ge 0 : Y_t \ge l\}$. Moreover, we have $Y_t \overset{\mathrm{d}}{=} X_t + \varepsilon\sqrt{t}W$ with W independent of X_t. It follows that

$$\begin{aligned}
\mathbb{E}(X_\tau - \hat{X}_\tau)_+ &= \mathbb{E}(\varepsilon\sqrt{\tau}W)_+ \\
&= \varepsilon\mathbb{E}(\sqrt{\tau})\mathbb{E}(W)_+ \\
&= \frac{\varepsilon}{\sqrt{2\pi}}\mathbb{E}(\sqrt{\tau}) \\
&\le \frac{\varepsilon}{\sqrt{2\pi}}\sqrt{\mathbb{E}(\tau)} \\
&\le \frac{\varepsilon}{\sqrt{2\pi}}\sqrt{\frac{l+2s+4}{s}}
\end{aligned} \tag{3.16}$$

where for the first inequality we used Jensen's inequality, and where the second inequality follows from Corollary 3.1.

Combining (3.16) with (3.15) ($c = 1$) yields

$$\mathbb{E}\left(\eta^{(1)} - \tau_l\right)_+ \le \frac{\varepsilon\sqrt{l+2s+4}}{\sqrt{2\pi s^3}} + \frac{s + 2\sqrt{1+\varepsilon^2}}{s}$$

which, together with (3.13), gives

$$\mathbb{E}\left|\eta^{(1)} - \tau_l\right| \le \frac{2\varepsilon\sqrt{l+2s+4}}{\sqrt{2\pi s^3}} + \frac{4(s + 1 + \sqrt{1+\varepsilon^2})}{s}. \tag{3.17}$$

Comparing (3.17) with (3.11) we note that for fixed $s > 0$, if $\varepsilon \ll 1$, then $\mathbb{E}\left|\eta^{(1)} - \tau_l\right| \ll \mathbb{E}\left|\eta^{(0)} - \tau_l\right|$ for large values of l.

General case $c \ge 0$: We compute a general upper bound on $\mathbb{E}(X_{\tau_l} - \hat{X}^{(c)}_{\tau_l})_+$, $c \ge 0$, and use (3.13) and (3.15) to obtain a bound on $\mathbb{E}|\eta^{(c)} - \tau_l|$.

Let U_i be the incremental step of the random walk $X_t - \hat{X}_t^{(c)}$, i.e.

$$U_i = (1-c)V_i - c\varepsilon W_i.$$

Given the fixed time horizon $n = \lfloor l/s \rfloor$, we have

$$X_{\tau_l} - \hat{X}_{\tau_l}^{(c)} = \sum_{i=1}^{n} U_i - \mathbb{1}\{\tau_l < n\} \sum_{i=\tau_l+1}^{n} U_i + \mathbb{1}\{\tau_l > n\} \sum_{i=n+1}^{\tau_l} U_i\,, \tag{3.18}$$

and therefore

$$\begin{aligned} \mathbb{E}(X_{\tau_l} - \hat{X}_{\tau_l}^{(c)})_+ \leq \mathbb{E}\Big(\sum_{i=1}^{n} U_i\Big)_+ + \mathbb{E}\Big(-\mathbb{1}\{\tau_l < n\} \sum_{i=\tau_l+1}^{n} U_i\Big)_+ \\ + \mathbb{E}\Big(\mathbb{1}\{\tau_l > n\} \sum_{i=n+1}^{\tau_l} U_i\Big)_+ . \end{aligned} \tag{3.19}$$

We bound each term on the right-side of (3.19). For the first term, since $\sum_{i=1}^{n} U_i \stackrel{\mathrm{d}}{=} \sqrt{n[(1-c)^2 + c^2\varepsilon^2]}V$, we have

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{n} U_i\right)_+ &= \sqrt{n[(1-c)^2 + c^2\varepsilon^2]}\mathbb{E}(V)_+ \\ &= \sqrt{\frac{n[(1-c)^2 + c^2\varepsilon^2]}{2\pi}} \\ &\leq \sqrt{\frac{l[(1-c)^2 + c^2\varepsilon^2]}{2\pi s}}\,. \end{aligned} \tag{3.20}$$

For the second term on the right-side of (3.19), since τ is independent of $U_{\tau+1}, U_{\tau+2}, \ldots$, we have

$$\begin{aligned} \mathbb{E}\Big(-\mathbb{1}\{\tau < n\} \sum_{i=\tau+1}^{n} U_i\Big)_+ &= \mathbb{E}\left[\sqrt{(n-\tau)_+[(1-c)^2 + c^2\varepsilon^2]}\,V_+\right] \\ &= \sqrt{\frac{(1-c)^2 + c^2\varepsilon^2}{2\pi}}\,\mathbb{E}\sqrt{(n-\tau)_+} \\ &\leq \sqrt{\frac{[(1-c)^2 + c^2\varepsilon^2]}{2\pi}\mathbb{E}(n-\tau)_+} \\ &\leq \sqrt{\frac{[(1-c)^2 + c^2\varepsilon^2]}{2\pi}\left[\sqrt{\frac{l}{2\pi s^3}} + 1 + \frac{2}{s}\right]} \end{aligned} \tag{3.21}$$

where the first inequality holds by Jensen's inequality and where the last inequality follows from (3.3).

For the third term on the right-side of (3.19), we have

$$\mathbb{E}\Big(\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} U_i\Big)_+ \leq c\varepsilon \mathbb{E}\left(\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} W_i\right)_+ + (1-c)_+ \mathbb{E}\left(\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} V_i\right)_+ . \tag{3.22}$$

Since τ and $\{W_i\}$ are independent, we have

$$\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} W_i \overset{\mathrm{d}}{=} \sqrt{[\tau - n]_+} W \, ,$$

and a similar calculation as for (3.21) shows that

$$\mathbb{E}\left[\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} W_i\right]_+ \leq \sqrt{\frac{1}{2\pi}\left[\sqrt{\frac{l}{2\pi s^3}} + 1 + \frac{2}{s}\right]}. \tag{3.23}$$

We now focus on the second expectation on the right-side of (3.22). Note first that, on $\{\tau > n\}$, we have

$$\sum_{i=n+1}^{\tau} V_i = (X_\tau - X_n) - s(\tau - n).$$

Therefore, to bound $\mathbb{E}\left(\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} V_i\right)_+$, we consider the 'shifted' sequence $\{S_t = X_t - X_n\}_{t \geq n}$, and its crossing of level $l - X_n$. Using (3.5) (with $l - X_n$ instead of l) we have

$$\begin{aligned}
\mathbb{E}\left(\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} V_i\right)_+ &\leq \mathbb{E}\left([X_\tau - X_n - s(\tau - n)]_+ ; X_n \leq l\right) \\
&\leq \mathbb{E}\sqrt{\frac{(l - X_n)_+}{2\pi s}} + 3s + 6 \\
&\leq \sqrt{\frac{\mathbb{E}(l - X_n)_+}{2\pi s}} + 3s + 6 \\
&\leq \frac{l^{1/4}}{(2\pi s)^{3/4}} + 3s + 6 \, ,
\end{aligned} \tag{3.24}$$

where the third inequality follows from Jensen's inequality. Combining (3.22) together with (3.23) and (3.24) yields

$$\mathbb{E}\Big(\mathbb{1}\{\tau > n\} \sum_{i=n+1}^{\tau} U_i\Big)_+ \leq c\varepsilon \sqrt{\frac{1}{2\pi}\left[\sqrt{\frac{l}{2\pi s^3}} + 1 + \frac{2}{s}\right]} + (1-c)_+ \left(\frac{l^{1/4}}{(2\pi s)^{3/4}} + 3s + 6\right) , \tag{3.25}$$

and from (3.15), (3.19), (3.20), (3.21), and (3.25) we get

$$\begin{aligned}\mathbb{E}\left(\eta^{(c)}-\tau\right)_+ \leq &\sqrt{\frac{l[(1-c)^2+c^2\varepsilon^2]}{2\pi s^3}}+c\varepsilon\sqrt{\frac{1}{2\pi s^2}\left[\sqrt{\frac{l}{2\pi s^3}}+1+\frac{2}{s}\right]}\\ &+\sqrt{\frac{[(1-c)^2+c^2\varepsilon^2]}{2\pi s^2}\left[\sqrt{\frac{l}{2\pi s^3}}+1+\frac{2}{s}\right]}\\ &+\frac{(1-c)_+}{s}\left[\frac{l^{1/4}}{(2\pi s)^{3/4}}+3s+6\right]\\ &+1+\frac{2c\sqrt{1+\varepsilon^2}}{s}. \end{aligned}\tag{3.26}$$

To minimize the first term on the right-side of (3.26), we set $c=\bar{c}=1/(1+\varepsilon^2)$ so that to minimize the factor $(1-c)^2+c^2\varepsilon^2$. With $c=\bar{c}$ we have $(1-c)^2+c^2\varepsilon^2=\varepsilon^2/(1+\varepsilon^2)$ and get

$$\begin{aligned}\mathbb{E}\left(\eta^{(\bar{c})}-\tau\right)_+ \leq &\sqrt{\frac{l\varepsilon^2}{2\pi(1+\varepsilon^2)s^3}}+\frac{\varepsilon}{1+\varepsilon^2}\sqrt{\frac{1}{2\pi s^2}\left[\sqrt{\frac{l}{2\pi s^3}}+1+\frac{2}{s}\right]}\\ &+\sqrt{\frac{\varepsilon^2}{2\pi(1+\varepsilon^2)s^2}\left[\sqrt{\frac{l}{2\pi s^3}}+1+\frac{2}{s}\right]}\\ &+\frac{\varepsilon^2}{s(1+\varepsilon^2)}\left[\frac{l^{1/4}}{(2\pi s)^{3/4}}+3s+6\right]\\ &+1+\frac{2}{s\sqrt{1+\varepsilon^2}}. \end{aligned}$$

We further upper bound $\varepsilon/(1+\varepsilon^2)$ and $\varepsilon^2/(1+\varepsilon^2)$ by one and get the weaker yet simpler bound

$$\mathbb{E}\left(\eta^{(\bar{c})}-\tau\right)_+ \leq \sqrt{\frac{l\varepsilon^2}{2\pi(1+\varepsilon^2)s^3}}+\frac{3}{s}\left(\frac{l}{(2\pi s)^3}\right)^{1/4}+\sqrt{\frac{2(s+2)}{\pi s^3}}+4+\frac{8}{s}.\tag{3.27}$$

Finally, combining (3.27) with (3.13) yields

$$\mathbb{E}|\eta^{(\bar{c})}-\tau| \leq \sqrt{\frac{2l\varepsilon^2}{\pi(1+\varepsilon^2)s^3}}+\frac{6}{s}\left(\frac{l}{(2\pi s)^3}\right)^{1/4}+\sqrt{\frac{8(s+2)}{\pi s^3}}+10+\frac{20}{s}$$

from which Theorem 2.1 follows. □

Proof of Theorem 2.2. We prove Theorems 2.2 by establishing a lower bound on $\mathbb{E}|\eta-\tau_l|$ for any estimator $\eta=\eta(Y_0^\infty)$ that has access to the entire observation sequence Y_0^∞.

Fix an arbitrary integer n such that $1 \leq n < l/s$ (by assumption $l/s \geq 2$), and let us break the minimization problem into two parts as

$$\begin{aligned}
\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau_l| &\geq \inf_{\eta(Y_0^\infty)} \mathbb{E}\left[\left|\left(\eta - n - \frac{l - X_n}{s}\right) + \left(n + \frac{l - X_n}{s} - \tau_l\right)\right|; Y_n \leq l\right] \\
&\geq \inf_{\eta(Y_0^\infty)} \mathbb{E}\left[\left|\eta - n - \frac{l - X_n}{s}\right|; Y_n \leq l\right] \\
&- \mathbb{E}\left[\left|n + \frac{l - X_n}{s} - \tau_l\right|; Y_n \leq l\right] \\
&= \frac{1}{s} \inf_{\eta(Y_0^\infty)} \mathbb{E}\left[|\eta - X_n|; Y_n \leq l\right] - \mathbb{E}\left[\left|n + \frac{l - X_n}{s} - \tau_l\right|; Y_n \leq l\right].
\end{aligned} \tag{3.28}$$

We first upperbound the second expectation on the right-side of (3.28). Using (3.4), we have for $X_n \leq l$

$$\mathbb{E}\left[\left|n + \frac{l - X_n}{s} - \tau_l\right| \middle| X_n, Y_n \leq l\right] \leq \sqrt{\frac{2(l - X_n)}{\pi s^3}} + 2 + \frac{4}{s}. \tag{3.29}$$

Since $X_n \overset{\mathrm{d}}{=} sn + \sqrt{n}\, V$ and since $l - sn > 0$ by assumption, we have

$$\begin{aligned}
\mathbb{E}(l - X_n)_+ &= \mathbb{E}(l - sn - \sqrt{n}\, V)_+ \\
&\leq l - sn + \sqrt{n}\, \mathbb{E}V_+ \\
&= l - sn + \sqrt{\frac{n}{2\pi}}\, .
\end{aligned}$$

Hence, from Jensen's inequality

$$\begin{aligned}
\mathbb{E}\sqrt{(l - X_n)_+} &\leq \sqrt{\mathbb{E}(l - X_n)_+} \\
&\leq \left(l - sn + \sqrt{\frac{n}{2\pi}}\right)^{1/2},
\end{aligned}$$

and therefore from (3.29)

$$\mathbb{E}\left[\left|n + \frac{l - X_n}{s} - \tau_l\right|; Y_n \leq l\right] \leq \sqrt{\frac{2}{\pi s^3}} \left(l - sn + \sqrt{\frac{n}{2\pi}}\right)^{1/2} + 2 + \frac{4}{s}. \tag{3.30}$$

To lower bound the first expectation on the right-side of (3.28), we proceed as follows. Since X_n and Y_n are jointly gaussian, we may represent X_n as

$$X_n \overset{\mathrm{d}}{=} \sqrt{n\varepsilon^2/(1 + \varepsilon^2)} V + c \cdot Y_n + d\, ,$$

where V is a standard Gaussian random variable independent of Y_n, and where c and d are (nonnegative) constants (that depend on s and ε). Using this alternative representation of X_n yields

$$
\begin{aligned}
\inf_{\eta(Y_0^\infty)} \mathbb{E}\left[|\eta - X_n|\,; Y_n \le l\right] &= \inf_{\eta(Y_0^\infty)} \mathbb{E}\left[\left|\eta - \sqrt{n\varepsilon^2/(1+\varepsilon^2)}V - c\cdot Y_n - d\right|; Y_n \le l\right] \\
&= \sqrt{\frac{n\varepsilon^2}{1+\varepsilon^2}} \inf_{\eta(Y_0^\infty)} \mathbb{E}\left[|\eta - V|\,; Y_n \le l\right] \\
&= \sqrt{\frac{n\varepsilon^2}{1+\varepsilon^2}} (\inf_e \mathbb{E}\,|e - V|)\mathbb{P}(Y_n \le l) \\
&= \sqrt{\frac{n\varepsilon^2}{1+\varepsilon^2}} (\mathbb{E}\,|V|)\mathbb{P}(Y_n \le l) \\
&= \sqrt{\frac{2n\varepsilon^2}{\pi(1+\varepsilon^2)}} \left(1 - Q\left(\frac{l - sn}{\sqrt{n(1+\varepsilon)}}\right)\right) \qquad (3.31)
\end{aligned}
$$

where the infimum on the right-side of the third equality is over constant estimators (i.e., independent of Y_0^∞), and where for the fourth equality we used the fact that the median of a random variable is its best estimator with respect to the average absolute deviation.

Combining (3.28), (3.30), and (3.31) we obtain

$$
\begin{aligned}
\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau_l| \ge{}& \sqrt{\frac{2n\varepsilon^2}{\pi s^2(1+\varepsilon^2)}} \left(1 - Q\left(\frac{l - sn}{\sqrt{n(1+\varepsilon)}}\right)\right) \\
&- \sqrt{\frac{2}{\pi s^3}} \left(l - sn + \sqrt{\frac{n}{2\pi}}\right)^{1/2} - 2 - \frac{4}{s}\,,
\end{aligned}
$$

yielding the desired result. □

Proof of Proposition 2.1. We prove the result only for the Gaussian random walk setting. The proof for the Brownian motion setting follows the same arguments and is therefore omitted.

Throughout the proof we fix some $\varepsilon > 0$, $l > 0$, and let $s = 0$.

To prove claim *i.*, we show that, for any $h > 0$, $\inf_{\eta(Y_0^\infty)} \mathbb{E}f(|\eta - \tau_l|)$ is lower bounded by $\mathbb{E}f(\tau_h/2)$ multiplied by some strictly positive constant.

The first step consists in removing the 'noise' in the observation process Y from time $t = 2$ onwards, i.e., instead of $\{Y_t\}_{t\ge 0}$, we consider the better observation process $\{Z_t\}_{t\ge 0}$ defined as

$$
\begin{aligned}
&Z_0 = 0 \\
&Z_1 = X_1 + \varepsilon W_1 = V_1 + \varepsilon W_1 \\
&Z_t = X_t - X_{t-1} = V_t \qquad t \ge 2.
\end{aligned}
$$

Clearly, it is easier to estimate τ_l based on Z_0^∞ than based on Y_0^∞; one gets $Y_t - Y_{t-1}$ by artificially adding the 'noise' εW_t to Z_t, $t \ge 1$. Therefore,

$$
\inf_{\eta(Y_0^\infty)} \mathbb{E}f(|\eta - \tau_l|) \ge \inf_{\eta(Z_0^\infty)} \mathbb{E}f(|\eta - \tau_l|)\,. \qquad (3.32)
$$

Given Z_0^∞, estimation errors on τ_l are only due to the unknown value of X_1 because of the unknown value of the noise εW_1. In turn, given Z_0^∞, it is sufficient to consider only Z_1 in order to estimate X_1 (Z_1 is a sufficient statistic for X_1).

Below, we are going to make use of the important property that the conditional density function of $X_1(=V_1)$ given Z_1 is not degenerated since it is given by

$$p\left(x|z\right)=\frac{\sqrt{1+\varepsilon^2}}{\varepsilon\sqrt{2\pi}}\exp\left\{-\frac{(1+\varepsilon^2)}{2\varepsilon^2}\left(x-\frac{z}{1+\varepsilon^2}\right)^2\right\},$$

and since $\varepsilon>0$ by assumption.

Define $C=C\left(Z_1\right)=Z_1/(1+\varepsilon^2)-h/2$ and $D=D\left(Z_1\right)=Z_1/(1+\varepsilon^2)+h/2$ where $h>0$ is some arbitrary constant. From the above non-degeneration property it follows that

$$\mathbb{P}\left(X_1\leq C\right)=\mathbb{P}\left(X_1\geq D\right)=\delta_1=\delta_1(h,\varepsilon)>0\,.$$

Using this, we lower bound

$$\inf_{\eta\left(Z_0^\infty\right)}\mathbb{E}f(|\eta-\tau_l|)$$

by considering the following three hypothesis problem: with probability $1-2\delta_1$, X_1 is known exactly (hence τ_l is known exactly as well), and with equal probability δ_1, X_1 is either equal to C or equal to D (and no additional information on X_1 is available). More specifically, denoting by τ_l^C the value of τ_l when $X_1=C$, and by τ_l^D the value of τ_l when $X_1=d$, we have

$$\begin{aligned}\inf_{\eta\left(Z_0^\infty\right)}\mathbb{E}f(|\eta-\tau_l|)&\geq\inf_{\eta\left(Z_0^\infty\right)}\left\{\mathbb{E}[f(|\eta-\tau_l|);X_1\leq C]+\mathbb{E}[f(|\eta-\tau_l|);X_1\geq D]\right\}\\&\geq\inf_{\eta\left(Z_0^\infty\right)}\left\{\mathbb{E}[f(|\eta-\tau_l^C|);X_1\leq C]+\mathbb{E}[f(|\eta-\tau_l^D|);X_1\geq D]\right\}\\&=\delta_1\inf_{\eta\left(Z_0^\infty\right)}\mathbb{E}\left[f(|\eta-\tau_l^C|)+f(|\eta-\tau_l^D|)\right]\\&\geq\delta_1\mathbb{E}f\left(\frac{\tau_l^C-\tau_l^D}{2}\right)\,,\end{aligned}\tag{3.33}$$

where the second and third inequalities follow from the assumption that $f(x)$ is non-negative and non-decreasing. Further, since $\tau_l^C\overset{\mathrm{d}}{=}\tau_{(l-C)_+}$ and since $\tau_{l_1}-\tau_{l_2}\overset{\mathrm{d}}{=}\tau_{l_1-l_2}$, $l_1\geq l_2$, from (3.33) we get

$$\begin{aligned}\inf_{\eta\left(Z_0^\infty\right)}\mathbb{E}f(|\eta-\tau_l|)&\geq\delta_1\mathbb{E}f\left(\frac{\tau_l^C-\tau_l^D}{2}\right)\\&=\delta_1\mathbb{E}f\left(\frac{\tau_{(l-C)_+}-\tau_{(l-D)_+}}{2}\right)\\&=\delta_1\mathbb{E}f\left(\frac{\tau_{(l-C)_+-(l-D)_+}}{2}\right)\,.\end{aligned}\tag{3.34}$$

Now, on $\{D \leq l\}$ we have

$$(l-C)_+ - (l-D)_+ = D - C = h\,,$$

therefore from (3.34) we get

$$\inf_{\eta(Z_0^\infty)} \mathbb{E}f(|\eta-\tau_l|) \geq \delta_1\delta_2\mathbb{E}f\left(\frac{\tau_h}{2}\right)\,, \tag{3.35}$$

where

$$\delta_2 = \delta_2(h,l,\varepsilon) = \mathbb{P}(D \leq l) > 0\,.$$

Claim i. follows from (3.35) and (3.32).

We now prove claim ii.. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion with $B_0 = 0$. For $l > 0$ introduce the crossing time

$$\tau_l^{(B)} = \inf\{t \geq 0 : B_t = l\}.$$

Since $\tau_l^{(B)} \leq \tau_l$ for all $l > 0$, had we proved that $\mathbb{E}f\left(\tau_h^{(B)}/2\right) = \infty$, $h > 0$, equation (2.5) would be satisfied since $f(x)$ is non-decreasing.

Now, using the reflection principle we get

$$\mathbb{P}(\tau_h^{(B)} \leq t) = 2\mathbb{P}(B_t \geq h) = 2Q\left(\frac{h}{\sqrt{t}}\right) \qquad h > 0,\ t > 0\,,$$

hence

$$\begin{aligned}
\mathbb{E}f\left(\tau_h^{(B)}/2\right) &= 2\int_0^\infty f(t/2)dQ\left(\frac{h}{\sqrt{t}}\right) \\
&= \frac{h}{\sqrt{2\pi}}\int_0^\infty \frac{f(t/2)}{t^{3/2}}\,e^{-h^2/2t}dt \\
&> \frac{he^{-h/2}}{\sqrt{2\pi}}\int_h^\infty \frac{f(t/2)}{t^{3/2}}\,dt.
\end{aligned}$$

Therefore, if $f(x) = x^r$ with $r \geq 1/2$, then $\mathbb{E}f\left(\tau_h^{(B)}/2\right) = \infty$ for all $h > 0$. Claim ii. follows.

□

4. Concluding Remarks

We considered the TST problem with two correlated Gaussian random walks (or two correlated Brownian motions with drift) and a threshold crossing time to be tracked τ_l. Non-asymptotic upper and lower bounds on $\inf_\eta \mathbb{E}|\eta - \tau_l|$ have been derived that coincide in certain asymptotic regimes.

Some analysis suggests that ideas used to obtain the upper and lower bounds given by Theorems 2.1 and 2.2 could be extended to higher order loss functions of the form $\mathbb{E}|\eta - \tau_l|^r$, $r > 1$. However, while a more refined estimate analysis may result in a tight asymptotic characterization of $\inf_\eta \mathbb{E}|\eta - \tau_l|^r$, simple non-asymptotic bounds as given by Theorems 2.1 and 2.2 may be more difficult to obtain.

Finally, extensions of our results to non-Gaussian random walks settings may be envisioned. Here a main difficulty appears to be the derivation of a good lower bound. In fact, a main step in the proof of Theorem 2.2 (see argument after equation 3.30) takes advantage of the fact that X_n and Y_n are jointly gaussian.

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