The Fourier Transform

DNA's double helix, the sunspot cycle and the sawtooth signals of electronics can be reduced mathematically to a series of undulating curves. This idea underlies a powerful analytical tool

by Ronald N. Bracewell

To calculate a transform, just listen. The ear automatically performs the calculation, which the intellect can execute only after years of mathematical education. The ear formulates a transform by converting sound—the waves of pressure traveling through time and the atmosphere—into a spectrum, a description of the sound as a series of volumes at distinct pitches. The brain turns this information into perceived sound.

Similar operations can be done by mathematical methods on sound waves or virtually any other fluctuating phenomenon, from light waves to ocean tides to solar cycles. These mathematical tools can decompose functions representing such fluctuations into a set of sinusoidal components—undulating curves that vary from a maximum to a minimum and back, much like the heights of ocean waves. The Fourier transform is a function that describes the amplitude and phase of each sinusoid, which corresponds to a specific frequency. (Amplitude describes the height of the sinusoid; phase specifies the starting point in the sinusoid's cycle.)

The Fourier transform has become a powerful tool in diverse fields of science. In some cases, the Fourier transform can provide a means of solving unwieldy equations that describe dynamic responses to electricity, heat or light. In other cases, it can identify the regular contributions to a fluctuating signal, thereby helping to make sense of observations in astronomy, medicine and chemistry.

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The world first learned about the technique from the mathematician for whom the transform is named, Baron Jean-Baptiste-Joseph Fourier. Fourier was not merely interested in heat; he was obsessed by it. He kept his home in Grenoble so uncomfortably hot that visitors often complained. At the same time he would cloak himself in heavy coats. Perhaps it was the lure of a warm climate that in 1798 drew Fourier to join the retinue of 165 savants that accompanied Napoleon's expedition to Egypt.

While Napoleon was fighting Syrians in Palestine, repelling the Turks from Egypt and hunting the Mameluke chief, Murad Bey, the French scientists undertook ambitious studies in geography, archaeology, medicine, agriculture and natural history. Fourier was appointed secretary of a scientific body known as the Institute of Egypt. He discharged administrative duties with such competence that he received many diplomatic assignments. Yet he was still able to conduct intensive research on Egyptian antiquities and contemplate a theory about the roots of algebraic equations.

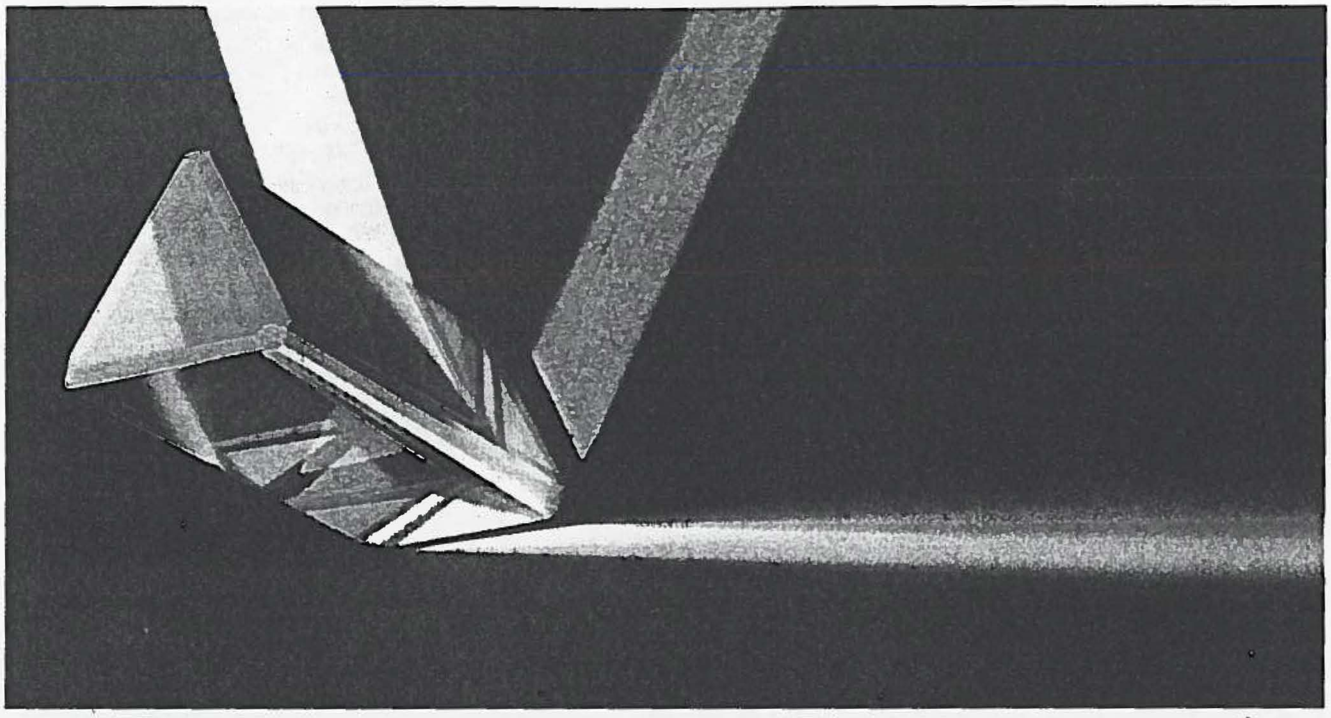
Shortly before the French were driven from Egypt in 1801, Fourier and his colleagues set sail for France. The commander of the British fleet, Admiral Sir Sidney Smith, promptly seized their ship along with its cargo of Egyptian documents and relics. In the honorable spirit of the time, Smith put the scientists ashore unharmed in Alexandria. The English commander eventually traveled to Paris to return the confiscated material—except for the Rosetta stone (the key to Egyptian hieroglyphics), which stands today in the British Museum as a monument to Napoleon's military defeat and his contribution to Egyptology.

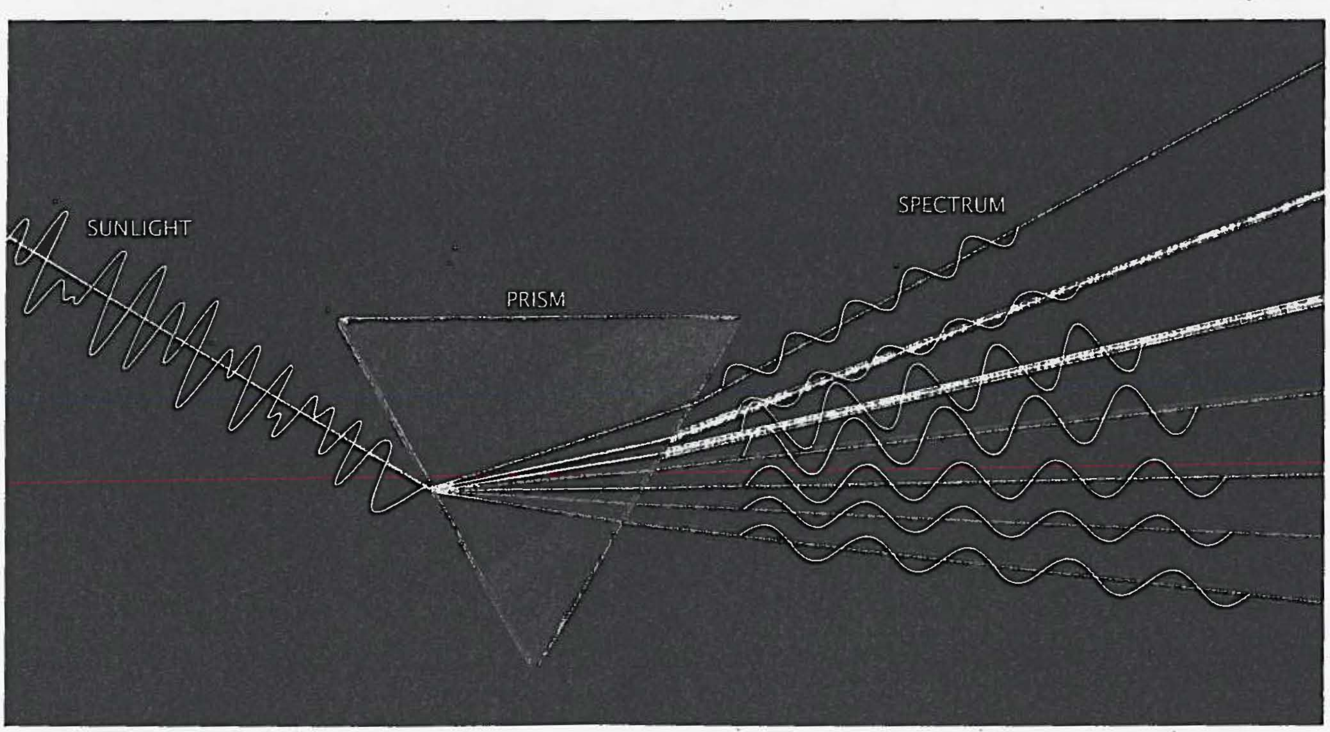
Returning to France relatively unscathed, Fourier focused on mathematical matters as professor of analysis at the Polytechnic School, but in 1802 he again entered Napoleon's service. Fourier became the prefect of the Isère department. While attempting to repair the disruptions remaining from the Revolution of 1789, he built the French section of the road to Turin and drained 80,000 square kilometers of malarial swamp. During this time he derived an equation that described the conduction of heat in solid bodies. By 1807 Fourier had invented a method for solving the equation: the Fourier transform.

Fourier applied his mathematical technique to explain many instances of heat conduction. A particularly instructive example that avoids computational complications is the flow of heat around an anchor ring—an iron ring that attaches a ship's anchor to its chain—that has been thrust halfway into a fire. When part of the circumference becomes red hot, the ring is withdrawn. Before much heat is lost to the air, the ring is buried in fine, insulating sand, and the temperature is measured around the outer curve [*see illustration on page 88*].

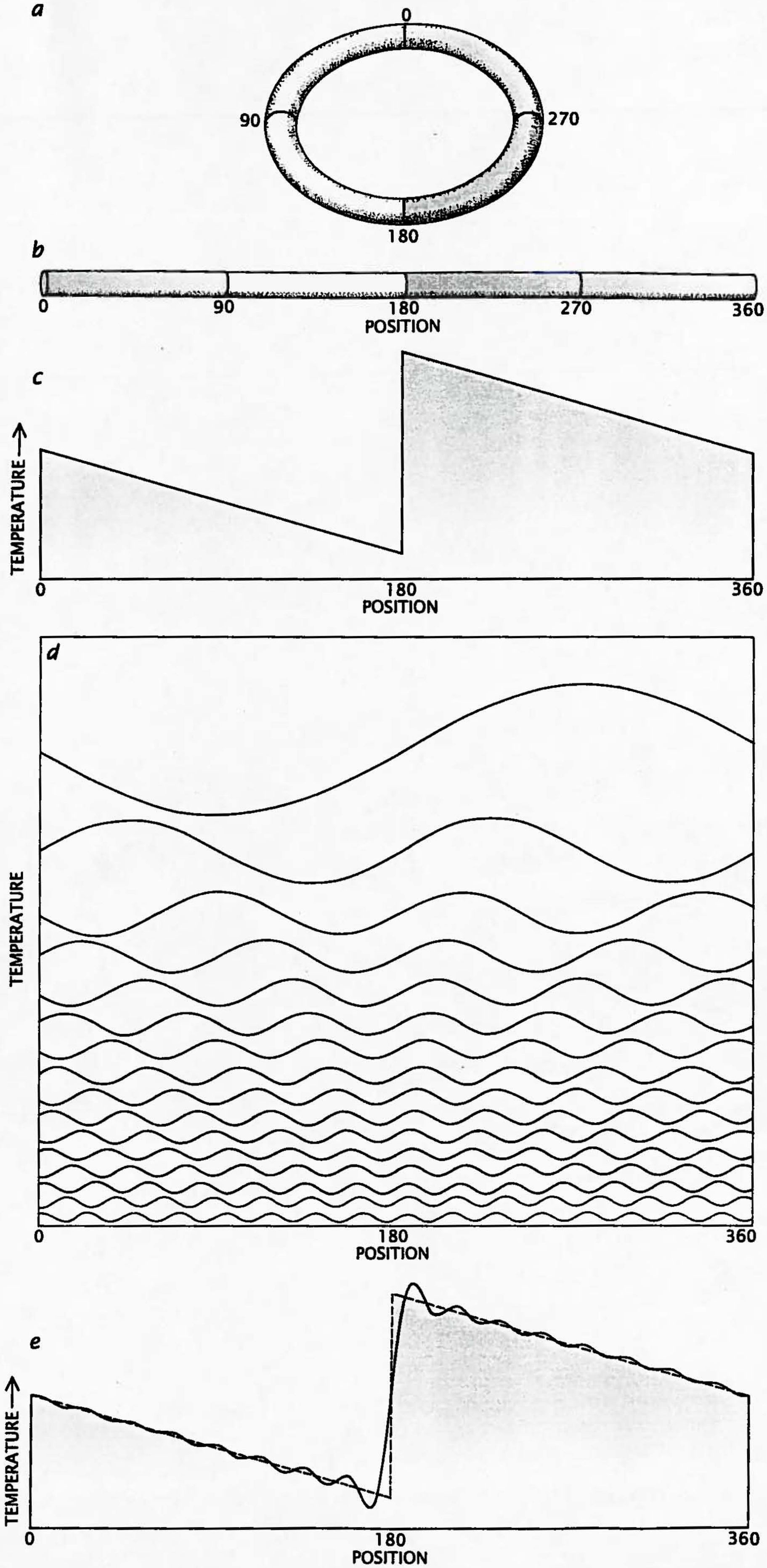
Initially the temperature distribution is irregular: part of the ring is uniformly cool, and part is uniformly hot; in between the temperature abruptly shifts. As heat is conducted from the hot region to the cool region, however, the distribution begins to smooth out. Soon the temperature distribution of heat around the ring reaches a sinusoidal form: a plot of the temperature rises and falls evenly, like an S curve, in exactly the way sine and cosine functions vary. The sinusoid gradually flattens until the whole ring arrives at a constant temperature.

Fourier proposed that the initial, irregular distribution could be broken down into many simple sinusoids that had their own maximum temperature and phase, that is, relative position around the ring. Furthermore, each sinusoidal component varied from a maximum to a minimum and back an integral number of times in a single rotation around the ring. The one-cycle variation became known as the fundamental harmonic, whereas variations with two, three or more cycles in a single rotation became the second, third and higher harmonics. The mathematical function that describes the maximum temperature and position, or phase, for each of the harmonics is the Fourier transform of the temperature distribution. Fourier had traded a single distribution that was difficult to describe mathematically for a more manageable series of full-period sine and cosine functions





SUNBEAM resolved into a spectrum provides a physical analogy for mathematical transforms (*top*). The sunlight entering the prism varies in strength from moment to moment (*bottom*). The light leaving the prism has been separated in space into pure colors, or frequencies. The intensity of each color implies an amplitude at each frequency. Thus, a function of strength versus time has been transformed into a function of amplitude versus frequency. The Fourier transform can represent a time-varying signal as a function of frequency and amplitude, but the transform also provides information about phase.



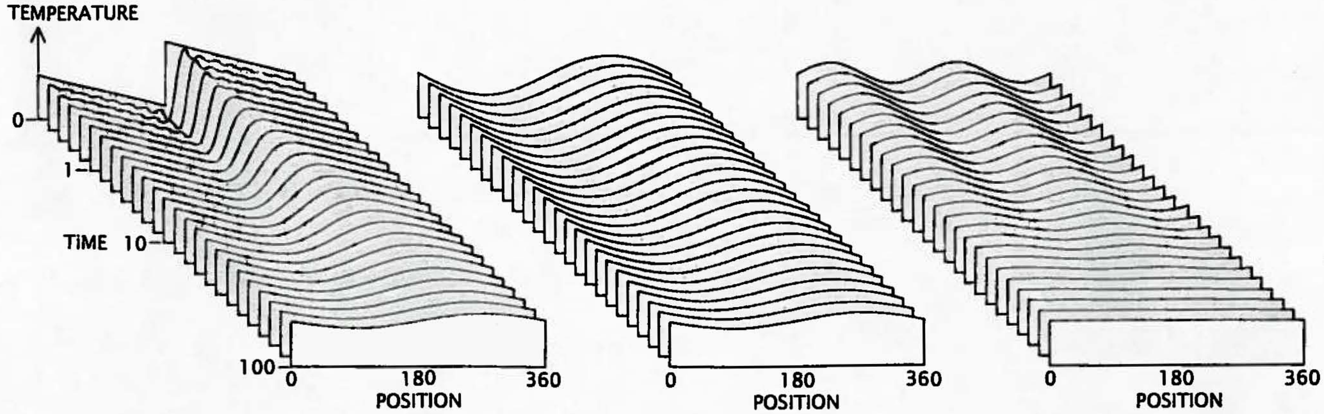
that when added together would make up the original distribution.

In applying this analysis to the conduction of heat around the ring, Fourier reasoned that the greater the number of periods of a sinusoidal component, the more rapidly it will decay. One can follow his reasoning by examining the relation between the fundamental and the second harmonic of the temperature distribution. The temperature of the second harmonic varies from hot to cool twice around the circumference of the ring, whereas the fundamental varies only once. Therefore, the distance that the heat must travel from hot peak to cool trough is only half as far for the second harmonic as it is for the fundamental. Furthermore, the temperature gradient in the second harmonic is twice as steep as it is in the fundamental variation. Because twice the heat flow occupies half the distance, the second harmonic will die out four times faster than will the fundamental.

Higher harmonics will decay even more rapidly. Hence, it is only a single sinusoidal distribution of the fundamental variation that persists as the temperature of the ring approaches equilibrium. Fourier believed that the evolution over time of any initial heat distribution could be computed by his technique.

Fourier's analysis challenged the mathematical theories to which his contemporaries adamantly adhered. In the early 19th century, many extraordinary Parisian mathematicians, including Lagrange, Laplace, Legendre, Biot and Poisson, could not accept Fourier's claim that any initial temperature distribution could be decomposed into a simple arithmetic sum that consisted of a fundamental variation and its higher-frequency harmonics. Leonhard Euler also found fault with Fourier's ideas, although he had already proposed that some

TEMPERATURE OF AN IRON RING was one of the first phenomena analyzed by Fourier's technique. One distribution of heat around a ring is shown (*a*); brighter color represents hotter areas. To begin the analysis, the ring is "uncoiled" (*b*), and the temperature is measured at every point, yielding a temperature distribution around the circumference (*c*). Then the temperature distribution is decomposed into many sinusoidal curves having one, two, three or more cycles (*d*). When 16 of the curves are simply added together (*solid line in e*), they yield a good approximation of the original temperature distribution (*broken line in e*).



CONDUCTION OF HEAT through an iron ring causes the temperature distribution to change over time (*left*). Just as the temperature distribution at any instant can be described as a series of sinusoidal curves, the evolution of a temperature distribution over time can be described in terms of changes in the sinusoids. The one-cycle distribution, or first harmonic (*middle*), and the two-cycle distribution, or second harmonic (*right*), are shown. Fourier determined that the second harmonic will decay four times faster than the first harmonic and higher harmonics will decay even faster. Because the first harmonic persists the longest, the overall temperature distribution approaches the sinusoidal shape of the first harmonic.

functions could be represented as a sum of sine functions. And so when Fourier made this claim at a meeting of the French Academy of Sciences, Lagrange stood up and held it to be impossible.

Even under these circumstances the Academy could not ignore the significance of Fourier's results, and it awarded him a prize for his mathematical theory of the laws of heat propagation and his comparison of the results of his theory with precise experiments. The award was announced, however, with the following caveat: "The novelty of the subject, together with its importance, has decided us to award the prize, while nevertheless observing that the manner in which the author arrives at his equations is not without difficulties, and that his analysis for integrating them still leaves something to be desired both as to generality and even as to rigor."

The great uneasiness with which Fourier's colleagues regarded his work caused its publication to be delayed until 1815. In fact, it was not completely described until the 1822 publication of his book, *The Analytical Theory of Heat.*

Objections to Fourier's approach focused on the proposition that an apparently discontinuous function could be represented by a sum of sinusoidal functions, which are continuous. Discontinuous functions describe broken curves or lines. For instance, a function called the Heaviside step function is zero on the left and jumps to one on the right. (Such a function can describe the flow of current when a switch is turned on.) Fourier's contemporaries had never seen a discontinuous function described as resulting from a combination of ordinary, continuous functions, such as linear, quadratic, exponential and sinusoidal functions. If Fourier was correct, however, a sum of an infinite number of sinusoids would converge to represent accurately a function with jumps, even with many jumps. At the time this seemed patently absurd.

In spite of these objections many workers, including the mathematician Sophie Germain and the engineer Claude Navier, began extending Fourier's work beyond the field of heat analysis. Yet mathematicians continued to be plagued by the question of whether a sum of sinusoidal functions would converge to represent a discontinuous function accurately.

The question of convergence arises whenever an infinite series of numbers is to be added up. Consider the classic example: will you ever arrive at a wall if with each step you travel half of the remaining distance? The first step will bring your toe to the halfway mark, the second, three quarters of the way, and at the end of the fifth step you are almost 97 percent of the way there. Clearly this is almost as good as reaching the wall, but no matter how many steps you take, you will never quite reach it. You could prove mathematically, however, that you would ultimately get closer to the wall than any distance nominated in advance. (The demonstration is equivalent to showing that the sum of a half, a fourth, an eighth, a 16th and so on approaches one.)

The question of the convergence of Fourier series emerged again late in the 19th century in efforts to predict the ebb and flow of the tides. Lord Kelvin had invented an analogue computer for providing information about the tides to the crews of merchant and naval vessels. First sets of amplitudes and phases were calculated manually from a record of tidal heights and corresponding times that had been painstakingly measured during the course of a year in a particular harbor.

Each amplitude and phase represented a sinusoidal component of the tidal-height function and revealed one of the periodic contributions to the tide. Then the results were fed into Lord Kelvin's computer, which synthesized a curve predicting the heights of the tide for the next year. Tidal curves were soon produced for ports all over the world.

It seemed obvious that a tide-predicting machine with more parts could process more amplitudes and phases and thus would make better predictions. This turned out not to be completely true if the mathematical function to be processed contained a steep jump, that is, it described an essentially discontinuous function.

Suppose such a function was reduced into a small set of amplitudes and phases—that is, just a few Fourier coefficients. The original function can then be reconstructed from the sinusoidal components corresponding to the coefficients, and the error between the original function and the reconstructed function can be measured at each point. The error-finding procedure is repeated, each time computing more coefficients and incorporating them into the reconstruction. In every case, the value of the maximum error does not diminish. On the other hand, the error becomes confined to a region that gradually shrinks around the discontinuity, so that ultimately at any given point the error approaches zero. Josiah Willard Gibbs of Yale University confirmed this result theoretically in 1899.

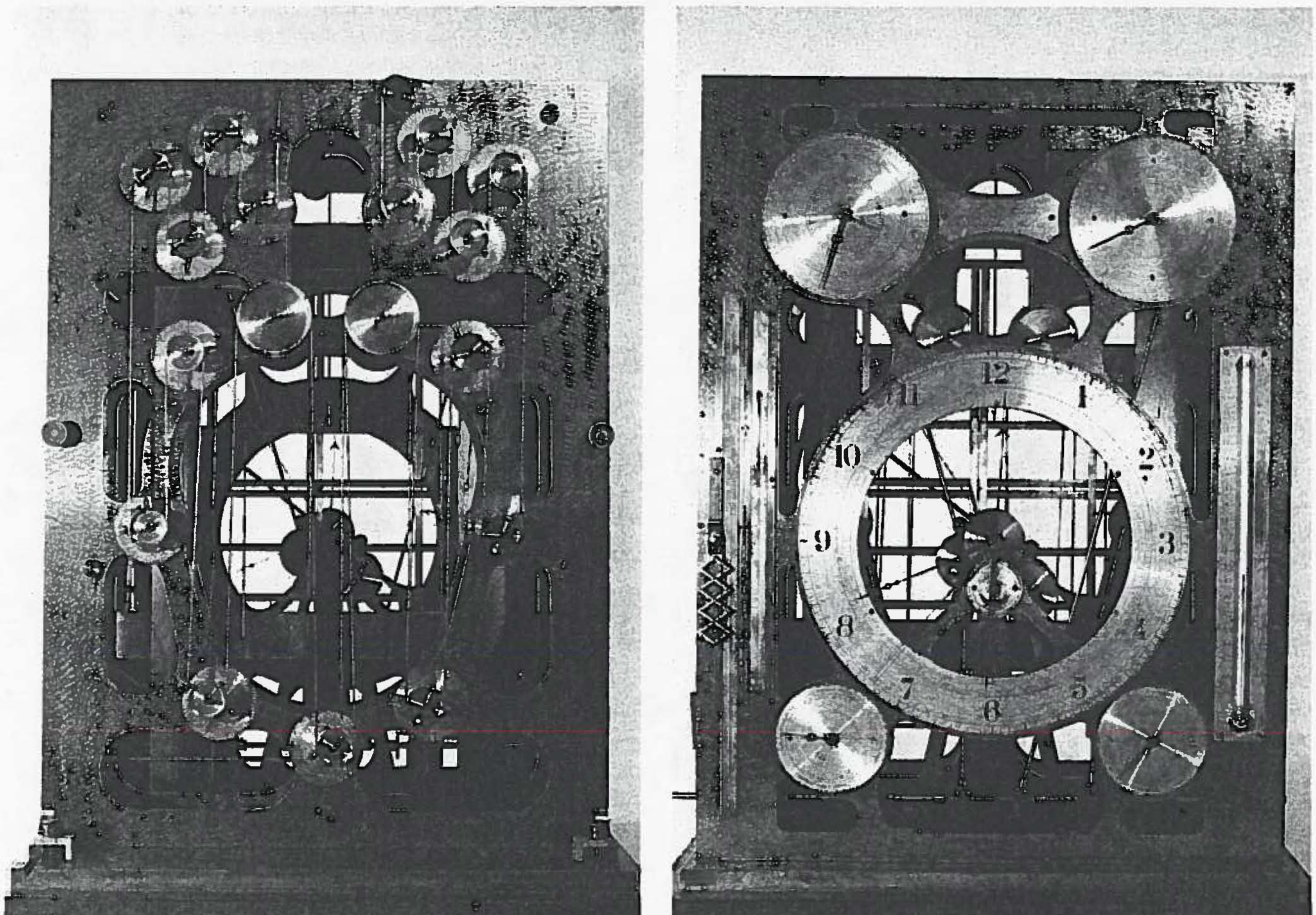
Fourier analysis is still not applicable to unusual functions, such as those possessing an infinite number of infinite jumps in a finite interval. By and large, however, a Fourier series will converge if its original function represents the measurement of a physical quantity.

Vast areas of new mathematics have been developed from investigations of whether the Fourier series of a particular function converges. One example is the theory of generalized functions, which is associated with George F. J. Temple of England, Jan G. Mikusiński of Poland and Laurent Schwartz of France. It established in 1945 a firm basis for the Heaviside step function and the Dirac delta function; the latter describes a unit of area concentrated at a point. The theory enabled the Fourier transform to be applied to solve equations that involved such intuitively accepted concepts as point mass, point charge, magnetic dipoles and the concentration of a load on a beam.

After almost two centuries of development, the theory behind the Fourier transform is firmly established and well understood. As we have seen, Fourier analysis breaks down a function in space or time into sinusoidal components that have varying frequencies, amplitudes and phases. The Fourier transform is a function that represents the amplitude and phase at each frequency. The transform can be derived by two different mathematical methods, one applied when the original function is continuous and the other when it consists of many discrete measurements.

If the function is made from a list of values at discrete intervals, it can be broken down into a series of sinusoidal functions at discrete frequencies, which range from a lowest frequency, the fundamental, through a series of frequencies that are two, three or more times the fundamental. Such a sum of sinusoids is called the Fourier series.

If the original function provides a value for every real number, then it is decomposed into sinusoidal functions at all frequencies, which are combined by means of an operation called the Fourier integral. The transform is neither the series nor the integral. In the case of the discrete function, it is the frequency-dependent list of amplitudes and phases appearing in the Fourier series; in the case of the



FERREL TIDE PREDICTOR, an analogue computer built in the late 19th century, performed Fourier synthesis to forecast the ebb and flow of the tides. Data that were collected on tidal heights at a particular harbor could be reduced by hand calculations into a set of numbers, each one representing a periodic contribution to the tide, such as the gravitational pull of the moon. The numbers for a specific port could then be fed into the Ferrel Tide Predictor by twisting knobs on the back of the machine (*left*). When a time was set on the front of the machine (*right*), the predicted height of the tide could be read off a dial.

unbroken function, it is the function of frequency that results when the Fourier integral is evaluated.

Regardless of the manner in which the transform is derived, it is necessary to specify two numbers at each frequency. These might be the amplitude and phase; however, other number pairs could encode the same information. These values can be expressed as a single complex number. (A complex number is the sum of one real number and another real number multiplied by the square root of negative one.) This representation is very popular because it invites the use of complex algebra. The theory of complex algebra and the Fourier transform have become indispensable in the numerical calculations needed to design electrical circuits, analyze mechanical vibrations and study wave propagation.

Representing an original function by its complex Fourier transform leads to computational advantages. A typical problem is to ascertain the current that flows when a known voltage is applied to a circuit. The direct method involves solving a differential equation that relates the voltage and current functions. The Fourier transforms of the voltage and current function, in contrast, can be related by an equation whose solution is trivial.

Today the study of Fourier transforms consists largely of acquiring techniques for moving freely between functions and their transforms. Analytical methods can be applied to evaluate the Fourier integral and produce the transform. Although these methods may be difficult for ordinary practitioners, many Fourier integrals have been found and are listed in tables of reference. These methods can be supplemented by learning a handful of theorems pertaining to transforms. With the aid of these theorems more or less complicated wave forms can be handled by reduction to simpler components.

Fortunately numerical methods are available for computing Fourier transforms of functions whose forms are based on experimental data or whose Fourier integrals are not easily evaluated and are not found in tables. Before electronic computers, numerical calculation of a transform was rather tedious, because such a large amount of arithmetic had to be performed with paper and pencil. The time required could be reduced somewhat by forms and schedules that guided investigators through the calculations, but the labor involved could still be daunting.

Just how much arithmetic had to be performed depended on the number of data points needed to describe the wave. The number of additions was comparable to the number of points, and the number of multiplications equaled the number of points squared. For example, analyzing a wave specified by 1,000 points taken at regular intervals required on the order of 1,000 additions and one million multiplications.

Such calculations became more feasible as computers and programs were developed to implement new methods of Fourier analysis. One was developed in 1965 by James W. Cooley of IBM's Thomas J. Watson Research Center and John W. Tukey of the Bell Telephone Laboratories in Murray Hill, N.J. Their work led to the development of a program known as the fast Fourier transform.

The fast Fourier transform saves time by decreasing the number of multiplications needed to analyze a curve. At the time, the amount of multiplication was emphasized simply because multiplication was slow with respect to other computer operations, such as addition and fetching and storing data.

The fast Fourier transform divides a curve into a large number of equally spaced samples. The number of multiplications needed to analyze a curve decreases by one half when the number of samples is halved. For example, a 16-sample curve would ordinarily take 16 squared, or 256, multiplications. But suppose the curve was halved into two pieces of eight points each. The number of multiplications needed to analyze each segment is eight squared, or 64. For the two segments the total is 128, or half the number required before.

If halving the given sequence yields a twofold gain, why not continue with the strategy? Continued subdivision leaves eight irreducible pieces of two points each. The Fourier transforms of these two-point pieces can be computed without any multiplications, but multiplication is required in the process of combining the two-point transforms to construct the whole transform. First, eight two-point transforms are combined into four four-point transforms, then into two eight-point transforms and finally into the desired 16-point transform. These three stages that combine the pieces each call for 16 multiplications, and so the total number of multiplications will be 48, which is 3/16 of the original 256.

This strategy for reducing the number of computations can be traced back long before Cooley and Tukey's work to the astronomer Carl Friedrich Gauss. Gauss wanted to calculate asteroidal and cometary orbits from only a few observations. After discovering a solution, he found a way to reduce the complexity of the calculations based on principles similar to those of the fast Fourier transform. In an 1805 paper describing the work, Gauss wrote: "Experience will teach the user that this method will greatly lessen the tedium of mechanical calculation." Thus, the challenge of celestial motions not only gave us calculus and the three laws of motion but also stimulated the discovery of a modern computing tool.

Physicists and engineers, indoctrinated with complex algebra early in their education, have become comfortable with the representation of sinusoids. The convenience of representing the Fourier transform as a complex function lets us forget that the underlying sinusoidal components are real and not necessarily complex. This habit of thought has obscured the significance of and retarded the adoption of a transform similar to Fourier's that was conceived by Ralph V. L. Hartley in 1942.

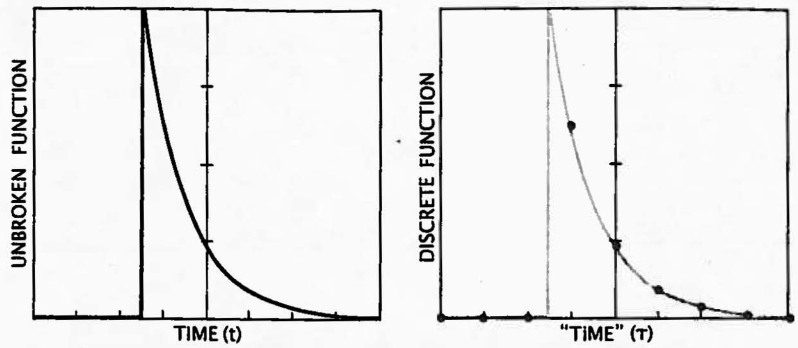
Working in the research laboratory of the Western Electric Company, Hartley directed the early development of radio receivers for a transatlantic radiotelephone and invented the Hartley oscillating circuit. During World War I Hartley investigated how a listener, through mechanisms in the ear and brain, perceives the direction from which a sound emanates. Working at Bell Laboratories after the war, Hartley was the first to formulate an important principle of information technology that states that the total amount of information a system can transmit is proportional to the product of the frequency range the system transmits and the time during which the system is available for transmission. In 1929 Hartley gave up the direction of his group because of illness. As his health improved he devoted himself to the theoretical studies that led to the Hartley transform.

The Hartley transform is an alternative means of analyzing a given function in terms of sinusoids. It differs from the Fourier transform in a rather simple manner. Whereas the Fourier transform involves real and imaginary numbers and a complex sum of sinusoidal functions, the Hartley transform involves only real numbers and a real sum of sinusoidal functions.

In 1984 I developed an algorithm

THE FOURIER AND HARTLEY TRANSFORMS

The Fourier and Hartley transforms convert functions of time into functions of frequency that encode phase and amplitude information. The graphs below represent the unbroken function $g(t)$ and the discrete function $g(\tau)$, where t is time and τ is a number designated at each data point.



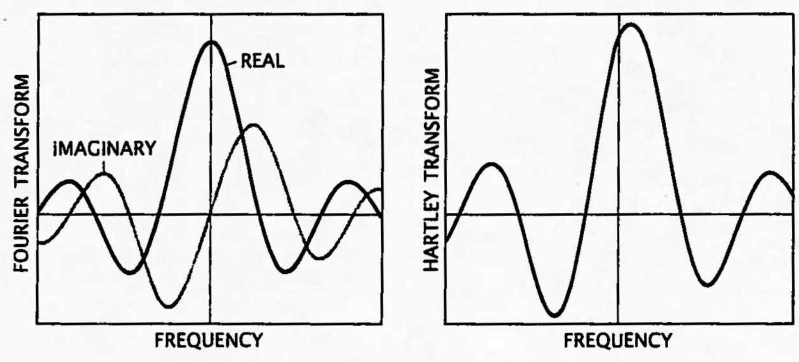
Both functions start at zero, jump to a positive value and then decay exponentially. The definition of the Fourier transform for the unbroken function is an infinite integral, $F(f)$, whereas the definition for the discrete function is a finite sum, $F(\nu)$.

$$F(f) = \int_{-\infty}^{\infty} g(t)\,(\cos 2\pi ft - i \sin 2\pi ft)\, dt \qquad F(\nu) = \frac{1}{n} \sum_{\tau=0}^{n-1} g(t)\,(\cos 2\pi\nu\tau - i \sin 2\pi\nu\tau)$$

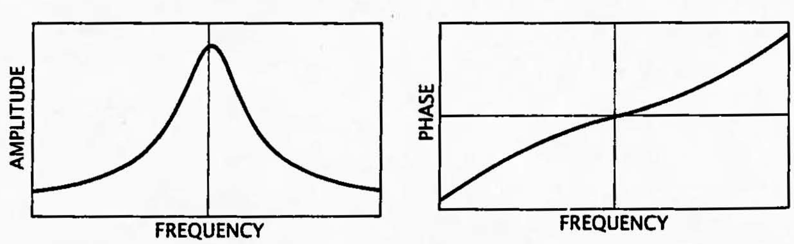
Here f is frequency, ν is related to frequency, n is the total number of samples and i is the imaginary number equal to the square root of -1. The integral representation is more suited to theoretical manipulations, whereas the finite-sum representation is more suited to computer applications. The Hartley transform and discrete Hartley transform have similar definitions.

$$H(f) = \int_{-\infty}^{\infty} g(t)\,(\cos 2\pi ft + \sin 2\pi ft)\, dt \qquad H(\nu) = \frac{1}{n} \sum_{\tau=0}^{n-1} g(t)\,(\cos 2\pi\nu\tau + \sin 2\pi\nu\tau)$$

Even though the only notational difference between the Fourier and Hartley definitions is a factor $-i$ in front of the sine function, the fact that the Fourier transform has real and imaginary parts makes the representations of the Fourier and Hartley transforms quite different. The discrete Fourier and discrete Hartley transforms have essentially the same shape as their unbroken counterparts.



Although the graphs look different, the phase and amplitude information that can be deduced from the Fourier and Hartley transforms is the same, as shown below.



Fourier amplitude is the square root of the sum of the squares of the real and imaginary parts. Hartley amplitude is the square root of the sum of the squares of $H(-\nu)$ and $H(\nu)$. Fourier phase is the arc tangent of the imaginary part divided by the real part, and Hartley phase is 45 degrees added to the arc tangent of $H(-\nu)$ divided by $H(\nu)$.

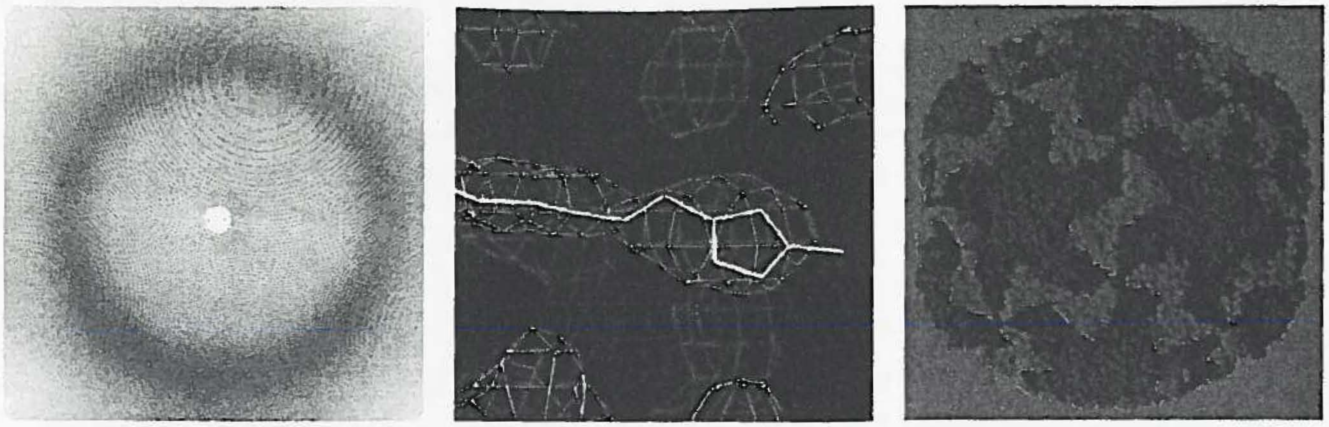
for a fast Hartley transform. The difference in computation time between the fast Hartley transform and the fast Fourier transform depends on the computer and the programming language and style. If these factors are kept constant and no oversights are made in the programming, programs for the fast Hartley transform run more quickly than those for the fast Fourier transform. Although both programs need the same amount of time to retrieve the data, provide for trigonometric functions and perform other preliminaries, the time spent on the stages of the Hartley transform is half that required by the Fourier.

It was not clear at first, however, that the Hartley transform provided the same information as the Fourier transform. Therefore, when the first programs were developed for computing the Hartley transform, an extra step was provided to convert it to the more familiar Fourier. Workers soon realized, however, that intensities and phases can be deduced directly from the Hartley transform without the need for the additional step. Further reflection revealed that either kind of transform furnishes at each frequency a pair of numbers that represents a physical oscillation in amplitude and phase.

Yet another reservation about the Hartley transform was that the Fourier transform described physical phenomena more naturally. Many phenomena, such as the response of a simple system to vibration, are commonly described by a complex sum of sinusoidal functions, which is the hallmark of the Fourier transform. It might seem, therefore, that Fourier transforms are more suitable for describing the behavior of nature.

Such a conclusion is in fact more a reflection of our mathematical upbringing than it is of nature. After all, when physical objects are measured, they provide data in real numbers, not complex ones.

The advent of the fast Hartley transform has made obsolete certain adaptations of the fast Fourier transform, such as those used for eliminating noise from digitally recorded music. These adaptations require two programs: one of them transforms real functions into the complex Fourier domain, whereas the other converts complex functions from the Fourier domain into real functions. High-frequency noise in digitally recorded music can be eliminated by filtering out portions of the transform produced by the first program. The second program then converts the changed trans-



FOURIER ANALYSIS can transform X-ray diffraction patterns into molecular models. X rays scatter off the electrons in a virus, for example, to produce patterns on film (*left*). These patterns represent part of the Fourier transform of the virus's molecular structure. If the process of transformation is reversed, the distribution of electrons, and therefore atoms, can be deduced (*middle*). From these distributions, models of the virus are made (*right*). Here colors indicate different proteins.

form back into an improved musical signal. Although these ingenious programs run individually at speeds rivaling the fast Hartley transform, a single Hartley program suffices for both transforming a real function into a Hartley transform and converting the transform, after the desired filtering, back to a real function. Therefore, extra computer memory for storing two programs is not required.

In the most general terms, Fourier and Hartley transforms have been applied in fields that contend with fluctuating phenomena. Their field of application is thus very broad indeed.

Many applications exist in biology. In fact, the double-helix form of DNA was discovered in 1962 through X-ray diffraction techniques and Fourier analysis. A beam of X rays was focused on a crystal of DNA strands, and the X rays were diffracted by the molecules of the DNA and recorded on film. This diffraction pattern provided the amplitude information of the crystal structure's Fourier transform. The phase information, which the photographs alone did not provide, was deduced by comparing the DNA diffraction pattern with patterns produced by similar chemicals. From the X-ray intensity and phase information in the Fourier transform, biologists worked back to a crystal structure—the original function. In recent years, X-ray diffraction studies combined with such "reverse" Fourier analysis have revealed the structure of many other biological molecules and more complex structures, such as viruses.

The National Aeronautics and Space Administration improves the clarity and detail of pictures of celestial objects taken in space by means of Fourier analysis. Planetary probes and earth-orbiting satellites transmit images to the earth as a series of radio impulses. A computer transforms these impulses by Fourier techniques. The computer then adjusts various components of each transform to enhance certain features and remove others—much as noise can be removed from the Fourier transforms of recorded music. Finally, the altered data are converted back to reconstruct the image. This process can sharpen focus, filter out background fog and change contrast.

The Fourier transform is also valuable in plasma physics, semiconductor physics, microwave acoustics, seismography, oceanography, radar mapping and medical imaging. Among the many applications in chemistry is the use of the Fourier-transform spectrometer for chemical analysis.

Fourier analysis has proved valuable in my own work in two-dimensional imaging. In 1956 I stumbled on a "projection slice" theorem that yielded a way to reconstruct images from strip integrals, a problem now widely known as tomographic reconstruction. Later, I hit on the "modified back-projection" algorithm, now universally used in computer-assisted X-ray tomography, or CAT scanning.

I was also interested in reconstructing images based on data from radio astronomy. I wanted to pinpoint sources of radio waves on the sun's surface, so I applied transform methods to the design of a scanning radio telescope that made daily microwave temperature maps of the sun for 11 years. The methods led to the first antenna with a beam sharper than the resolution of the human eye and have since diffused into general antenna technology. NASA commended the maps of the sun for contributing to the safety of the lunar astronauts.

I have also applied the Hartley transform to other studies. Recently my colleague John D. Villasenor and I described an optical method for finding the Hartley transform, a development that enables Fourier phase and amplitude to be encoded in a single real image. We have also developed a device that constructs the Hartley transform using microwaves. I am now writing papers on solar physics in which transform techniques underlie new ways of analyzing data from sunspot counts and from the thickness of sedimentary layers on the earth.

The wide use of Fourier's method and related analytical techniques makes what Lord Kelvin said in 1867 just as true today: "Fourier's theorem is not only one of the most beautiful results of modern analysis, but it may be said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics."

FURTHER READING

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