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The Power Model of Fitts' Law Does Not Encompass the Logarithmic Model

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Abstract

Whether Fitts' law, a well-known model of human pointing movement, is a logarithmic law or a power law has remained unclear so far. In two widely cited papers, Meyer *et al.* have claimed that the power model they derived from their celebrated stochastic optimized-submovement theory encompasses the logarithmic model as a limiting case. We show that Meyer *et al.*'s theory implies in fact a quasi-logarithmic, rather than quasi-power model, the two models being not equivalent.

Keywords: mathematical models in psychology, pointing, simple aimed movement.

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1 Introduction

Fitts' law [1,2] is a well-known empirical regularity which predicts the average time T it takes people, under time pressure, to reach with some pointer a target of width W located at distance D (Fig. 1). The law states that movement

Fig. 1. One-dimensional single-shot movement paradigm settled by Fitts [2]. Starting at origin o, one moves towards the center c of the target as fast as possible.

time $T = a + b \cdot \mathrm{ID}$ is linearly dependent (with adjustable constants a and $b > 0$) on an *index of difficulty* [1] $\mathrm{ID} = f(D/W)$, which in turn is a strictly increasing function f of relative distance D/W. Many different formulations have been proposed for f in the literature, none of which have been empirically disproved. Perhaps the most favorably received are:

$$\mathrm{ID} = \begin{cases} \log_2 \dfrac{2D}{W} & \text{Fitts (1954) [1]} \quad (1a) \\ \log_2(1 + \dfrac{D}{W}) & \text{McKenzie (1989) [5]} \quad (1b) \\ \sqrt{\dfrac{D}{W}} & \text{Meyer } et\ al. \text{ (1988)[6]} \quad (1c) \\ \left(\dfrac{D}{W}\right)^{1/n} & \text{Meyer } et\ al. \text{ (1990)[7] [2]} \quad (1d) \end{cases}$$

In particular, whether Fitts' law is a logarithmic (1a)–(1b) or a power law (1c)–(1d) has remained unclear so far. One plausible explanation is that the practical range of relative distances D/W that can be actually investigated in the laboratory is rather narrow [4]:

$$3 \lesssim \frac{D}{W} \lesssim 33. \tag{2}$$

For $D/W \ll 3$ experimenters have a floor effect on T (speed saturation) while for $D/W \gg 33$ the error rate tends to explode (accuracy saturation). Within the range $(3, 33)$, the curves (1a)–(1d) are indeed similar (see Fig. 2), making it difficult to decide empirically between the logarithmic and the power model. A Fitts' law formula being a *conceptual model*, not just a tool for the numerical

[2] Here n is an integer representing the maximum 'number of submovements', and (1c) corresponds to $n = 2$.

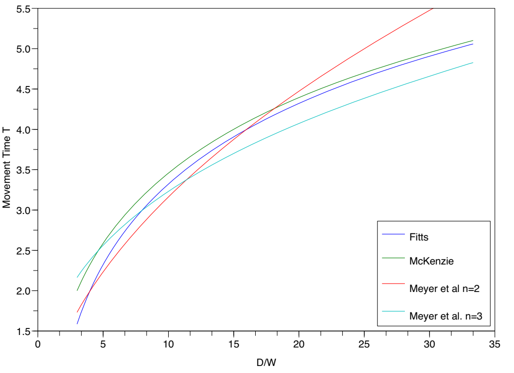

Fig. 2. Movement time T vs. D/W for the four formulations of (1) in the range (2). Eq. (1c) yields a square-root law while Eq. (1d) is plotted with $n = 3$ (cube-root law).

simulation of empirical data, such an irresolution about the exact mathematical form of Fitts' law is problematic.

In two widely cited papers [6,7], Meyer *et al.* argued that $a + b \cdot (D/W)^{1/n}$ tends to $a' + b' \cdot \log(D/W)$ as the maximum number n of submovements tends to infinity. This suggests that there is no real logarithmic vs. power issue about Fitts' law: the power model they derived from their celebrated stochastic optimized-submovement theory would encompass the logarithmic model as a limiting case. We show, however, that Meyer *et al.*'s theory does not predict a genuine power law but rather some quasi-logarithmic law, the two classes of candidate mathematical descriptions of Fitts' law being not equivalent.

2 A Disproof

2.1 Meyer et al.'s [6] Stochastic Optimized-Submovement (SOSM) Theory

This theory assumes a sequence of n submovements toward the target, and a random spread of submovement endpoints. The authors follow Woodworth's 1899 suggestion [10] that the movement involves $n = 2$ successive phases: There is an initial 'ballistic' submovement whose endpoint spread is proportional to velocity, and whose duration is given by $T_i = \frac{D/W-1/2}{s}$ where s is some spread parameter [3]. The resulting distance Δ to target center is modeled by a random variable (following, e.g., a Gaussian distribution), whose standard deviation is proportional to s. Next, if the target is not reached yet ($|\Delta| > W/2$), a

[3] Following [8], a term $1/2$ is introduced so that $T = 0$ when the origin already lies in the target.

secondary submovement occurs. The total average time (within a constant multiplicative factor) is the sum:

$$T = \min_s \left\{ \frac{D/W - 1/2}{s} + \mathbf{E}_{|\Delta|>W/2}\left(\frac{|\Delta|}{W}\right) \right\} \tag{3}$$

where the expectation $\mathbf{E}$ is with respect to Δ's distribution over the region $|\Delta| > W/2$. The optimization takes the form of a minimization over s, the only free variable left [6, Appendix]. Meyer *et al.* found a closed-form solution that can be closely approximated by a square-root law of the form (1c).

2.2 Meyer et al.'s Claim [6,7] for Multiple Submovements

Meyer *et al.* stated that for n submovements, their model yields a solution that can likewise be approximated by a nth root law of the form (1d). The claim is now as follows (see Fig. 3): "*Mathematically,* $\log_2(D/W)$ *is equivalent to the limiting case of a power function* $(D/W)^x$ *of* D/W *whose positive exponent* x *tends to zero" [6, Footnote 13].*

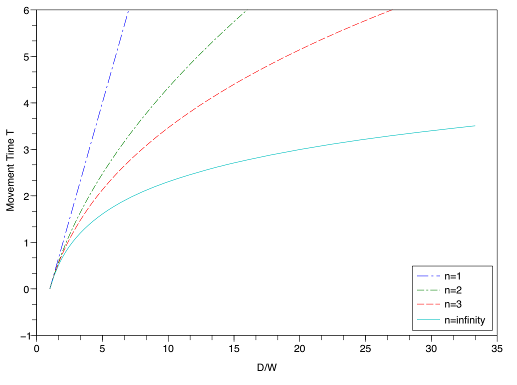

Fig. 3. A remake of Meyer *et al.*'s original Fig. 6.13 [7] giving their hypothetical T vs. D/W. They claim that as n grows large, their power relation "*approaches a logarithmic function, paralleling Fitts' Law*"[7].

Meyer *et al.*'s widely cited papers [6,7] have convinced the Fitts' law research community that their power law encompasses the logarithmic law as an extreme case $n = \infty$: see e.g. [9, Eq. (18)] for a recent account. It turns out, however, that the above claim is mathematically questionable. When the exponent $x = 1/n$ tends to zero, $\sqrt[n]{D/W} = (D/W)^{1/n} = \exp\left(\frac{1}{n}\log_e(D/W)\right)$ has first order approximation $1 + \frac{1}{n}\log_e(D/W)$ but tends to $\exp 0 = 1$ as $n \to +\infty$. Hence the limit is a 'constant law', which is not even a strictly increasing function of D/W and is, therefore, inadequate as a model for Fitts' law, as noticed by one of us [3, Section 4 (footnote)].

Since we took (1d) for granted, our disproof raises the question of the actual validity of the power law. Also, since a square-root or cube-root function is certainly not constant, it makes the dependence on n questionable. In the next section, we attempt to solve these problems.

3 Detailed Analysis for n Submovements

We now review the submovement theory of Meyer *et al.* for multiple submovements [7,8] to explain analytically why the nth-root model model fails.

3.1 Derivation

Let $T = f_n(D/W)$ be the average movement time required to reach the target after n submovements. After the initial 'ballistic' submovement, there remain $n-1$ submovements to reach the target located at random distance $|\Delta|$. The SOSM model (3) then predicts:

$$f_n\Big(\frac{D}{W}\Big) = \min_s \Big\{ \frac{D/W - 1/2}{s} + \mathbf{E}_{|\Delta|>W/2} f_{n-1}\Big(\frac{|\Delta|}{W}\Big) \Big\} \tag{4}$$

for any $n > 1$. To simplify the notation let δ denote any value of D/W and let t denote any time value. To simplify the calculations we follow Smith's [8] assumption that Δ's distribution is uniform in the interval $(-Ws/2, Ws/2)$ [4]:

$$f_n(\delta) = \min_s \Big\{ \frac{\delta - 1/2}{s} + \frac{2}{s} \int_{1/2}^{s/2} f_{n-1}(\delta) \mathrm{d}\delta \Big\} \tag{5}$$

Note that the factor 2 accounts for undershoots as well as overshoots. It is easily seen by induction that f_n is well defined, regular (indefinitely continously differentiable) and strictly increasing in the range $\delta > 1/2$. We can, therefore, define its inverse function $\delta = g_n(t)$. It turns out [8] that the determination of relative distance vs. time (that is, of g_n) is easier than the direct determination of f_n, that is, of time vs. relative distance as in the classical formulation of Fitts' law.

We now derive a simple proof leading to Smith's solution to (5). Making the first derivative of (5) vanish, the optimal $s = s(\delta)$ satisfies $-\frac{\delta-1/2}{s^2} -$

[4] Calculations run similarly for other distributions (e.g., Gaussian), with just more intricate results.

$\frac{2}{s^2}\int_{1/2}^{s/2} f_{n-1}(\delta)\,\mathrm{d}\delta + \frac{1}{s} f_{n-1}(\frac{s}{2}) = 0$, which boils down to the condition

$$f_n(\delta) = f_{n-1}(s/2). \tag{6}$$

Now, by inverting arguments according to $\delta = g_n(t)$, (5) can be rewritten as

$$t = \frac{g_n(t) - 1/2}{s} + \frac{2}{s}\int_{g_{n-1}(0)}^{g_{n-1}(t)} f_{n-1}(\delta)\mathrm{d}\delta \tag{7}$$

$$t = \frac{g_n(t) - 1/2}{s} + \frac{2}{s}\Big(\frac{s}{2}\cdot t - \int_0^t g_{n-1}(\tau)\mathrm{d}\tau\Big) \tag{8}$$

where we have used (6) in the form $g_{n-1}(t) = s/2$ in (7) and the inverse function integration theorem in (8). After subtracting t on both sides of (8) we end up with a simple recursion relation which is easily solved by induction. One finds:

$$g_n(t) = \frac{1}{2} + 2\int_0^t g_{n-1}(\tau)\mathrm{d}\tau = \frac{E_n(2t)}{2}, \tag{9}$$

where

$$E_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} \tag{10}$$

denotes the nth partial sum of the Taylor series of the exponential $\exp(t)$. Letting $L_n = E_n^{-1}$ its inverse function, one arrives at the following law:

$$T = \frac{1}{2} L_n\Big(2\frac{D}{W}\Big). \tag{11}$$

3.2 Quasi-Power vs. Quasi-Exponential Laws

In our derivation, the case $|\Delta| < W/2$ (target is reached) implies a total number of submovements strictly less than n. Therefore, n appears as the *maximum* number of permitted submovements and it would be desirable to let $n \to +\infty$ to obtain a general formulation of Fitts' law accounting for any number of multiple submovements.

Similarly to the case $n = 2$ [6], one could argue that when D/W is large (hence T is large) (10) can be approximated by its highest-degree term: $\frac{D}{W} = \frac{1}{2}E_n(2T) \approx \frac{1}{2}\frac{(2T)^n}{n!}$ so that (11) is indeed approximated by an *nth root law*:

$$T = \frac{1}{2}\sqrt[n]{n!\frac{D}{W}}. \tag{12}$$

However, this is not a genuine power model since as $n \to +\infty$, the multiplying slope factor explodes: $\sqrt[n]{n!} \sim n/e \to +\infty$ (see Fig. 4 (a)). In contrast, as Smith noticed [8], for any value of D/W (including small ones), the partial sum (10) rapidly converges to the exponential as $n \to +\infty$: $\frac{D}{W} = \frac{1}{2}E_n(2T) \longrightarrow \frac{1}{2}\exp(2T)$, and so the final result is logarithmic (see Fig. 4 (b)):

$$T = \frac{1}{2}\log_e\left(2\frac{D}{W}\right). \tag{13}$$

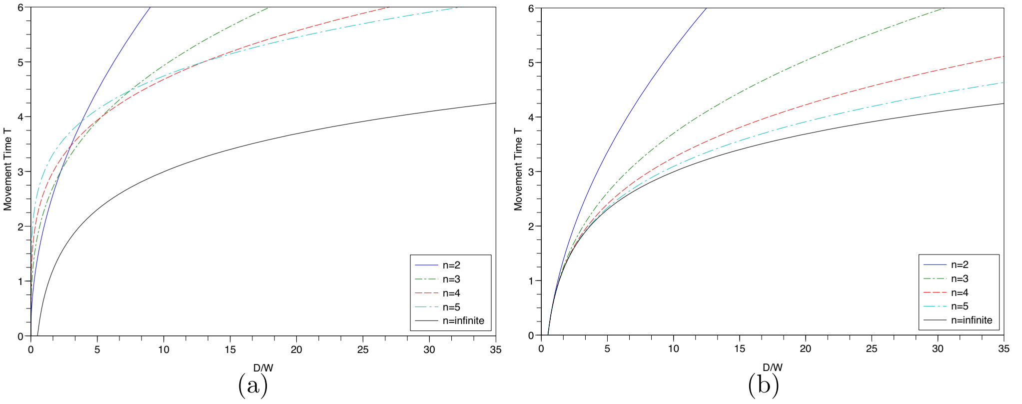

Fig. 4. (a) Approximated nth-root laws (12) for increasing values of n. (b) Exact laws (11) rapidly converging to the logarithmic law (13) as $n \to \infty$.

4 Conclusion

Not only does the power model fail to encompass the logarithmic model for multiple submovements, but the SOSM theory yields a quasi-logarithmic law (11) which rapidly converges, not on a genuine power law of the form (1d), but rather on a logarithmic law of the form (1a) or (13)—reminiscent of Fitts' original formulation. Meyer et al.'s SOSM theory [6,7] being admittedly the best explanation of Fitts' law to date, and the two classes of candidate mathematical descriptions of Fitts' law being not equivalent, we believe our finding may usefully stimulate experimental research on the subject.

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