# The Power Model of Fitts' Law Does not Encompass the Logarithmic Model

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EMPG 2011
Paris, France
Aug. 29–31, 2011



## Plan

1. What is Fitts' Law ?
2. Submovement Theories
3. Meyer et al.'s Claim
4. Detailed Computation & Conclusion
5. Open Issues

Power vs. Logarithmic Fitts' Law
O. Rioul & Y. Guiard
Fitts' law
Submovements
False claim
Details
Open Issues

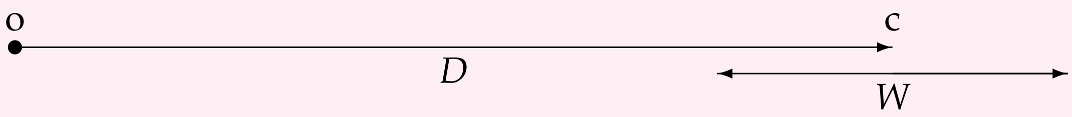
# Fitts' Law



A model of human pointing movement in HCI

## One-dimensional single-shot movement paradigm

$T$ time required to rapidly move to a target interval
$D$ distance to the target
$W$ size of the target






# Fitts' Law

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Fitts' law
Submovements
False claim
Details
Open Issues


## Mathematical formulation

- $T$ is linearly dependent on an *index of difficulty* $ID$ :

$$T = a + b \cdot ID$$

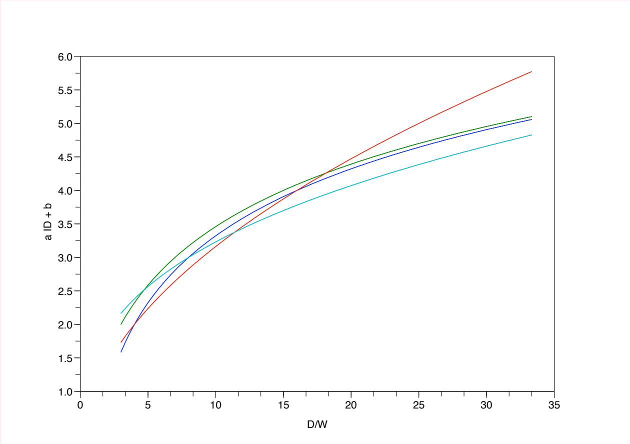
- $ID$ is a strictly increasing function of $\frac{D}{W}$ :

$$ID = \begin{cases} \log_2 \frac{2D}{W} & \text{Fitts (1954)} \\ \log_2 \frac{D}{W} & \text{Crossman (1956)} \\ \log_2 (1 + \frac{D}{W}) & \text{McKenzie (1992)} \\ \sqrt{\frac{D}{W}} & \text{Meyer } et\ al. \text{ (1988)} \\ \left(\frac{D}{W}\right)^{1/n} & \text{Meyer } et\ al. \text{ (1990)} \end{cases}$$

Practical range of relative distances is narrow (Guiard *et al.*, in press) :

$$\text{(speed saturation)} \quad 3 \lesssim \frac{D}{W} \lesssim 33 \quad \text{(accuracy saturation)}$$

$$ID = \begin{cases} \log_2 \frac{D}{W} \\ \log_2 (1 + \frac{D}{W}) \\ \sqrt{\frac{D}{W}} \\ (\frac{D}{W})^{1/n} \end{cases}$$




*"A failure to agree for 50 years is public advertisement of a failure to disprove" Platt (1964, Strong inference, p. 351)*



# Mathematical Derivation of Fitts' Law



Assuming a sequence of $n$ submovements toward the target

## Deterministic Iterative-Corrections Model

Crossman & Goodeve (1963)

- each submovement requires a constant time $\Delta T$
- and moves a constant proportion $\lambda < 1$ of the remaining distance

$$T = n\,\Delta T \qquad \text{s.t.} \qquad \lambda^n D = W/2$$

so $T = a + b \log_2 \frac{D}{W}$.

# Mathematical Derivation of Fitts' Law



## Stochastic Optimized-Submovement Model

Meyer *et al.* (1988,1990)

- random submovement endpoint $\Delta$ (Gaussian r.v.)
- $n = 2$ (Woodworth, 1899)
- initial ballistic submovement $T_i = \frac{D}{W \cdot s}$
  where dispersion parameter $s \propto \sigma_\Delta$
- if $|\Delta| > W/2$, secondary submovement, averaged over $\Delta$

$$T = \min_s \{T_i + E_{|\Delta|>W/2}(\tfrac{|\Delta|}{W})\}$$

- Result (approx.) :

$$T \propto \frac{2\theta\sqrt{D/W} - \sqrt{W/D}}{\theta\sqrt{\theta - W/D}} \qquad \text{where} \qquad \theta \propto \exp\frac{1}{2(\theta D/W - 1)}$$



# Meyer *et al.*'s Claim

Power vs. Logarithmic Fitts' Law
O. Rioul & Y. Guiard
Fitts' law
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Details
Open Issues

for $n = 2$ :

$$T = a + b\sqrt{\frac{D}{W}}$$

for any $n$ :

$$T = a_n + b_n\sqrt[n]{\frac{D}{W}}$$

let $n \to +\infty$ :

$$T = a' + b' \ln(\frac{D}{W})$$

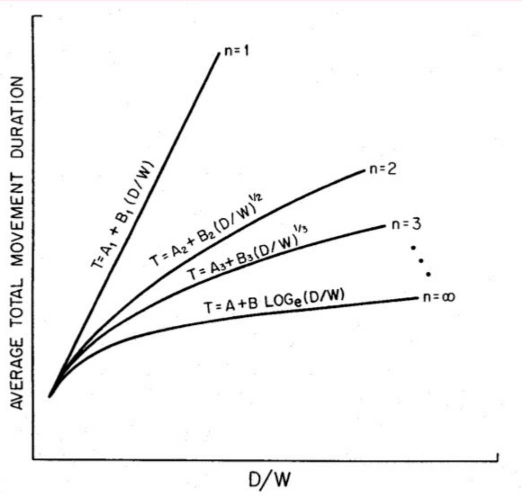



FIG. 6.13. Hypothetical average total movement durations (*T*) versus the target distance–width ratio, *D/W*. (The solid curves show predictions made by the optimized multiple-submovement model as a function of the maximum submovement number, *n*. In each case for which *n* has a finite value, the predicted relation is a quasi power function whose exponent equals 1/*n*. As *n* grows larger, this relation approaches a logarithmic function, paralleling Fitts' Law.)

Conclusion : The power law encompasses the logarithmic law.

# Meyer *et al.*'s Claim is False



## Proof

As $n \to +\infty$ (Guiard et al., 2001)

$$\sqrt[n]{\frac{D}{W}} = \exp \frac{\ln \frac{D}{W}}{n} \to 1$$

So if there exists sequences $(a_n, b_n)$ s.t.

$$T = a_n + b_n \sqrt[n]{\frac{D}{W}} \to a' + b' \ln(\frac{D}{W})$$

then $b' = 0$ (contradiction) Q.E.D.

## Question

Validity of the power law $T = a_n + b_n \sqrt[n]{\frac{D}{W}}$ ?



# Detailed Calculation



Let $T = f_n(D/W)$ after $n$ submovements. Then by the stochastic optimized-submovement model, for any $n > 1$ :

$$f_n(D/W) = \min_s \{ \underbrace{\tfrac{D/W - 1/2}{s}}_{T_i} + E_{|\Delta| > W/2} \big( f_{n-1}(\tfrac{|\Delta|}{W}) \big) \}$$

To simplify, let $\Delta$ be *uniformly* distributed in $(-\frac{Ws}{2}, \frac{Ws}{2})$ (Smith, 1988)

$$f_n(y) = \min_s \{ \tfrac{y - 1/2}{s} + \frac{2}{s} \int_{1/2}^{s/2} f_{n-1}(x) \mathrm{d}x \}$$

By induction $f_n$ vanishes at $y = 1/2$ and is strictly increasing and regular for $y > 1/2$.

# Detailed Calculation (cont'd)



Making the first derivative vanish, when $s = s(y)$ is optimal :

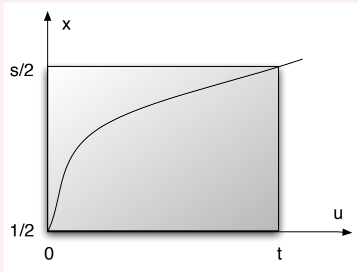
$$0 = -\tfrac{y-1/2}{s^2} - \frac{2}{s^2}\int_{1/2}^{s/2} f_{n-1}(x) + \frac{1}{s} f_{n-1}(\tfrac{s}{2}) \iff f_n(y) = f_{n-1}(\tfrac{s}{2})$$

so letting $y = g_n(t)$ denote the inverse function of $t = f_n(y)$ :

$$f_n(y) = \tfrac{y-1/2}{s} + \frac{2}{s}\int_{1/2}^{s/2} f_{n-1}(x)\mathrm{d}x$$

$$t = \tfrac{g_n(t)-1/2}{s} + \frac{2}{s}\int_{g_{n-1}(0)}^{g_{n-1}(t)} f_{n-1}(x)\mathrm{d}x$$

$$t = \tfrac{g_n(t)-1/2}{s} + \frac{2}{s}\left(\frac{s}{2}\cdot t - \int_0^t g_{n-1}(u)\mathrm{d}u\right)$$




by the inverse function integration theorem.



# Detailed Calculation (cont'd and ending)

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One obtains the relative distance $y = \frac{D}{W} = g_n(t)$ as a function of time $t$ :

$$g_n(t) = \frac{1}{2} + 2\int_0^t g_{n-1}(u)\mathrm{d}u$$

$$= \frac{1}{2} + \frac{1}{2}(2t) + \frac{1}{2}\frac{(2t)^2}{2} + \cdots + \frac{1}{2}\frac{(2t)^n}{n!} \quad \text{by induction}$$

Therefore, $T$ is the root of the $n$th order equation $\frac{D}{W} = \frac{1}{2}e_n(2T)$

where $\boxed{e_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}$ is the $n$th order truncated exponential. That is,

$$\boxed{T = \frac{1}{2} l_n(2\frac{D}{W})}$$

where $\boxed{l_n = e_n^{-1}}$ is the inverse function of $e_n$.

# Examples (closed form expressions)



$n = 2$ quasi square root law (Meyer *et al.*, 1988)

$$l_2(x) = \sqrt{2x-1} - 1 \qquad T = \sqrt{\frac{D}{W} - \frac{1}{4}} - \frac{1}{2}$$

$n = 3$ quasi cube root law

$$l_3(x) = \sqrt[3]{3x + \sqrt{9x^2 - 6x + 2} - 1} - \frac{1}{\sqrt[3]{3x + \sqrt{9x^2 - 6x + 2} - 1}} - 1.$$

$$T = \frac{1}{2}\sqrt[3]{6\frac{D}{W} + \sqrt{36(\frac{D}{W})^2 - 12\frac{D}{W} + 2} - 1} - \frac{1}{2\sqrt[3]{6\frac{D}{W} + \sqrt{36(\frac{D}{W})^2 - 12\frac{D}{W} + 2} - 1}} - \frac{1}{2}.$$

$n = 4$

$$l_4(x) = \frac{1}{2\sqrt{\frac{16(2x-1)}{\sqrt[3]{192x+32\sqrt{32x^3-12x^2+12x-3}-32}} - \sqrt[3]{192x+32\sqrt{32x^3-12x^2+12x-3}-32} - \frac{16}{\sqrt{-\frac{32x-\sqrt[3]{(192x+32\sqrt{32x^3-12x^2+12x-3}-32)^2}+4\sqrt[3]{192x+32\sqrt{32x^3-12x^2+12x-3}-32}-16}{\sqrt[3]{192x+32\sqrt{32x^3-12x^2+12x-3}-32}}}} - 8}} + \frac{1}{2\sqrt{-\frac{32x-\sqrt[3]{(192x+32\sqrt{32x^3-12x^2+12x-3}-32)^2}+4\sqrt[3]{192x+32\sqrt{32x^3-12x^2+12x-3}-32}-16}{\sqrt[3]{192x+32\sqrt{32x^3-12x^2+12x-3}-32}}}} - 1$$



# A Power Law ?



Meyer *et al.* would argue that when $\frac{D}{W}$ is large (when $T$ is large),

$$\frac{D}{W} = \frac{1}{2}e_n(2T) \approx \frac{1}{2}\frac{(2T)^n}{n!}$$

so that we obtain a *nth root (power) law* :

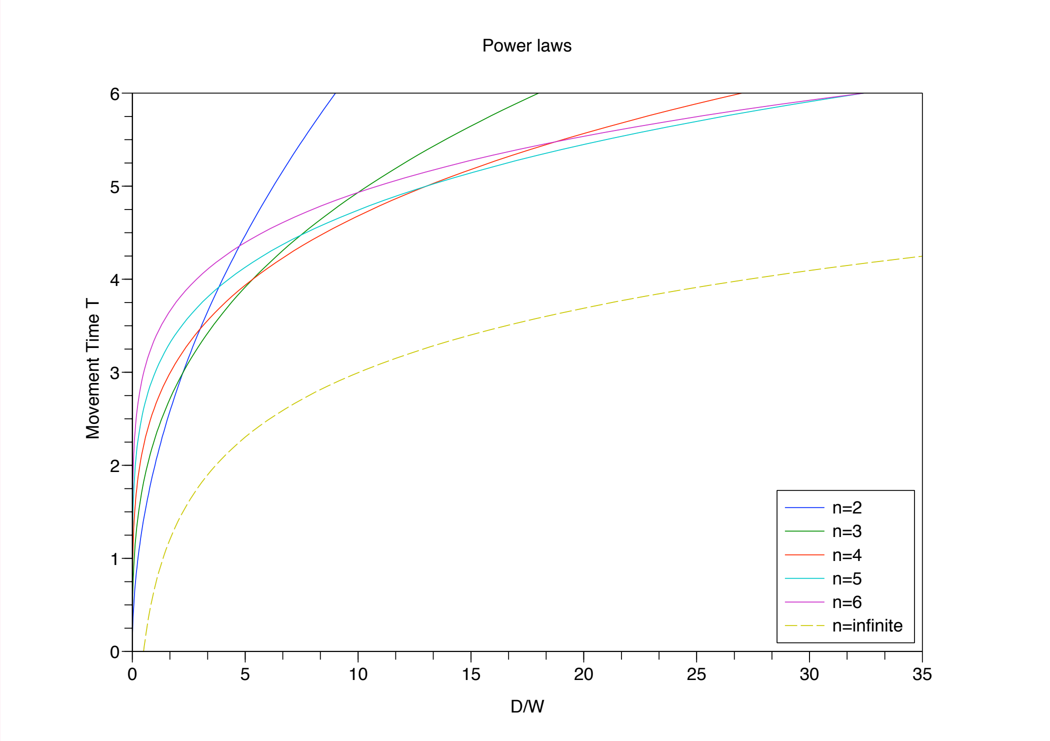
$$T = \frac{1}{2}\sqrt[n]{n!\frac{D}{W}}$$

However, this is not a genuine power model since as $n \to +\infty$,

$$\sqrt[n]{n!} \sim \frac{n}{e} \to +\infty!$$

# A Power Law ?








# A Quasi Exponential Law

Power vs. Logarithmic Fitts' Law
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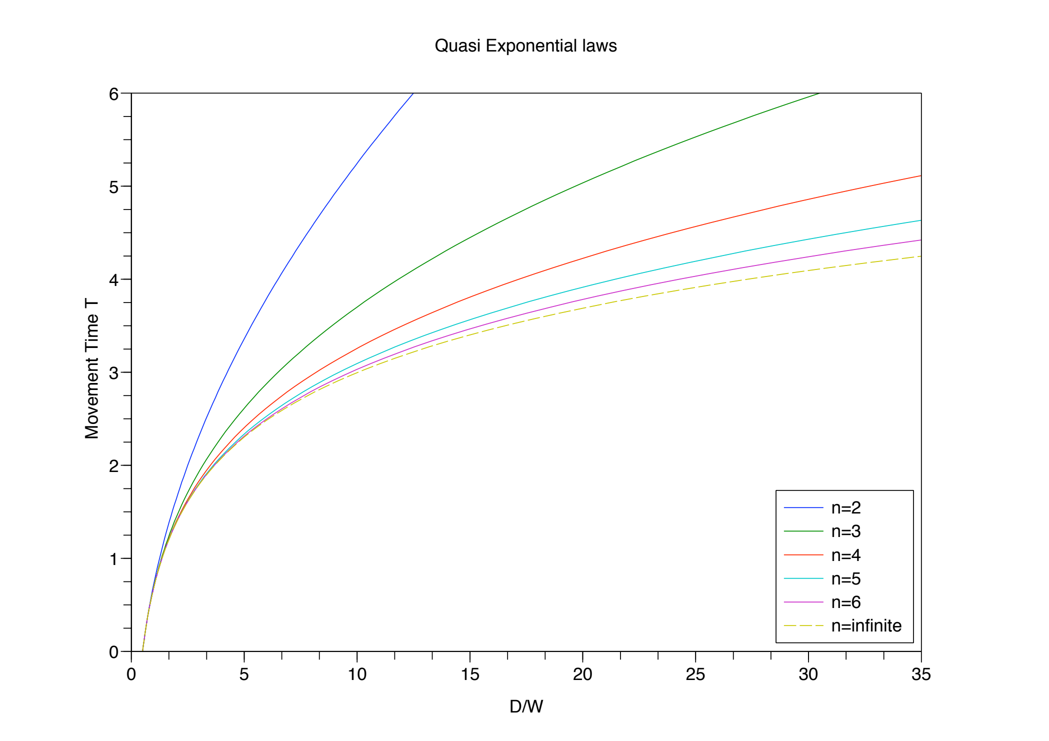
Rather, as $n \to +\infty$,

$$\frac{D}{W} = \frac{1}{2} e_n(2T) \to \frac{1}{2} \exp(2T)$$

and we end up with a *logarithmic law* :

$$T = \frac{1}{2} \ln(2\frac{D}{W})$$

# A Quasi Exponential Law








# Open Issues

- robustness to endpoint $\Delta$'s distribution (uniform vs. Gaussian)
- sensitivity to the number $n$ of submovements
- a submovement theory that is well adapted to information theoretic principles (channels with feedback)
- an experimental method to determine which fits best (non linear regression)

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# Thank you




Comments & Questions