SIMPLE REGULARITY CRITERIA FOR SUBDIVISION SCHEMES. II. THE RATIONAL CASE*

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Abstract. We study regularity properties of special functions obtained as limits of "p/q-adic subdivision schemes." Such "rational" schemes generalize—in a flexible way—binary (or dyadic) subdivision schemes, used in computer-aided geometric design and in functional analysis to construct compactly supported wavelets.

This finds natural applications in the signal processing area, where it may be desirable to decompose a signal into compactly supported wavelets over fractions of an octave. This results in a finer decomposition than in the dyadic case, which corresponds to an octave by octave decomposition.

The main difficulty here, as compared to the dyadic case, is the lack of shift invariance of the limit functions. In this case, a direct extension of Daubechies and Lagarias ideas concerning regularity order estimation becomes impossible, because what they call "two-scale difference equations" cannot be obtained.

Using another, "discrete approach", originally proposed in an earlier work for the dyadic case, we extend most results on regularity properties of limit functions. In particular, we obtain sharp Hölder regularity estimates. We also interpret these results in a Daubechies and Lagarias fashion, by proposing a matrix-based approach.

As opposed to the dyadic case, it is interesting, in the p/q-adic case, to emphasize that the limit function regularity order is *equal* to the maximum convergence rate of its associated subdivision scheme. This new result leads to a simpler and more powerful presentation.

Key words. subdivision algorithms, Hölder regularity, wavelets

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1. Introduction. This is the second part of a series of papers investigating regularity properties of functions, obtained as limits of iterative procedures called subdivision schemes. The preceding paper [17] was devoted to the "binary" or "dyadic" case, in which the subdivision scheme is an infinite collection of sequences g_n^j ($n \in \mathbb{Z}$), labelled by $j \in \mathbb{N}$, and computed using the recursion

$$g_n^{j+1} = \mathcal{G}\{g_n^j\} \tag{1.1}$$

where $\mathcal{G}$ is a dyadic operator which interpolates discrete sequences by convolving them after a change of scale [17].

$$u_n \xrightarrow{\mathcal{G}} v_n = \sum_{k \in \mathbf{Z}} u_k \, g_{n-2k}. \tag{1.2}$$

All sequences considered in [17] and this paper are real-valued and of finite length. The choice of the (finitely many) "subdivision mask" coefficients g_n governs the behavior of a function $\varphi(x)$, obtained as the limit of the discrete "curves" g_n^j plotted against $n2^{-j}$, as $j \to \infty$ (see [17] for a rigorous definition). In [17], one of us characterized the existence and regularity properties of $\varphi(x)$ by equivalent conditions on the g_n^j's and derived optimal Hölder regularity estimates for $\varphi(x)$ given any subdivision mask g_n. These results can be easily extended to "p-adic" subdivision schemes (the "integer case"), where the number 2 in (1.2) is replaced by any integer $p \geq 2$.

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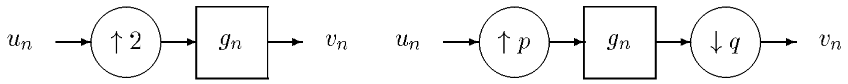


(a) Dyadic interpolation operator (b) p/q-adic interpolation operator

FIG. 1.1. *Flow-graph representations of the interpolation operator $\mathcal{G}$. The squares represent convolution with g_n, while up-sampling and down-sampling operators by a factor n are denoted $\uparrow n$ and $\downarrow n$, respectively. Up-sampling by p inserts $p-1$ zeroes between samples of the input, i.e., maps u_n to $u_{n/p}$ if p divides n, and to 0 otherwise. Down-sampling by q maps u_n to u_{qn}.*

This paper investigates a "rational" extension to (1.1), (1.2): "p/q-adic" subdivision schemes. The only change is that the interpolation operator is now

$$u_n \xrightarrow{\mathcal{G}} v_n = \sum_{k \in \mathbf{Z}} u_k \, g_{qn-pk} \tag{1.3}$$

where p and q are positive integers such that $p > q$. The dyadic case is, of course, recovered by setting $p = 2$ and $q = 1$.

1.1. Motivation. Our motivation comes from the importance of p/q-adic subdivision schemes in filter bank decomposition for signal processing applications [3, 12, 13]. To get an idea of why p/q-adic subdivision schemes constitute an improvement over dyadic ones, consider the flow-graph representation of Fig. 1.1, which will be useful throughout the paper. In the dyadic case, $\mathcal{G}$ (Fig. 1.1 (a)) is the building block used for the construction of wavelet bases, onto which a given signal is decomposed into a set of multiresolution components [7]. This turns out to be an "octave-by-octave" decomposition because the length of g_n^j is roughly multiplied by a factor two at each iteration.

The rational case allows more flexibility since this factor becomes p/q as illustrated in Fig. 1.1 (b). This leads to a decomposition on "fractions" $\log_2 p/q$ of an octave [1, 11, 12, 13]. For $1 < p/q < 2$, the decomposition is thus finer and is a promising technique for applications such as signal compression and analysis of music [11]. It has long been observed that the auditory system performs a third of an octave analysis; this led one of us [2, 3] to implement a perceptual algorithm based on a rational multiresolution analysis with scale factor $p/q = 6/5 \approx \sqrt[3]{2}$.

Being also a natural extension to dyadic subdivision schemes which have long been used in computer-aided geometric design [10], rational subdivision schemes may also find application in this area.

Kovačević and Vetterli [11, 12, 13] were the first to investigate the existence of limit functions of p/q-adic subdivison schemes. They noticed, using an argument of Cohen and Daubechies [6], that for $q > 1$, p/q-adic subdivision schemes could not lead to a wavelet basis for subdivision masks of finite length, as opposed to the dyadic case. This negative result led them to think in [11] that limit functions cannot be obtained. However, one of us [1] showed that limit functions could in fact be obtained, yet they do not satisfy the "shift invariance" property, which we now explain.

In the dyadic case, shifts by $s \in \mathbb{Z}$ are preserved by repeated application of $\mathcal{G}$ in the sense that

$$g_n^j = \mathcal{G}^j\{g_n^0\} \text{ implies } g_{n-2^j s}^j = \mathcal{G}^j\{g_{n-s}^0\}.$$

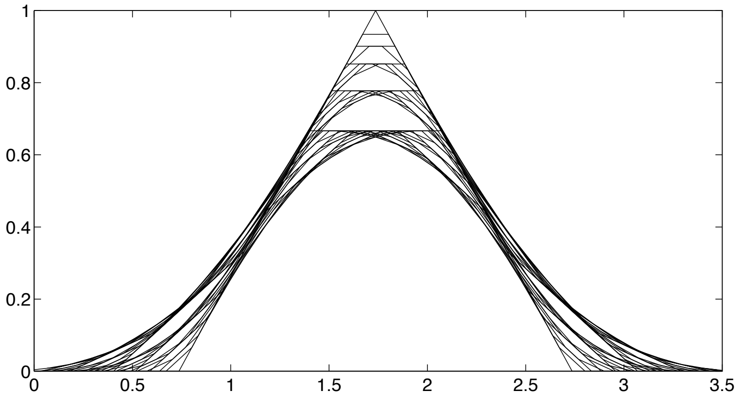


FIG. 1.2. *"Eiffel Tower" example of a converging* $3/2$*-adic subdivision scheme: The subdivision mask of length* 5 *is given by* $\{g_n\} = \frac{1}{3}(1,2,3,2,1)$. *The discrete sequences* $g_n^{j,s}$ *are plotted, joined by segments, against* $n(3/2)^{-j}$ *(here* $j=5$*) for* 32 *distinct values of* s. *Each value of* s *yields a different limit function* $\varphi_s(x)$. *In this picture, the* $\varphi_s(x)$ *have been re-shifted so as to emphasize the differences between the various* $\varphi_s(x+s)$.

This amounts to shifting the resulting limit function $\varphi(x)$ by s [17]. Therefore, as far as regularity properties of compactly supported limit functions are concerned, we can always restrict ourselves to the inital sequence $g_n^0 = \delta_n$ defined by $\delta_n = 1$ if $n = 0$, 0 otherwise.

The situation is different in the rational case, whenever $q > 1$. We may indeed define, similarly as in the dyadic case, functions $\varphi_s(x)$ that are limits, as $j \to \infty$, of the sequences

$$g_n^{j,s} = \mathcal{G}^j\{\delta_{n-s}\}. \tag{1.4}$$

plotted against $n(p/q)^{-j}$ (see Fig. 1.2—a precise definition is given in § 3). This leads us to consider [1] an infinite set of *distinct* compactly supported limit functions $\varphi_s(x)$ labelled by a shift parameter $s \in \mathbb{Z}$. The $\varphi_s(x)$'s reduce to $\varphi_0(x-s)$ when $q = 1$, but are never shifted versions of each other whenever $q > 1$ (at least for subdivision masks of finite length). This is exactly the remark made by Daubechies and Cohen in [6] which shows that rational subdivision schemes cannot lead to a shift invariant wavelet basis.

Even though shift invariance cannot be obtained, the existence and regularity of limit functions carries over, more or less easily, from the dyadic case [17] to the rational one. Achieving regularity should be important in practice because it imposes smooth evolutions of iterated sequences $g_n^{j,s}$, a property that should be useful for signal processing applications for the same reasons as in the dyadic case [12, 15].

The primary aim of this paper is to find the conditions on the subdivision mask g_n under which the associated limit functions $\varphi_s(x)$ exist and are regular. As in the preceding paper [17], regularity is quantified using Hölder spaces. We refer to the dyadic case throughout this paper, pointing out similarities and differences, and stressing the reasons for which some techniques derived in [17] cannot be applied directly. Moreover, this paper is organized similarly as [17] so that the reader can easily compare the two.

1.2. Regularity and shift invariance. There is also an important behavior of p/q-adic subdivision schemes which finds no equivalent in the dyadic case: By selecting very regular limit functions, it was observed numerically [1] that shift invariance was almost satisfied within a small error. In other words, it seems that regularity is beneficial to shift invariance. This might have interesting consequences for implementing an almost shift-invariant "rational" wavelet transform efficiently using p/q-adic subdivision algorithms [1].

This behavior was implicitly discovered by Kovačević and Vetterli in [12], who plotted coefficients of an "iterated filter" g_n^j defined below in § 2, *instead* of the decimated sequences $g_n^{j,s} = g^j_{q^j n - p^j s}$ defined by (1.4). The global behavior of g_n^j led them to conjecture (wrongly) that g_n^j tends to a regular limit function. However, as Kovačević noticed later [11], the obtained curve g_n^j presents rapid oscillations of small amplitude which preclude convergence for g_n^j.

All these observations can be explained as follows. As shown in § 11, they chose an example which in fact corresponds to almost three times differentiable limit functions $\varphi_s(x)$, and shift invariance was almost satisfied. The small oscillations in g_n^j are precisely due to the fact that the decimated curves $g_n^{j,s} = g^j_{q^j n - p^j s}$ converge to different limit functions $\varphi_s(x)$ which are almost, but not quite, shifted versions of each other.

1.3. Organization of the paper. This paper is organized as follows: First, § 2 describes p/q-ary subdivision schemes using the convenient polynomial notation. Then, uniform convergence of discrete sequences towards functions is defined (§ 3) and basic properties of these limit functions are described (§ 4). Continuity is connected to uniform convergence for which a necessary and sufficient condition is derived (§ 5).

To tackle the Hölder regularity problem, we use an original approach which is based on the evaluation, made in § 6 of the convergence rate of modified subdivision schemes, when an interpolation function $\chi(x)$ is appropriately chosen. This is an improvement over what was presented in [17] because it leads to a simpler and more powerful presentation of Hölder regularity estimation (§ 7), which shows that the limit function regularity order is *equal* to the maximum convergence rate.

Based on these theoretical estimates, we derive a practical, sharp Hölder regularity order estimation algorithm (§ 8). We also interpret this result in a Daubechies and Lagarias fashion, by proposing a matrix-based approach (§ 9), which provide alternate upper and lower bounds (§ 10). We conclude the paper with examples.

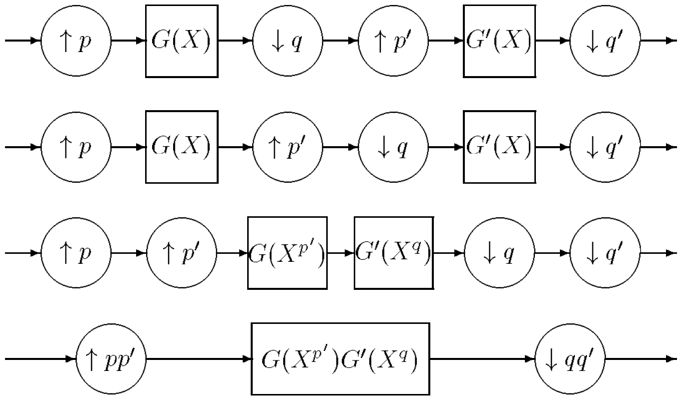
Contrary to the dyadic case [17], we were not able to prove that our estimates are optimal. This is because a simple condition, similar to "stability" as defined in [17], can no longer be derived in the p/q-adic case. However it is proven in [2, Thm. V.8] that our estimates are indeed optimal under conditions that are usually met in signal processing applications. This leads us to conjecture that these estimates are fairly sharp in general.

2. Polynomial notation and fundamental properties. In this paper, we adopt the notations and terminology given in [17]. In particular, to any finite causal sequence u_n $(n = 0, \ldots, L-1)$ we associate the polynomial

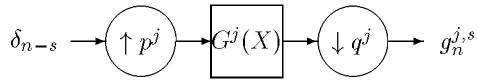
$$U(X) = \sum_{n=0}^{L-1} u_n X^n$$

and use l^1 and l^∞-norms of sequences in terms of polynomials:

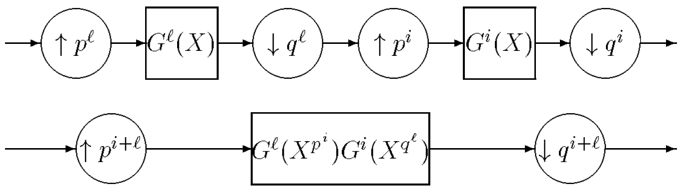
$$||U(X)||_1 = \sum_n |u_n| \text{ and } ||U(X)||_\infty = \max_n |u_n|.$$



(a) Composition property



(b) Flow-graph giving $g_n^{j,s}$



(c) Basic recursion

FIG. 2.1. *Composition properties for iterated p/q-adic subdivision schemes. (a) Four steps in flow-graph notation (using polynomials) that are necessary to rewrite the composition of two interpolation operators $\mathcal{G}$ and $\mathcal{G}'$, with two different rational factors, in a simpler form. All these steps are easy to prove using* (1.3)*. The first one assumes that p' and q are mutually prime. (b) is the resulting flow-graph for $\mathcal{G}^j$, where $G^j(X)$ is given by* (2.2)*. The basic recursion* (2.5)*,* (2.6)*, illustrated in (c), immediately comes from the composition property of (a).*

A useful norm inequality is $||U(X)V(X)||_\infty \leq ||U(X)||_1||V(X)||_\infty$. In this section, we describe p/q-adic schemes using polynomials, and review fundamental recursion formulæ which were derived in [1].

First, whenever p and q are not coprime, we observe that only coefficients g_{dn} are present in (1.3), where d is the greatest common divisor of p and q. We may, therefore, replace p, q, and g_n by p/d, q/d, and g_{dn}, respectively. In this manner, we may always assume that the fraction p/q is written in irreducible form, i.e., p and q are coprime.

Second, we have a simple composition property for $\mathcal{G}$ [1], described graphically in Fig. 2.1 (a). The iterated sequences $g_n^{j,s}$ (1.4) can thus be written in the form

$$g_n^{j,s} = g^j_{q^j n - p^j s}. \tag{2.1}$$

This is illustrated in Fig. 2.1 (b).

To the sequence g_n^j corresponds the polynomial

$$
\begin{aligned}
G^j(X) &= G(X^{q^{j-1}})\,G(X^{pq^{j-2}})\cdots G(X^{p^{j-2}q})\,G(X^{p^{j-1}}) \\
&= \prod_{i=0}^{j-1} G(X^{p^i q^{j-1-i}}). \qquad (2.2)
\end{aligned}
$$

This is a product of "up-sampled" polynomials; the first term is up-sampled by q^{j-1}, and the up-sampling factor is multiplied by p/q from one term to the next. In the dyadic case, we have $q = 1$, and the resulting product has a very simple structure, namely $G(X)\,G(X^2)\,G(X^4)\ldots$.

Two useful recursive forms of $G^j(X)$ can be easily written in polynomial notation. They are derived, similarly as in the dyadic case [17], from the operator recursions $\mathcal{G}^{j+1} = \mathcal{G}\cdot\mathcal{G}^j$ and $\mathcal{G}^{j+1} = \mathcal{G}^j\cdot\mathcal{G}$, respectively. We obtain

$$G^{j+1}(X) = G(X^{q^j})\,G^j(X^p) \qquad (2.3)$$

$$= G^j(X^q)\,G(X^{p^j}) \qquad (2.4)$$

Note that one equation is obtained from the other by exchanging p and q.

The most general recursion, which comes from $\mathcal{G}^{j+\ell} = \mathcal{G}^j\cdot\mathcal{G}^\ell$, is

$$G^{i+\ell}(X) = G^\ell(X^{p^i})G^i(X^{q^\ell}). \qquad (2.5)$$

This is illustrated in Fig. 2.1 (c). Noting that $\mathcal{G}^j\{u_n\} = \sum_k u_k g_n^{j,k}$, (2.5) can be easily written, in terms of sequences, as

$$g_n^{i+\ell,s} = \sum_k g_k^{\ell,s} g_n^{i,k} \qquad (2.6)$$

In particular, we obtain

$$g_n^{j+1,s} = \sum_k g_k^{j,s} g_{qn-pk} \qquad (2.7)$$

$$= \sum_k g_n^{j,k} g_{qk-ps} \qquad (2.8)$$

as rewritings of (2.3) and (2.4), respectively.

All of these recursions are very useful in the sequel, and are easily recovered using the powerful flow-graph notation of Fig. 2.1. Of course, they can be applied to any iterated polynomial of the form

$$U^j(X) = U(X^{q^{j-1}})\,U(X^{pq^{j-2}})\cdots U(X^{p^{j-2}q})\,U(X^{p^{j-1}}) \qquad (2.9)$$

associated to $U(X)$.

3. Definition of convergence. In [17] we discussed various definitions of convergence found in the literature, and showed that all such definitions of *uniform* convergence of the g_n^j's to $\varphi(x)$ are equivalent. Moreover, under very weak conditions [17], uniform convergence in the dyadic case holds whenever $\varphi(x)$ is continuous. By analogy with [17], we restrict ourselves to uniform convergence of p/q-adic subdivision schemes, and adopt the following flexible definition.

DEFINITION 3.1. *For a given shift parameter* $s \in \mathbb{Z}$, *the* p/q*-adic subdivision scheme* $g_n^{j,s}$ converges uniformly *to a limit function* $\varphi_s(x)$ *if, for any sequence of integers* n_j *satisfying*

$$\left| n_j - \left(\frac{p}{q}\right)^j x \right| \leq c \tag{3.1}$$

(where c *is a constant independent of* j*), we have*

$$\sup_x |\varphi_s(x) - g_{n_j}^{j,s}| \to 0 \text{ as } j \to \infty. \tag{3.2}$$

In (3.2), $g_{n_j}^{j,s}$ may be regarded as a function of x, and the flexibility comes from the choice of n_j. A typical exemple is $n_j = \lfloor (p/q)^j x \rfloor$, for which the $g_{n_j}^{j,s}$'s are stepwise constant functions. By (3.1), n_j is chosen such that $n_j(p/q)^{-j}$ stays very close to x as $j \to \infty$, hence $\varphi_s(x)$ can be thought of as uniform limits of discrete curves $g_n^{j,s}$ plotted against $n(p/q)^{-j}$, as mentionned in § 1 and illustrated in Fig. 1.2. Definition 3.1 also implies other ones, such as uniform convergence of linear interpolations of $g_n^{j,s}$ (see § 5) and of smoother interpolations (see § 6).

It is also possible to define *pointwise* convergence of $g_n^{j,s}$ by $\varphi_s(x) = \lim_{j \to \infty} g_{n_j}^{j,s}$. In [1], an example of non-uniform convergence is described taking $G(X) = 1 + X + X^2 + \cdots + X^{p-1}$: Pointwise convergence holds to rectangular functions of different supports, except possibly at the edges. Notice, however, that these functions are not even continuous, whereas this paper is mainly concerned with regular limit functions. Whether regular, non-uniform limit functions exist is an open question.

4. Basic properties of limit functions. This section states several basic properties and simplifications, some of which follow easily from §§ 2 and 3. Most of the material presented here can also be found in [1] and [2].

4.1. Compact support. To prove that limit functions of p/q-adic subdivision schemes are compactly supported, consider the degree of the polynomial $G^j(X)$ (2.2), which reads

$$(L-1)\left(q^{j-1} + pq^{j-2} + \cdots + p^{j-1}\right) = (L-1)\frac{p^j - q^j}{p - q}$$

where L is the length of the subdivision mask g_n. We remark, in passing, that since we assume $p/q > 1$, the length of iterated polynomials $G^j(X)$ is always bounded by $c\,p^j$, a bound we shall often use in this paper.

Now, non-vanishing points of $g_n^{j,s}$ (2.1) are such that

$$0 \leq q^j n - p^j s < (L-1)\frac{p^j - q^j}{p - q}, \tag{4.1}$$

and if $g_n^{j,s}$ converges (pointwise is enough) to a limit function $\varphi_s(x)$, combining (3.1) and (4.1) gives, as $j \to \infty$, the following estimation of the support of $\varphi_s(x)$:

$$\text{support}(\varphi_s(x)) \subseteq \left[s, s + \frac{L-1}{p-q}\right]. \tag{4.2}$$

This will be enough for our purposes. In fact, the lengths of $\varphi_s(x)$ always vary for different values of s, and we refer the interested reader to [1] for refinements.

4.2. Initial sequence. So far, we considered only the "impulse responses" of $\mathcal{G}^j$ to δ_{n-s} (1.4). To justify this restriction, consider a different initial sequence h_n of finite length in the p/q-adic subdivision scheme. By linearity of $\mathcal{G}^j$, the iterated sequence becomes $\sum_s h_s g_n^{j,s}$, and, if we assume convergence of $g_n^{j,s}$ for all s, the resulting limit function is

$$\psi(x) = \sum_s h_s \varphi_s(x). \tag{4.3}$$

Clearly, regularity properties obtained for $\varphi_s(x)$, *globally for all* s, carries over to $\psi(x)$. We need to go the other way round to justify our restriction to the study of $\varphi_s(x)$. Unfortunately, due to the lack of shift invariance, the argument developped in [17, § 4] fails. However, we have the following

PROPOSITION 4.1. *The functions $\psi_s(x) = \sum_k h_{k-s}\varphi_k(x)$ are all regular of some order r if and only if the $\varphi_s(x)$'s are all regular of order r.*

Proof. We have just seen the converse implication. To prove the direct part, consider $(h^{-1})_n$, the convolutional inverse of h_n, which satisfies $\sum_k (h^{-1})_k h_{n-k} = \delta_n$. It is easy to check that $\varphi_s(x) = \sum_k (h^{-1})_k \psi_{k+s}(x)$, the sum being finite because both $\varphi_s(x)$ and $\psi_s(x)$ are compactly supported. Therefore, $\varphi_s(x)$, written as a finite linear combination of the $\psi_k(x)$'s, has regularity order r. The regularity order may be defined using any of the usual spaces, e.g., the spaces of N-times continuously differentiable functions C^N or the Hölder spaces (§ 7). □

An exemple of infinite initial sequence is given in § 4.3.

As a consequence to this proposition, we can restrict ourselves to the study of the $g_n^{j,s}$ as far as the *lowest* regularity order in the family of functions $\psi_s x$ $(s \in \mathbb{Z})$ is concerned. Whether we can characterize regularity properties for a fixed initial sequence via the study of $\varphi_s(x)$ is an open problem, related to the question that the $\varphi_s(x)$ may have different regularity orders for different values of s.

4.3. A necessary condition for convergence. Convergence of the $g_n^{j,s}$ requires an important condition [1] to be fulfilled by g_n.

PROPOSITION 4.2. *If uniform convergence of the $g_n^{j,s}$'s to $\varphi_s(x) \not\equiv 0$ holds, then the subdivision mask meets the constraints*

$$\sum_k g_{n-pk} = 1 \text{ for all } n. \tag{4.4}$$

This condition is equivalent to

$$G(1) = p \tag{4.5}$$

and

$$\frac{1-X^p}{1-X} \text{ divides } G(X). \tag{4.6}$$

Notice that a stronger constraint, although not necessary for convergence, has been used by Kovačević and Vetterli [12], namely the divisibility of $G(X)$ by the factor $\frac{1-X^p}{1-X}\frac{1-X^q}{1-X}$.

The proof of this proposition is an immediate extension of the dyadic case [17, Prop. 4.1]. We sketch it here for completeness.

Proof. In fact, pointwise convergence for some $x \in \mathbb{R}$ is enough. First one easily obtains

$$\sum_k g_{qn-pk} = 1 \text{ for all } n \tag{4.7}$$

from the basic recursion (2.7), where we set $n = n_j$, a sequence of integers satisfying (3.1), and let $j \to \infty$. Since p and q are mutually prime, when n takes the values $0, \ldots, p-1$, qn takes the same values modulo p, possibly in a different order. Therefore we can replace qn by n in (4.7), which gives (4.4). Equation (4.6) immediately follows, e.g. by considering $G(e^{2ki\pi/p})$. □

Note that (4.5) simply normalizes $G(X)$ such that the order of magnitude of $g_n^{j,s}$ is preserved as $j \to \infty$. On the other hand, (4.6), where q has disappeared, is a much deeper condition. It is interpreted by Kovačević and Vetterli [12] as the spectral condition that the "frequency response" $G(e^{i\omega})$ vanishes at the "aliasing frequencies" $\omega = 2k\pi/p$, $k = 1, \ldots, p-1$. This clearly generalizes the dyadic case where the aliasing frequency is π [15], but condition (4.6) is perhaps better understood when considering derivatives of $\varphi_s(x)$ (see § 4.5).

We remark that condition (4.4) can be used to show that Proposition 4.1 fails for infinite initial sequences: If e.g. $h_n = 1$ for all n, we immediately have [1], using (2.7), $\psi(x) = \sum_s \varphi_s(x) \equiv 1$, which is C^∞ whatever the regularity order of the $\varphi_s(x)$'s.

4.4. A two-scale functional equation. In the dyadic case, the limit function $\varphi(x)$ satisfies a "two-scale difference equation" [8, 9] which was used as a starting point by Daubechies and Lagarias for deriving regularity estimates. In the rational case, this approach becomes impossible because of the lack of shift invariance mentionned in § 1.

Indeed, we have the following two-scale equation [1],

$$\varphi_s(x) = \sum_k g_{qk-ps} \varphi_k(\tfrac{p}{q}x) \tag{4.8}$$

which involves an infinite set of distinct limit functions. This equation is easily obtained using (2.8) for $n = n_j$ (3.1) and applying definition 3.1. Now, using (4.8) it is easily proven that the $\varphi_s(x)$'s are not shifted versions of each other. If they were, we would get

$$\varphi(x-s) = \sum_k g_{qk-ps} \varphi(\tfrac{p}{q}x - k)$$

for all s, which, after taking Fourier transforms, and recalling that g_n is a finite-length sequence, leads to a contradiction. This negative statement was first pointed out by Cohen and Daubechies [6].

4.5. Derivatives. In this section, we show that the rational fraction

$$R(X) = \frac{q}{p}\frac{1-X^p}{1-X^q} = \frac{q}{p}\frac{1+X+\cdots+X^{p-1}}{1+X+\cdots+X^{q-1}} \tag{4.9}$$

plays the role of a "regularity factor" in $G(X)$. The precise sense of this is given below. Note that in the dyadic case, $R(X)$ reduces to the polynomial $\frac{1+X}{2}$, which plays the same role in [17].

PROPOSITION 4.3. *Assume that $G(X)$ is of the form $G(X) = R(X)F(X)$, where $R(X)$ is defined by (4.9), and that the p/q-adic schemes associated to $F(X)$ converge (pointwise is enough) towards functions $f_s(x)$. Then, the $g_n^{j,s}$ converge towards differentiable functions $\varphi_s(x)$, and*

$$\partial\varphi_s(x) = f_s(x) - f_{s+1}(x). \tag{4.10}$$

More generally, if $G(X) = R(X)^N F(X)$, under the same assumption for $F(X)$, the $g_n^{j,s}$ converge towards N times differentiable functions $\varphi_s(x)$, and

$$\partial^N\varphi_s(x) = \sum_{k=0}^{N} \binom{N}{k} (-1)^k f_{s+k}(x) \tag{4.11}$$

Here ∂ is the differentiation operator. The proof of the first part ($N = 1$) is given in [1]. The second part follows easily by induction. Note that this result generalizes the result known in the dyadic case [8, 17], where e.g. $\partial\varphi(x) = f(x) - f(x-1)$ for $N = 1$.

In words, this proposition states that the rational schemes generated by $F(X)$ converge towards the Nth-derivatives of the limit functions $\varphi_s(x)$ generated by $G(X)$, provided we choose $h_k = (-1)^k\binom{N}{k}$ as the initial sequence in § 4.1. This provides a very simple way to obtain the derivatives of the limit functions.

In the dyadic case [17], it was always possible to generate arbitrary regular limit functions from an initial kernel $F(x)$ by repeated multiplication of the regularity factor $R(X)$. Now, whenever $q > 1$, $R(X)$ is not a polynomial anymore and the situation becomes more complicated: Since p and q are coprime, so are $1 + X + X^2 + \cdots + X^{p-1}$ and $1 + X + X^2 + \cdots + X^{q-1}$. Therefore, multiplying by $R(X)$ requires that $F(X)$ be divisible by $1 + X + X^2 + \cdots + X^{q-1}$.

Remark. Just as in the dyadic case [17], the presence of N regularity factors in the polynomial $G(X)$, that is:

$$R(X)^N \text{ divides } G(X),$$

is equivalent to the fact that the polynomial

$$\left(\frac{1-X^p}{1-X}\right)^N \text{ divides } G(X).$$

This relates the possibility to differentiate the limit functions and the necessary condition for uniform convergence (Proposition 4.2), which is recovered by setting $N = 1$.

4.6. Sum rules. A simple sum rule property follows from the presence of regularity factors in $G(X)$. This property will be very useful in § 6.

THEOREM 4.4. *Assume that the p/q-adic subdivision scheme converges uniformly, and that $\left(\frac{1-X^p}{1-X}\right)^N$ divides $G(X)$. Then, there exist N real numbers $a_0, a_1, \ldots a_{N-1}$ such that*

$$\sum_s (x-s)^m \varphi_s(x) = a_m \tag{4.12}$$

for all $x \in \mathbb{R}$ and $m = 0, 1, \ldots, N-1$.

Even in the dyadic case, where this sum rule reads

$$\sum_s (x-s)^m \varphi(x-s) = a_m,$$

this result is not widely known in the wavelet literature—possibly except for $N=1$ [19, p. 298]. However, sum rules of this kind are better known in approximation theory and several authors, working in both fields, have already noticed the connection to dyadic wavelets [18, p. 230].

Proof. Since $(\frac{1-X^p}{1-X})^N$ divides $G(X)$, so does $R(X)^N$ by the remark in § 4.5. Using (2.2), it is easily checked that $(R^j(X))^N = (\frac{1-X^{p^j}}{1-X^{q^j}})^N$ divides $G^j(X)$. By the same remark in § 4.5, we have

$$\left(\frac{1-X^{p^j}}{1-X}\right)^N \text{ divides } G^j(X).$$

We now express this relation on the discrete sequences $g_n^{j,s}$. Clearly it implies that

$$\frac{1-X^{p^j}}{1-X} \text{ divides } \frac{d^m}{dX^m} G^j(X).$$

for $m = 0, 1, \ldots, N-1$. Now, to $\frac{d^m}{dX^m} G^j(X)$ corresponds a sequence of the form $P_m(n) g_n^j$, where P_m is a polynomial of degree m. Therefore, the latter relation can be written $\sum_s P_m(n-p^j s) g^j_{n-p^j s} = c_m$, where c_m is independent of n. Since this equality is valid for $m = 0, 1, \ldots, N-1$, it follows that $\sum_s P(n-p^j s) g^j_{n-p^j s}$ is independent of n for every polynomial P of degree $\leq N-1$. In particular $\sum_s (n-p^j s)^m g^j_{n-p^j s}$ does not depend on n for $m = 0, 1, \ldots, N-1$.

We now let $j \to \infty$ by taking $n = q^j n_j$, where n_j is chosen according to (3.1) for a given $x \in \mathbb{R}$. We observe that

$$\sum_s (n_j \tfrac{q^j}{p^j} - s)^m g_{n_j}^{j,s} \text{ does not depend on } x$$

and that this sum is finite independently of j because the support of $g_n^{j,s}$ is bounded. Therefore, letting $j \to \infty$, we obtain the announced formula. □

The a_n may be computed using a recursion formula given in [2, Prop. IV.7].

5. Continuous limit functions. This section derives a simple, but fundamental equivalent condition for uniform convergence of p/q-adic subdivision schemes. This is the first step needed to estimate Hölder regularity orders (§§ 7–8). First we note that, as an immediate generalization of [17, Thm. 7.1], uniform convergence leads to continuous limit functions.

Proposition 5.1. *If, for a given s, $g_n^{j,s}$ converges uniformly, then the limit function $\varphi_s(x)$ is continuous.*

Proof. Let $\varphi_{\mathcal{L}}^{j,s}(x)$ be a sequence of piecewise linear functions obtained by joining the $g_n^{j,s}$ by "segments" as in Fig. 1.2. These functions take $g_n^{j,s}$ as values for $x = n(p/q)^{-j}$, are continuous and compactly supported. Note that the construction of such a function is equivalent to writing

$$\varphi_{\mathcal{L}}^{j,s}(x) = \sum_k g_k^{j,s} \chi(\tfrac{p^j}{q^j} x - k)$$

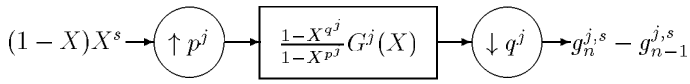


FIG. 5.1. *Finite differences expressed in flow-graph notation.*

where χ is the second order B-spline function (piecewise linear, continuous, and taking the values $0, 1, 0$ for $x = -1, 0, 1$).

Therefore, if we prove that uniform convergence, in the sense of definition 3.1, implies uniform convergence of the $\varphi_{\mathcal{L}}^{j,s}(x)$'s to $\varphi_s(x)$, then $\varphi_s(x)$ is continuous and the proposition is proven.

Choose $n_j = \lfloor (p/q)^j x \rfloor$ as a sequence of integers satisfying (3.1). Since $\varphi_{\mathcal{L}}^{j,s}(x)$ is monotonous on each interval $[n(p/q)^{-j}, (n+1)(p/q)^{-j}]$, we have

$$|\varphi_s(x) - \varphi_{\mathcal{L}}^{j,s}(x)| \leq |\varphi_s(x) - g_{n_j}^{j,s}| + |g_{n_j+1}^{j,s} - g_{n_j}^{j,s}|$$

which, taking suprema over x, and applying (3.2), proves uniform convergence of $\varphi_{\mathcal{L}}^{j,s}(x)$ to $\varphi_s(x)$. □

As mentionned above (§ 3), we do not know whether continuous limit functions could be obtained as non-uniform limits. Following the dyadic case [17], we conjecture, however, that such functions would correspond to very special conditions on g_n, which are not often encountered in practice.

We now derive a necessary and sufficient condition for uniform convergence of the $\varphi_s(x)$ for all s. We need the following fundamental lemma, the rational equivalent to [17, Lemma 7.2], which will also be useful for deriving regularity estimates in § 8.

LEMMA 5.2. *Assume that* $\frac{1-X^p}{1-X}$ *divides* $G(X)$*, and let* $F(X)$ *be the polynomial*

$$F(X) = \frac{1-X^q}{1-X^p} G(X). \tag{5.1}$$

The sequence of first-order differences $g_n^{j,s} - g_{n-1}^{j,s}$ *follows a* p/q*-adic subdivision scheme, with initial sequence* $(1-X)X^s$ *and subdivision mask* $F(X)$*, and we have*

$$\max_n |g_n^{j,s} - g_{n-1}^{j,s}| \leq c \left(\max_{0 \leq n < p^j} \sum_k |f_{n-p^i k}^i| \right)^{j/i} \tag{5.2}$$

where f_n^i *is defined by* (2.9) *and* c *is a constant independent of* j.

The proof is the same as in [17, Lemma 7.2], except that the recursions are somewhat more complicated; we include here to describe the role played by the shift parameters s.

Proof. First, we have

$$\begin{aligned} F^j(X) &= G^j(X) \frac{1-X^{q^j}}{1-X^{pq^{j-1}}} \frac{1-X^{pq^{j-1}}}{1-X^{p^2q^{j-2}}} \cdots \frac{1-X^{p^{j-1}q}}{1-X^{p^j}} \\ &= G^j(X) \frac{1-X^{q^j}}{1-X^{p^j}} \end{aligned}$$

from which the first part of the lemma is obvious considering the flow-graph depicted in Fig. 5.1.

Now, noting $d_n^{j,s} = g_n^{j,s} - g_{n-1}^{j,s}$, and using the general recursion (2.6) applied to f_n^j with initial sequence $(1-X)X^s$, we obtain

$$d_n^{i+\ell,s} = \sum_k d_k^{\ell,s} f^i_{q^i n - p^i k}.$$

Hence,

$$\max_n |d_n^{i+\ell,s}| = \left(\max_n \sum_k |f^i_{q^i n - p^i k}| \right) \max_k |d_k^{\ell,s}|,$$

which, by induction for $j = \ell + ni$, $0 \le \ell < i$, gives (5.2), where $q^i n$ can be replaced by n for the same reason as in the proof of Proposition 4.2. □

THEOREM 5.3. *A p/q-adic subdivision scheme $g_n^{j,s}$ converges uniformly, for all $s \in \mathbb{Z}$, to (continuous) limit functions $\varphi_s(x)$ if and only if $G(X)$ satisfies the basic conditions* (4.5), (4.6), *and*

$$\max_n |g_{n+1}^{j,s} - g_n^{j,s}| \to 0 \text{ as } j \to \infty. \tag{5.3}$$

Moreover, there exists $\alpha > 0$ such that

$$\max_n |g_{n+1}^{j,s} - g_n^{j,s}| \le c \left(\frac{p}{q}\right)^{-j\alpha}. \tag{5.4}$$

Proof. The proof is, again, a variation of [17, Thm. 7.1]. The direct part follows immediately from definition 3.1 and proposition 4.2. Note that this implication works for a given s.

Before proving the converse implication, we prove that (5.4) is implied by (4.4) and (5.3). With the notations of lemma 5.2, we have $(1 - X^{p^j})F^j(X) = (1 - X^{q^j})G^j(X)$, hence, $g_n^{j,s} - g_{n-1}^{j,s} = d_n^{j,s} = f^j_{q^j n - p^j s} - f^j_{q^j n - p^j s - p^j}$. We inverse this relation as

$$f^j_{q^j n - p^j s} = d_n^{j,s} + d_n^{j,s+1} + d_n^{j,s+2} + \cdots$$

The number of terms in the right-hand side is bounded as j increases because the length of $d_n^{j,s}$ is bounded by cp^j. Now, since we assume (5.3) for all s, it follows that $f^j_{q^j n - p^j s}$ tends uniformly to 0, for all s. Applying (5.2) from lemma 5.2, where the sum contains a bounded number of terms, we immediately find (5.4), where we have chosen $(p/q)^{-i\alpha} = \max_n \sum_s |f^i_{q^i - p^i s}|$ and i large enough so that $\alpha > 0$.

We now prove the converse part of the theorem from (5.4). This will work for a given s if (5.4) is satisfied for this s. Consider n_j satisfying (3.1) and let $m_j = qn_{j+1} - pn_j$. This number is bounded by a constant c for all j because of (3.1), and we have

$$\max_{n_j} |g_{n_{j+1}}^{j+1,s} - g_{n_j}^{j,s}| \le \max_{|m_j| \le c} \max_{n_j} |g^{j+1,s}_{(pn_j + m_j)/q} - g_{n_j}^{j,s}|$$

Now, from (2.7), we have (dropping subscripts j for convenience)

$$g^{j+1,s}_{(pn+m)/q} = \sum_k g^{j,s}_{n-k} g_{pk+m}.$$

Therefore, $g^{j+1,s}_{(pn+m)/q} - g^{j,s}_n$ is a convolved version of $g^{j,s}_n$, corresponding to a polynomial multiplication by $U^m(X)$, where $U^m(1) = \sum_k g_{pk+m} - 1 = 0$ by (4.4). Hence, $U^m(X)$ can be written $U^m(X) = (1 - X)V^m(X)$ where $||V^{m_j}(X)||_1$ is bounded. Using the classical norm inequality we obtain

$$\max_{|m_j|\leq c} \max_{n_j} |g^{j+1,s}_{(pn_j+m_j)/q} - g^{j,s}_{n_j}| \leq c' \max_n |g^{j,s}_n - g^{j,s}_{n-1}|,$$

hence, from (5.4), $\max_{n_j} |g^{j+1,s}_{n_{j+1}} - g^{j,s}_{n_j}| \leq c''(p/q)^{-j\alpha}$. Iterating gives, for any $\ell > 0$,

$$\max_{n_j} |g^{j+\ell,s}_{n_{j+\ell}} - g^{j,s}_{n_j}| \leq c'''(p/q)^{-j\alpha}. \tag{5.5}$$

This shows that $g^{j,s}_{n_j}$ is a uniform Cauchy sequence and thus, converges uniformly. □

Condition (5.3) intuitively means that no jumps between two consecutive values of the discrete curves $g^{j,s}_n$ are allowed as $j \to \infty$, and hence, the resulting limit functions are continuous. Note that this theorem was proved globally (for all s), not for a given value of s.

Lemma 5.2 is powerful as far global continuity (for all s) is concerned: It is sufficient that $\max_n \sum_k |f^i_{n-q^ik}| < 1$, for some i, to ensure that all $\varphi_s(x)$'s are continuous. In fact, as seen in § 7, (5.4) implies more than just continuity, namely, that all $\varphi_s(x)$ are Lipschitz of order α.

6. Convergence rate. From the preceding section we know under which conditions $g^{j,s}_n$ converge to limit functions $\varphi_s(x)$. This section is concerned with the convergence rate toward the $\varphi_s(x)$. Our motivation is that a higher convergence rate may be useful in filter bank decomposition for signal processing applications.

At this point, we can mimic the dyadic case [17] to show that the convergence rate of the $g^{j,s}_n$ is of the form $(\frac{p}{q})^{-j\alpha}$, where α cannot be greater than 1. The maximum rate ($\alpha = 1$) is obtained for differentiable limit functions.

However, we can find other discrete sequences which achieve faster convergence ($\alpha > 1$), and it is the purpose of this section to build them. To this aim, it is a good idea to consider interpolating functions. Another nice feature of this construction is that it can be used to simplify proofs concerning regularity in § 7.

6.1. Interpolation. A natural way to relate functions to discrete sequences is to use of an interpolating function $\chi(x)$, as exemplified by the definition of $\varphi^{j,s}_{\mathcal{L}}(x)$ in section 5 and by the approach taken e.g. in [7, 8, 10].

For every scale j we thus define

$$\varphi_{j,s}(x) = \sum_k g^{j,s}_k \chi\left(\frac{p^j}{q^j}x - k\right) \tag{6.1}$$

and consider the convergence of the functions $\varphi_{j,s}(x)$ toward $\varphi_s(x)$ as $j \to \infty$.

Here we assume that $\chi(x)$ is continuous and compactly supported. In particular, if we choose $\chi(x)$ as a piecewise linear centered B-spline function then $\varphi_{j,s}(x) = \varphi^{j,s}_{\mathcal{L}}(x)$ takes the values of the discrete sequence $g^{j,s}_k$ at points $x = k\frac{q^j}{p^j}$, and the convergence of the discrete schemes can be recovered from the convergence of the $\varphi_{j,s}(x)$. We shall, however, use more general interpolating functions.

6.2. Regularity factors and approximation error. Assume that $G(X)$ contains $N+1$ regularity factors and define $F_n(X)$ as

$$F_n(X) = \left(\frac{p}{q}\right)^n \left(\frac{1-X^q}{1-X^p}\right)^{n+1} G(X) \tag{6.2}$$

for $n = -1 \dots N$. $F_m(X)$ follows the recursion $F_{m+1}(X) = R(X)F_m(X)$, where $R(X)$ is defined by (4.9). In terms of discrete sequences, we have

$$\Delta(f_0)_n^{j,s} = g_n^{j,s} - g_{n-1}^{j,s} \tag{6.3}$$

$$\Delta(f_{m+1})_n^{j,s} = (\frac{p}{q})^j \left((f_m)_n^{j,s} - (f_m)_{n-1}^{j,s}\right) \tag{6.4}$$

where for convenience we have used the notation

$$\Delta u_n^{j,s} = u_n^{j,s} - u_n^{j,s+1}$$

for any subdivision scheme $u_n^{j,s}$.

As seen below, it is possible to benefit from factorization (6.2) so that the convergence can be made faster than $\propto (\frac{p}{q})^{-j}$. This will require a particular choice for $\chi(x)$ that will be specified later on.

Now consider the approximation error

$$\varepsilon_{j,s}(x) = \varphi_{j+1,s}(x) - \varphi_{j,s}(x) \tag{6.5}$$

between two consecutive scales. This quantity is essential to our derivation because its exponential decay toward 0 will readily give the convergence rate of the functions $\varphi_{j,s}(x)$ toward $\varphi_s(x)$ as $j \to \infty$. Note that for $j = 0$, we have

$$\varepsilon_{0,s}(x) = \varphi_{1,s}(x) - \varphi_{0,s}(x) = \varphi_{1,s}(x) - \chi(x-s)$$

and for general j, using the recursion equation (2.7) for $g_n^{j,s}$, $\varepsilon_{j,s}(x)$ can be written

$$\varepsilon_{j,s}(x) = \sum_k g_k^{j,s} \varepsilon_{0,k}(\tfrac{p^j}{q^j}x).$$

In order to rewrite $\varepsilon_{j,s}(x)$ in such a way as to exploit the regularity factors in $G(X)$, assume tentatively that it is possible to build a sequence of functions $u_s^m(x)$ such that the following "backward" recursion scheme holds.

$$u_s^0(x) = \varepsilon_{0,s}(x) \tag{6.6}$$

$$u_s^m(x) = u_s^{m+1}(x) - u_{s+1}^{m+1}(x) \tag{6.7}$$

for $m = 0 \dots N$. Then, neglecting possible problems due to infinite summation for the moment, we would have

$$\begin{aligned}
\varepsilon_{j,s}(x) &= \sum_k g_k^{j,s}(u_k^1(\tfrac{p^j}{q^j}x) - u_{k+1}^1(\tfrac{p^j}{q^j}x)) \\
&= \sum_k (g_k^{j,s} - g_{k-1}^{j,s})u_k^1(\tfrac{p^j}{q^j}x) \\
&= \sum_k \Delta(f_0)_k^{j,s}\ u_k^1(\tfrac{p^j}{q^j}x)
\end{aligned}$$

$$= \sum_k (\frac{p}{q})^{-j} \Delta^2 (f_1)_k^{j,s} \; u_k^2(\frac{p^j}{q^j}x)$$

$$\vdots$$

(6.8) $$= \sum_k (\frac{p}{q})^{-jN} \; \Delta^{N+1}(f_N)_k^{j,s} \;\; u_k^{N+1}(\frac{p^j}{q^j}x)$$

Here we have successively used (6.7), a change of variables $k+1 \to k$, (6.3), and repeated application of (6.4).

Equation (6.8) provides an essential information: it directly provides an exponential decay factor $(\frac{p}{q})^{-jN}$ into the expression for $\varepsilon_{j,s}(x)$.

Let us assume, for the sake of the discussion, that $u_s^{N+1}(x)$ *is compactly supported around* s. In the next subsection, we show that this requirement can be satisfied under some specific condition on $\chi(x)$, namely (6.14).

Now, if $u_s^{N+1}(x)$ is compactly supported around s, so are all the $u_s^m(x)$ for $m = 0, \ldots, N+1$ by (6.7). Then, all summations in equations leading to (6.8) are finite, so that (6.8) is valid in full rigor. Moreover, it is easily seen by induction from (6.7), starting from $u_s^0(x)$, that $u_s^{N+1}(x)$ is uniformly bounded over s and we obtain

(6.9) $$\sup_{s,x} |\varepsilon_{j,s}(x)| \leq c(\frac{p}{q})^{-jN} ||F_N^j||_\infty,$$

where c is a constant.

To tackle the term $||F_N^j||_\infty$ in the r.h.s. of (6.9), we need the following

LEMMA 6.1. *Assume that the p/q-adic schemes $g_n^{j,s}$ converge uniformly and let $F_N^j(X)$ be defined as above. Then, there exist two real numbers $C > 0$ and $\alpha > -N$ such that*

(6.10) $$||F_N^j||_\infty \leq C\left(\frac{p}{q}\right)^{-j\alpha}.$$

Proof. In the proof of Theorem 5.3 we have proved, under the same hypothesis, that there exist two positive real numbers C_0 and α_0 such that $||F_0^j||_\infty \leq C_0(\frac{p}{q})^{-j\alpha_0}$.

Owing to (6.2) and to its iterated form $F_N^j(X) = (\frac{p}{q})^{jN}(\frac{1-X^{q^j}}{1-X^{p^j}})^{N+1}G^j(X)$, we can mimic the proof of Theorem 5.3 N times to show that $||F_N^j||_\infty \leq c\,(\frac{p}{q})^{jN}||F_0^j||_\infty$, where c is a constant. We end up with $||F_N^j||_\infty \leq C(\frac{p}{q})^{-j\alpha}$, where C is a constant and $\alpha = -N + \alpha_0 > -N$, which proves the lemma. □

Applying lemma 6.1 to the r.h.s. of (6.9) we obtain

(6.11) $$\sup_{s,x} |\varepsilon_{j,s}(x)| \leq c'(\frac{p}{q})^{-j(N+\alpha)},$$

so that $\varepsilon_{j,s}$ may decrease faster than $(\frac{p}{q})^{-j}$, if we're lucky. From this bound, the convergence rate of the functions $\varphi_{j,s}(x)$ toward $\varphi_s(x)$ will follow immediately.

In order to obtain (6.11), it remains to show that it is possible to build the interpolating function $\chi(x)$ such that $u_s^{N+1}(x)$ is compactly supported around s. Such a construction is given next.

6.3. Conditions on $\chi(x)$. To prove the required compact support property for the $u_s^{N+1}(x)$, it is sufficient to require that the $u_s^0(x)$ satisfy certain sum rules, as described by the following

LEMMA 6.2. *Let $u_s^0(x)$ be a compactly supported function, whose support is contained within $[s+a, s+b]$, where $a < b$ are two real numbers, and such that $u_s^0(x)$ obey the following sum rules*

$$\sum_s (x-s)^m u_s^0(x) = 0 \tag{6.12}$$

for all $m = 0 \dots N$. Then there exists a sequence of functions $u_s^m(x)$ satisfying (6.7), *such that $u_s^{N+1}(x)$ is compactly supported and its support is contained within $[s+a, s+b-N-1]$.*

Proof. The proof is by finite induction on $0 \le n \le N$. We show that the double property for $u_s^n(x)$, i.e., support within $[s+a, s+b-n]$ and sum rules

$$\sum_s (x-s)^m u_s^n(x) = 0$$

for $m = 0 \dots N-n$, propagates to a similar double property for $u_s^{n+1}(x)$, i.e. support within $[s+a, s+b-n-1]$ and sum rules for $m = 0 \dots N-n-1$. Let us see how the induction propagates from $n = 0$ to $n = 1$.

Thanks to the compact support property of $u_s^0(x)$, (6.7) can be inverted to yield a solution

$$u_s^1(x) = \sum_{k \ge 0} u_{s+k}^0(x) = -\sum_{k<0} u_{s+k}^0(x)$$

The second equality follows from the first one and the sum rule property for $m = 0$.

From the first equality, the support of $u_s^1(x)$ is contained within $\cup_{k\ge 0}[s+k+a, s+k+b] = [s+a, +\infty[$, while from the second it is contained within $\cup_{k<0}[s+k+a, s+k+b] =]-\infty, s+b-1]$. This clearly shows that the support of $u_s^1(x)$ is contained within $[s+a, s+b-1]$.

To show how the sum rules propagate, consider, for $m = 0 \dots N$, the polynomials $P_m(X) = (X-m+1)(X-m+2)\cdots(X)$, which obey the recursion $P_m(X) - P_m(X+1) = -mP_{m-1}(X)$. Then from (6.7), sum rules become

$$\begin{aligned} 0 = \sum_s P_m(x-s)u_s^0(x) &= \sum_s P_m(x-s)(u_s^1(x) - u_{s+1}^1(x)) \\ &= -m \sum_s P_{m-1}(x-s)u_s^1(x) \end{aligned}$$

for $m = 0 \dots N$, by the induction hypothesis. Since the $P_m(X)$ form a basis of the polynomials of degree $\le N$, we end up with $\sum_s (x-s)^m u_s^1(x) = 0$ for all $m = 0 \dots N-1$.

Thus the induction from $n = 0$ to $n = 1$ is complete. The very same argument easily carries over for each $n = 1, 2, \dots, N$, which proves the lemma. ☐

It remains to find conditions on $\chi(x)$ under which (6.12) holds. To this end, we translate the sum rules property for $u_s^0(x)$ into similar sum rules for $\chi(x)$. We need the following "propagation" lemma.

LEMMA 6.3. *Let $v_s(x)$ and $w_s(x)$ be two functions, compactly supported around s, satisfying the two-scale equation*

$$w_s(x) = \sum_k g_{kq-sp} v_k(\tfrac{p}{q}x). \tag{6.13}$$

If $v_s(x)$ obeys the sum rules $\sum_s (x-s)^m v_s(x) = 0$ for $m = 0 \dots N$, then $w_s(x)$ obeys the same sum rules $\sum_s (x-s)^m w_s(x) = 0$ for $m = 0 \dots N$.

Proof. Consider any polynomial $P(X)$ of degree m less than or equal to N. We have, by the Taylor expansion formula,

$$\begin{aligned}\sum_s P(x-s) w_s(x) &= \sum_{s,k} g_{kq-sp} P(x - k\tfrac{q}{p} + \tfrac{kq-sp}{p}) v_k(\tfrac{p}{q}x) \\ &= \sum_{n=0}^{m} \sum_k \frac{1}{n!} P^{(n)}(x - k\tfrac{q}{p}) v_k(\tfrac{p}{q}x) \textstyle\sum_s g_{kq-sp} \left(\tfrac{kq-sp}{p}\right)^n\end{aligned}$$

The sum over s does not depend on k due to the factorization (6.2) of $G(X)$. Using the sum rule property for $v_s(x)$, the sum on the right-hand side vanishes, which proves the lemma. □

Now assume that the interpolating function satisfies

$$\sum_s (x-s)^m \chi(x-s) = a_m \tag{6.14}$$

for $n = 0..N$, where the a_m are the same as in (4.12). We refer to [2, Thm. V.2] for a proof that it is always possible to find such functions (one possibility is to use a combination of B-spline functions of order greater than or equal to $N+1$).

We set $v_s(x) = \varphi_s(x) - \chi(x-s)$, which satisfies the assumptions of lemma 6.3 by (4.12) and (6.14). By lemma 6.3, $w_s(x) = \varphi_s(x) - \varphi_{1,s}(x)$ satisfies the same sum rules as for $v_s(x)$. It follows that $u_s^0(x) = \varphi_{1,s}(x) - \varphi_{0,s}(x) = v_s(x) - w_s(x)$ also satisfies the same sum rules, as it was to be shown.

6.4. Main theorem. From the preceding discussion it follows, by lemma 6.2, that $u_s^{N+1}(x)$ has bounded support around s, and therefore, that (6.11) is satisfied. We can now state the main theorem of this section.

THEOREM 6.4. *Assume that the p/q-adic schemes $g_n^{j,s}$ converge uniformly and define $F_N(X)$ by (6.2). Choose a compactly supported, continuous function χ satisfying (6.14) and define the approximating functions $\varphi_{j,s}(x)$ by (6.1).*

Then, the convergence rate of the functions $\varphi_{j,s}(x)$ towards $\varphi_s(x)$ when j tends to infinity is given by

$$\sup_{x,s} |\varphi_s(x) - \varphi_{j,s}(x)| \le C \left(\frac{p}{q}\right)^{-j(N+\alpha)} \tag{6.15}$$

where C' is a constant.

Proof. From the preceding discussion we know that (6.11) holds. For convenience, write (6.11) in the form

$$\sup_{s,x} |\varphi_{i+1,s}(x) - \varphi_{i,s}(x)| \le c(\frac{p}{q})^{-i(N+\alpha)}$$

Summing over $i \ge j$ yields (6.15), where $C = \frac{c}{1-(q/p)^{N+\alpha}}$. □

Notice that the result of Theorem 6.4 is optimized when N is the maximum number in (6.2) and (6.14). This is because if we choose $N' < N$ instead of N, we have by the same argument as in the proof of Theorem 6.4, $||F_N^j||_\infty \le c(\frac{p}{q})^{j(N-N')}||F_{N'}^j||_\infty$.

6.5. Fast computation of limit functions. The following corollary shows how to build, in practice, iterated schemes from the $g_n^{j,s}$ whose convergence rate is increased up to a maximum of $(\frac{p}{q})^{-j(N+\alpha)}$.

COROLLARY 6.5. *With the same assumptions as Theorem 6.4, define the discrete sequences* $\gamma_n^{j,s}$ *by the convolution*

$$\gamma_n^{j,s} = \sum_k \chi(n-k) g_k^{j,s}. \tag{6.16}$$

Then we have

$$\sup_{n,s} \left| \varphi_s(n\tfrac{q^j}{p^j}) - \gamma_n^{j,s} \right| \le C' \left(\frac{p}{q}\right)^{-j(N+\alpha)} \tag{6.17}$$

The proof follows easily by replacing x by $n\frac{q^j}{p^j}$ in (6.15).

This corollary is very useful, in practice, for the computation of limit functions $\varphi_s(x)$. The sequence $\gamma_n^{j,s}$ is simply obtained by convolving the iterated sequences $g_n^{j,s}$ by $\chi(n)$, and the convergence toward limit functions $\varphi_s(x)$ at values $x = n\frac{q^j}{p^j}$ can be much faster than that of the $g_n^{j,s}$ themselves (whose convergence rate cannot exceed $(p/q)^{-j}$). This is particularly useful when p/q is close to 1. An illustration is given in Fig. 6.1 for $p/q = 3/2$ and $G(X) = \frac{1}{27}(1 + X + X^2)^4$.

We notice that when we apply the above section to the dyadic case, we obtain, when $\chi(x) = \varphi(x)$, an exact computation of limit functions as described in [17, § 5].

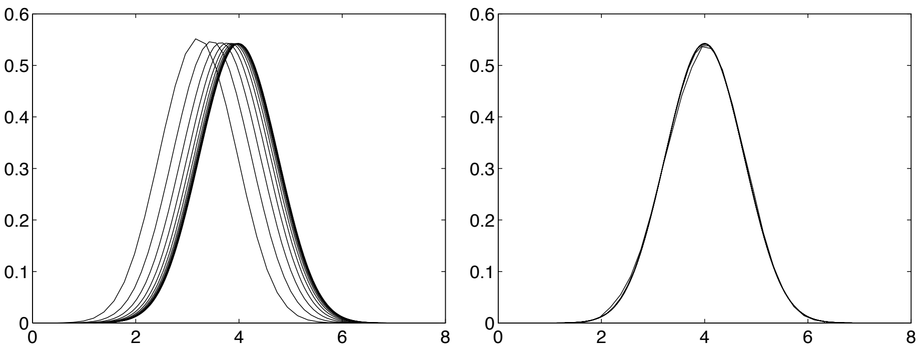


FIG. 6.1. *The discrete sequences* $g_n^{j,0}$ *and* $\gamma_n^{j,0}$ *are plotted against* $n\frac{q^j}{p^j}$ *for* $j = 4, \ldots, 15$ *and 20, at the left and right of the figure, respectively. The convolving sequence* $\chi(n)$ *is given by* $\{\chi(n)\} = \frac{1}{15}(-23, 88, -122, 72)$ *(Indications on how to build this sequence is given in [2, Thm. V.5]). According to Corollary 6.5, we have* $N = 3$ *and the convergence rate of* $\gamma_n^{j,0}$ *is* $1.5^{-j \times 3.709\ldots} \approx 4.5^{-j}$*, whereas for* $g_n^{j,0}$ *it cannot exceed* 1.5^{-j}*.*

In the next section, we show that the convergence rate $N + \alpha$ is, in fact, a sharp Hölder regularity estimate for the limit functions $\varphi_s(x)$. There is an interesting connection between regularity and convergence rate of rational subdivision schemes. This observation was made, in the dyadic case, in [17, Thm. 8.1] for $N = 0$ (only one sum rule was considered). In fact, this connection holds in general as shown next.

7. Hölder regularity of limit functions. In order to quantify regularity accurately, consider the Lipschitz spaces $\dot{C}^\alpha$, $0 < \alpha \le 1$, of functions $f(x)$, defined by the condition $|f(x+h) - f(x)| \leq c|h|^\alpha$ for all x and h, where c is a constant. We say that $\varphi_s(x)$ is *regular of order* α if $\varphi_s(x) \in \dot{C}^\alpha$, even though it is well known that $\dot{C}^1$ corresponds to "almost" differentiable functions only. This definition can be extended to regularity orders greater than 1: a function $f(x)$ is in $\dot{C}^\alpha$, $\alpha > 0$, iff $f(x)$ is $n = (\lceil \alpha \rceil - 1)$ times continuously differentiable, and $f^{(n)}(x)$ is in $\dot{C}^{\alpha - n}$.

Based on the framework of § 6, the following theorem provides an accurate value of the Hölder regularity of the limit functions $\varphi_s(x)$.

THEOREM 7.1. *Assume that the discrete schemes $g_n^{j,s}$ converge uniformly toward $\varphi_s(x)$, and that $(\frac{1-X^p}{1-X})^{N+1}$ divides $G(X)$. Define $F_N(X)$ by (6.2). From lemma 6.1, there exist $C > 0$ and $\alpha > -N$ such that (6.10) holds.*

Then the limit functions $\varphi_s(x)$ are $\dot{C}^{N+\alpha}$.

Proof. Choose an $N+1$ times continuously differentiable, compactly supported interpolating function $\chi(x)$ and define $N_0 = (\lceil \alpha \rceil + N - 1)$ which is ≥ 0 since $\alpha > -N$.

Since $F_N^j(1) = F_N(1)^j = q^j$ and $\deg(F_N^j) = \frac{p^j - q^j}{p-q}\deg(F_N)$, one has $||F_N^j(X)||_\infty \geq |F_N^j(1)|/(\deg(F_N^j)+1) \geq \text{Const} \times (\frac{q}{p})^j$ and therefore, $\alpha \leq 1$ which implies $N_0 \leq N$.

Because $\chi(x)$ is $N+1$ times continuously differentiable, so are $\varepsilon_{j,s}(x)$ and $u_s^{N+1}(x)$ in the fundamental relation (6.8). We now differentiate (6.8) N_0 times to obtain

$$\partial^{N_0}\varepsilon_{j,s}(x) = \sum_k (\frac{p}{q})^{-j(N-N_0)}\ \Delta^{N+1}(f_N)_k^{j,s}\ \ \partial^{N_0} u_k^{N+1}(\frac{p^j}{q^j}x)$$

where ∂ is the differentiation operator and $\varepsilon_{j,s}(x) = \varphi_{j+1,s}(x) - \varphi_{j,s}(x)$. Using (6.10) we bound this relation as follows.

$$|\partial^{N_0}\varphi_{j+1,s}(x) - \partial^{N_0}\varphi_{j,s}(x)| \leq K\left(\frac{p}{q}\right)^{-j(N-N_0+\alpha)} \tag{7.1}$$

where K is a constant.

Because of the choice of N_0, $N - N_0 + \alpha$ is > 0, and the sequence of continuous functions $\partial^{N_0}\varphi_{j,s}(x)$ is a uniform Cauchy sequence in j, which converges uniformly to a continuous limit function when $j \to \infty$.

Since $\varphi_{j,s}(x)$ converges uniformly to $\varphi_s(x)$ and $\partial^{N_0}\varphi_{j,s}(x)$ converges uniformly, it follows that the limit functions $\varphi_s(x)$ are N_0 times continuously differentiable, and that their N_0th derivatives $\partial^{N_0}\varphi_s(x)$ are uniform limits of the $\partial^{N_0}\varphi_{j,s}(x)$. Moreover, by iterating (7.1), we have

$$|\partial^{N_0}\varphi_s(x) - \partial^{N_0}\varphi_{j,s}(x)| \leq K'\left(\frac{p}{q}\right)^{-j(N-N_0+\alpha)} \tag{7.2}$$

where K' is a constant.

It remains to show that $\partial^{N_0}\varphi_s(x)$ is in $\dot{C}^{N-N_0+\alpha}$. Let $0 < h < p/q$, and choose j such that $(p/q)^{-j} < |h| < (p/q)^{-(j-1)}$. From (7.2) and the triangle inequality, we obtain

$$\begin{aligned}\left|\partial^{N_0}\varphi_s(x+h) - \partial^{N_0}\varphi_s(x)\right| &\leq 2K'\left(\frac{p}{q}\right)^{-j(N-N_0+\alpha)} + \left|\partial^{N_0}\varphi_{j,s}(x+h) - \partial^{N_0}\varphi_{j,s}(x)\right| \\ &\leq 2K'|h|^{N-N_0+\alpha} + \left|\partial^{N_0}\varphi_{j,s}(x+h) - \partial^{N_0}\varphi_{j,s}(x)\right|\end{aligned}$$

To prove that $\partial^{N_0}\varphi_s(x)$ is in $\dot{C}^{N-N_0+\alpha}$, it remains to show that the second term in the r.h.s. of this equation is $\leq c|h|^{N-N_0+\alpha}$. Because $\chi(x)$ is $N+1$ times continuously differentiable, this term is $\leq |h| \sup_{x,s} |\partial^{N_0+1}\varphi_{j,s}(x)|$. We would like to bound this (N_0+1)th derivative.

Differentiating the fundamental relation (6.8) N_0+1 times yields (similarly as for (7.1))

$$|\partial^{N_0+1}\varphi_{j'+1,s}(x) - \partial^{N_0+1}\varphi_{j',s}(x)| \leq K'' \left(\frac{p}{q}\right)^{-j'(N-N_0-1+\alpha)}$$

where K'' is a constant. Summing both sides of this latter inequality for $j'=0$ to $j-1$ we find

$$\left|\partial^{N_0+1}\varphi_{j,s}(x)\right| \leq K'' \left(\frac{p}{q}\right)^{-j(N-N_0-1+\alpha)} + \sup_x |\partial^{N_0+1}\chi(x)|$$

Therefore, using the inequality $|h| < (p/q)^{-(j-1)}$, we have

$$\begin{aligned} |h| \left|\partial^{N_0+1}\varphi_{j,s}(x)\right| &\leq K'' \frac{p}{q} |h|^{N-N_0+\alpha} + |h| \sup_x |\partial^{N_0+1}\chi(x)| \\ &\leq K'''|h|^{N-N_0+\alpha} \end{aligned}$$

This shows that $\partial^{N_0}\varphi_s(x)$ is in $\dot{C}^{N-N_0+\alpha}$, which ends the proof. ☐

Comparing our approach to the one taken in [17, Thm. 10.3] for the dyadic case, we see that it is no longer necessary to introduce the set of "almost $\dot{C}^N$" functions as an exception to the equivalent of our Theorem 7.1 for integer regularity orders. This simplification results from the fact that we have used interpolating functions as in the framework of § 6.

We emphasize once more that this theoretical estimate is not proven to be optimal: a condition such as the stability constraint in [17] cannot be extended to the rational case due to the lack of shift invariance of the limit functions. However it is proven in [2, Thm. V.8] that our estimates are indeed optimal under conditions that are usually met in signal processing applications. This leads us to conjecture that these estimates are fairly sharp in general.

8. A practical Hölder regularity estimate. Theorem 7.1 serves as the basis for deriving a practical Hölder regularity estimate for the limit functions $\varphi_s(x)$, given any finite sequence g_n. The term "practical" here means that this estimate should be computed within a finite number of operations. To do this, we transform Theorem 7.1 using Lemma 5.2 to obtain the following theorem, whose proof, omitted here, can be found in [17, Thm. 11.1].

THEOREM 8.1. *Let us recall assumptions and definitions we have already met. Assume that $G(X)$ satisfies* (4.4) *and* (4.6), *that is*

$$G(1) = p$$

and

$$\left(\frac{1-X^p}{1-X}\right)^{N+1} \quad \textit{divides } G(X).$$

with $N \geq 0$. Define $F_N(X)$ by (6.2), *i.e.,*

$$F_N(X) = \left(\frac{p}{q}\right)^N \left(\frac{1-X^q}{1-X^p}\right)^{N+1} G(X)$$

and let $F_N^j(X)$ be the iterated polynomial (2.9) *corresponding to $F_N(X)$, whose associated sequence is $(f_N^j)_n$.*

Now define the Hölder regularity estimate $N + \alpha_N^j$ by the formula

$$\left(\frac{p}{q}\right)^{-j\alpha_N^j} = \max_{0 \le n < p^j} \sum_k |(f_N^j)_{n-p^j k}| \tag{8.1}$$

and let $\alpha_N = \sup_j \alpha_N^j$.

If there exists j such that $N + \alpha_N^j > 0$, then all limit functions $\varphi_s(x)$ are $\dot{C}^{N+\alpha_N^j}$. Moreover, α_N^j tends to α_N as $j \to \infty$ and the $\varphi_s(x)$'s are $\dot{C}^{N+\alpha_N-\varepsilon}$ for any $\varepsilon > 0$.

The point here is that sharp regularity estimates for limit functions can be obtained for a finite number of iterations j, leading to an algorithm with a finite number of steps, which can be easily implemented on a computer. The main difference with the dyadic case [17] is that these estimates are not proven to be optimal for a large class of g_n in this paper, because of technical difficulties (due to the lack of shift property) as mentionned in the introduction. Another difference is that the length of the sequence $(f_N^j)_n$ is now proportional to p^j rather than 2^j. This soon requires much more computation in the rational case, for the same number of iterations. Moreover, the convergence of α_N^j was numerically found slower than in the dyadic case. This is related to the fact that the actual length of $g_n^{j,s}$ grows as $(p/q)^j$ (instead of 2^j) whereas the numerical complexity grows as p^j.

As an illustration, consider the Kovačević and Vetterli example [12] for which $p/q = 3/2$. The first 11 iterations are plotted in figure 8.1. Here the best estimate for regularity r is slightly more than 2.943. This can be compared with an upper bound derived later in § 9, which is a little less than 2.950. More about this example can be found in §11.4.

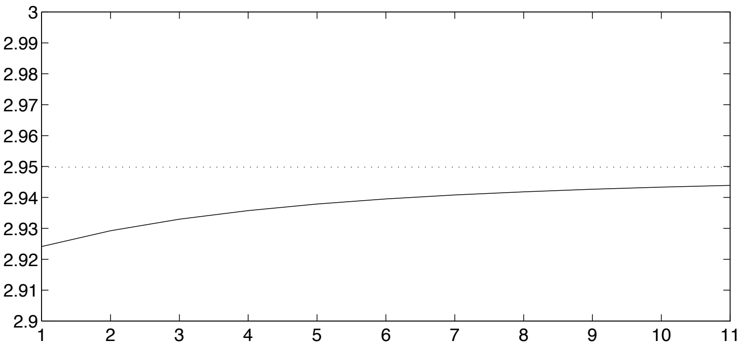
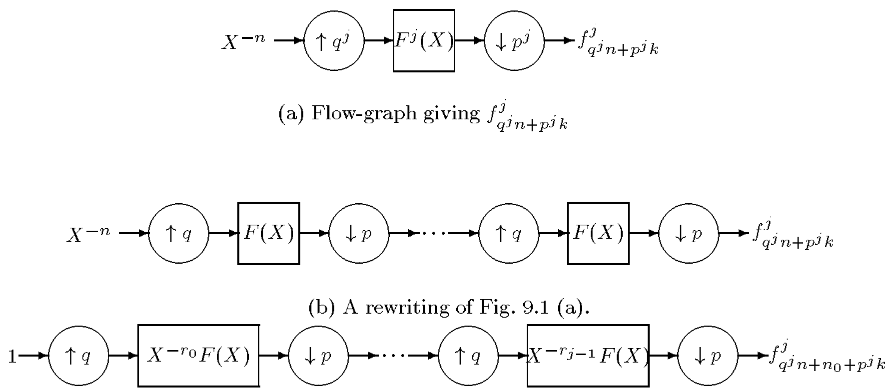


FIG. 8.1. *11 iterations of the algorithm defined in Theorem 8.1 for Kovačević and Vetterli example* $G(X) = \frac{1}{2^3 3^2}(1+X)^3(\frac{1-X^3}{1-X})^3$. *The dotted line shows the upper estimate as computed from* (10.2) *below.*

Notice that in this particular example, convergence of the α_N^j is relatively fast. This is not always the case, as will be seen for the orthonormal example shown in § 11.5, for which 11 iterations—which involves the computation of approximately $8 \times 3^{11} \approx 1{,}400{,}000$ coefficients—are not enough to obtain a fair estimate.

Especially when a high number of iterations is needed to obtain a good estimate, it is important to derive alternate bounds which, although less accurate, will be easier to compute. This is done in the next two sections. As a useful preliminary, we first formulate our algorithm using matrices.

9. Matrix formulation of Hölder regularity estimates. Daubechies and Lagarias [8, 9] proposed a sharp Hölder regularity estimate based on matrix products in the dyadic case. Their work was based on two-scale difference equations, and we have seen in § 4.4 that this approach cannot be applied directly in the rational case. However, one of us [17] showed that Daubechies and Lagarias estimates could be recovered (with slight changes) from the discrete approach that is followed in this paper. Therefore, it should be possible to extend Daubechies and Lagarias method to the rational case using the discrete approach. This is the aim of this section.



(a) Flow-graph giving $f^j_{q^j n+p^j k}$

(b) A rewriting of Fig. 9.1 (a).

(c) A rewriting of Fig. 9.1 (b).

FIG. 9.1. *Derivation of the matrix formulation of Hölder regularity estimates using flow-graph notation (see text).*

We first rewrite formula (8.1) using the flow-graph notation. This derivation is different from the one explained in [17] in the dyadic case and is perhaps easier to understand. First, we consider $\max_n \sum_k |(f^j)_{n+p^j k}|$, where we have dropped the subscript N for convenience. It will be also convenient to rewrite this sum as $\max_n \sum_k |(f^j)_{q^j n+p^j k}|$, which is justified in the proof of lemma 5.2. Now, for a fixed value of n, $(f^j)_{q^j n+p^j k}$, *considered as a sequence in* k, is the output of the flow-graph depicted in Fig. 9.1 (a). This figure looks very similar to Fig. 2.1 (b), which corresponds to the operator $\mathcal{G}^j$. In fact, it is a dual form of Fig. 2.1 (b), where p and q have been interchanged. It is easy to see that Fig. 9.1 (a) can be decomposed as shown in Fig. 9.1 (b), just like $\mathcal{G}^j$ can be decomposed into products of $\mathcal{G}$ as in Fig. 1.1 (b).

Now, we move the initial sequence X^{-n} to the right, by decomposing n into a sequence of integers n_i and r_i, $0 \leq r_i < p$, using the following recursion.

$$\begin{aligned} n &= (pn_0 + r_0)/q \\ n_0 &= (pn_1 + r_1)/q \\ n_1 &= (pn_2 + r_2)/q \\ &\vdots \end{aligned}$$

In this decomposition, n_{i+1} and r_{i+1} are the quotient and remainder, respectively, of the integer division of qn_i by p. The resulting flow-graph, fed by input δ_n (whose associated polynomial is 1), is given in Fig. 9.1 (c). Thus, we end up with a product of j operators labelled by r_i, $i = 0, \ldots, j-1$, each of them corresponding to the following actions:

1. Up-sampling the input u_n by a factor q.
2. Shifting the result by r_i and convolving it by f_n, then
3. Down-sampling by a factor p to produce the output v_n.

Such an operator can be easily rewritten as

$$u_n \xrightarrow{\mathcal{F}^r} v_n = \sum_{k \in \mathbf{Z}} u_k\, f_{pn-qk+r} \tag{9.1}$$

There are p distinct operators of this form, depending on the value of r, $r = 0, \ldots,$ $p-1$.

In matrix form, they are obtained by keeping every pth line and every qth column of a convolution matrix. Thus, the matrix corresponding to $\mathcal{F}^r$ is

$$\mathbf{F}^r = \begin{pmatrix} f_r & f_{r-q} & f_{r-2q} & \cdots & \\ f_{r+p} & f_{r+p-q} & f_{r+p-2q} & f_{r+p-3q} & \cdots \\ f_{r+2p} & f_{r+2p-q} & f_{r+2p-2q} & f_{r+2p-3q} & \cdots \\ f_{r+3p} & f_{r+3p-q} & f_{r+3p-2q} & f_{r+3p-3q} & \cdots \\ \vdots & \vdots & & & \end{pmatrix}, \tag{9.2}$$

whose entry in the ith line ($i \geq 0$) and jth column ($j \geq 0$) is $f_{r+in-jq}$. Of course, this reduces to two different matrices in the dyadic case [17]. In the p-adic case [9], there are p different matrices. Notice that the number of these matrices does not depend on q.

It is to be emphasized that the size of these matrices can be taken *finite*, owing to the fact that the sampling ratio q/p involved is < 1. More precisely, all matrices (9.2) will have size $K \times K$, where K is the minimum number such that an input x_n of length K ($x_n = 0$ for $n < 0$ and $n \geq K$) yields an output of the same length. From (9.1), it is easily seen that

$$K = \left\lfloor \frac{L-1-q}{p-q} \right\rfloor + 1 \tag{9.3}$$

and L is the length of the sequence f_n. In the dyadic case ($p = 2$, $q = 1$), this size is $L-1$ [17]. In the case $p/q = 3/2$, we obtain matrices of size $L-2$. More generally, if $p - q = 1$, matrices have size $L - q$, and are even smaller when $p - q$ is larger[1].

Thus, we have rewritten (8.1) as the l^1-norm of a product of square matrices of size (9.3) and form (9.2). In fact we have

$$\left(\frac{p}{q}\right)^{-j\alpha_N^j} = \max_{0 \leq r_i \leq p-1} \left\| \prod_{i=0}^{j-1} \mathbf{F}_N^{r_i} \right\|_1 . \tag{9.4}$$

This equation can be justified by the following. In what we obtained so far, only the l^1-norm of the *first column* of the matrix product should appear, because the initial

[1]However, for the same length of subdivision mask g_n, L is larger as q is taken large, due to multiplication by powers of $(1 - X^q)$ in (6.2)).

sequence in Fig. 9.1 (c) is δ_n, which corresponds to the vector $(1, 0, \ldots, 0)^t$. However, the other columns are obtained by shifting the initial sequence, which, as we just showed, amounts to changing the values of r_i. Hence, we can replace the l^1-norm of the first column of the matrix product by the l^1-norm over all columns, i.e., the l^1-norm of the matrix product itself, which gives (9.4).

10. Simple lower and upper bounds. We can extend (9.4) to any other matrix norm. Then the α_N^j depend on the choice of the norm, but their upper limit α_N as $j \to \infty$ does not since in finite dimension—here K as defined by (9.3)—all norms are equivalent.

In particular, we consider matrix norms which satisfy the inequality $||\mathbf{AB}|| \leq ||\mathbf{A}||.||\mathbf{B}||$. A useful example is the ℓ^2-norm $||\mathbf{A}||_2 = \sqrt{\rho(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$ where $\rho(\mathbf{M})$ denotes the spectral radius (maximum of moduli of eigenvalues) of $\mathbf{M}$.

In this case we are able to derive simple lower bounds for α. Indeed, collecting products of J consecutive matrices $\mathbf{F}_N^{r_i}$ in (9.4), using the norm inequality and letting $j \to \infty$ one easily finds, for any $J > 0$, the following lower bound for the regularity estimate:

$$N - \frac{1}{J} \log_{p/q} \left(\max_{0 \leq r_i \leq p-1} \left\| \prod_{i=0}^{J-1} \mathbf{F}_N^{r_i} \right\| \right). \tag{10.1}$$

We also derive an upper bound for α_N by considering a matrix norm which derives from a vector norm:

$$||\mathbf{M}|| = \sup_{||\mathrm{u}||=1} ||\mathbf{M}\mathrm{u}||.$$

In particular, this is how ℓ^p-norms $||\mathbf{A}||_p$ are defined.

To obtain this upper bound, let $\mathbf{M} = \prod_{i=0}^{J-1} \mathbf{F}_N^{r_i}$ where $r_i \in (0, 1, \ldots p-1)$. It is easy to see that $||\mathbf{M}|| \geq \rho(\mathbf{M})$ by taking u as an eigenvector associated to the largest eigenvalue of $\mathbf{M}$. Using this inequality in (9.4) we find $\alpha_N \leq -\log_{p/q}(\rho(\mathbf{M}))/J$. Taking all the possibilities for the choice of $(r_0, r_1, \ldots r_{J-1})$ into account, an upper bound of $N + \alpha_N$, which is the best regularity estimate as provided by Theorem 8.1, is

$$N - \frac{1}{J} \log_{p/q} \left(\max_{0 \leq r_i \leq p-1} \rho(\prod_{i=0}^{J-1} \mathbf{F}_N^{r_i}) \right). \tag{10.2}$$

Of course, our lower and upper bounds become sharper as J increases. It is also possible, following the approach taken in [17, Thm 12.1] to generalize Daubechies and Lagarias method [9] for determining a closed-form expression for the best possible regularity order $N + \alpha_N$ of Theorem 8.1.

11. Examples. In this section, we illustrate the matrix formulation (9.4) by applying formulæ (10.1) and (10.2) to simple examples of g_n. The most simple example we can think of is

$$G(X) = \frac{1}{p^N} \left(\frac{1 - X^p}{1 - X} \right)^{N+1} \tag{11.1}$$

for which

$$F_N(X) = \frac{1}{q^N} \left(\frac{1 - X^q}{1 - X} \right)^{N+1} \tag{11.2}$$

In the dyadic case ($p/q = 2$), we obtain B-spline functions (see e.g. [17]), which are $\dot{C}^N$. In the rational case, stepwise, non-continuous limit functions are obtained (see [1]) for $N = 0$. In the following, we concentrate on the case $N > 0$, $p/q = 3/2$.

11.1. Case $N = 1$. This is the "Eiffel tower" example illustrated in Fig. 1.2. We have $F_1(X) = 0.5 + X + 0.5X^2$, which gives three 1×1 matrices $\mathbf{F}_1^r$, $r = 0$, 1 and 2, having entries 1/2, 1 and 1/2, respectively. Here, the lower bound (10.1) and upper bound (10.2) are equal to 1. Therefore, limit functions are $\dot{C}^1$, and this is the best possible regularity order that Theorem 8.1 can provide.

11.2. Case $N = 2$. The next example gives $F_2(X) = \frac{1}{4}(1 + X)^3$. We obtain three 2×2 matrices,

$$\begin{pmatrix} 1/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}, \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix} \text{ and } \begin{pmatrix} 3/4 & 1/4 \\ 0 & 1/4 \end{pmatrix}.$$

In this case, the lower and upper bounds are again equal to $2 - \log(3/4)/\log(3/2) = 2.709\ldots$, and limit functions are $\dot{C}^{2.709\ldots}$, which is optimal for Theorem 8.1.

11.3. Case $N = 3$. We obtain

$$\begin{pmatrix} 1/8 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/8 & 3/4 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 0 & 1/2 \end{pmatrix} \text{ and } \begin{pmatrix} 3/4 & 1/8 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/8 \end{pmatrix}.$$

as matrices, whose norms and spectral radii are all equal to 3/4. Hence, the Hölder regularity order is $3 - \log(3/4)/\log(3/2) = 3.709\ldots$ and the functions are in $\dot{C}^{3.709\ldots}$, an optimal result for Theorem 8.1.

11.4. Kovačević and Vetterli example ($p/q = 3/2$). Kovačević and Vetterli considered the example $G(X) = \frac{1}{72}(1 + X)^3(\frac{1-X^3}{1-X})^3$ in [12]. We find $2.9439 \le r \le 2.9498$ as is shown in figure 8.1. (The l^2 lower estimate with $j = 4$ in (10.1) gives a poorer estimate $r \ge 2.717$.) The limit functions are thus almost three times differentiable. The second derivative $\partial^2\varphi_0(x)$ is plotted in figure 11.1. Although not differentiable, it appears quite smooth.

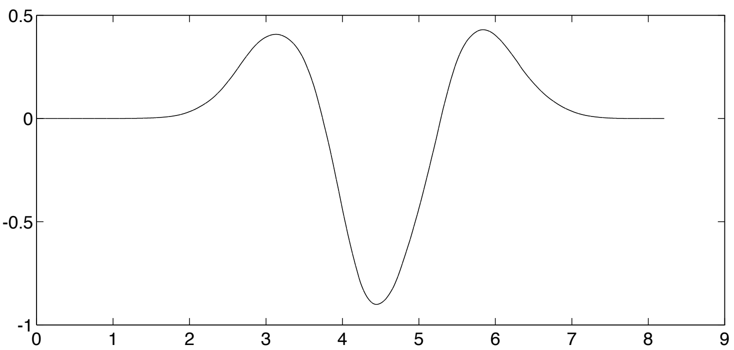


FIG. 11.1. *Second derivative of the limit function $\varphi_0(x)$ generated by $G(X) = \frac{1}{72}(1 + X)^3(\frac{1-X^3}{1-X})^3$*

11.5. Orthonormal sequence. As a last example, we consider an *orthonormal* subdivision mask, i.e., such that $\int \varphi_s(x)\varphi_{s'}(x)\,dx = \delta_{s-s'}$. It was designed using an algorithm presented in [2, 4]. The coefficients g_n are given in table 11.1.

TABLE 11.1
Subdivision orthonormal mask g_n

n	g_n
0	0.0366962809
1	0
2	-0.1576250675
3	-0.2061216574
4	0.1792525378
5	0.8853741953
6	1.1694253766
7	0.8207474622
8	0.2722508722

The first 11 iterations of our estimation algorithm (§ 8) are shown in figure 11.2. After 4 iterations, the estimate is still negative, while the lower bound (10.1) computed with the ℓ^2 norm gives a much better estimate $r = 0.1015$. Although more accurate, (10.1) requires heavier computation.

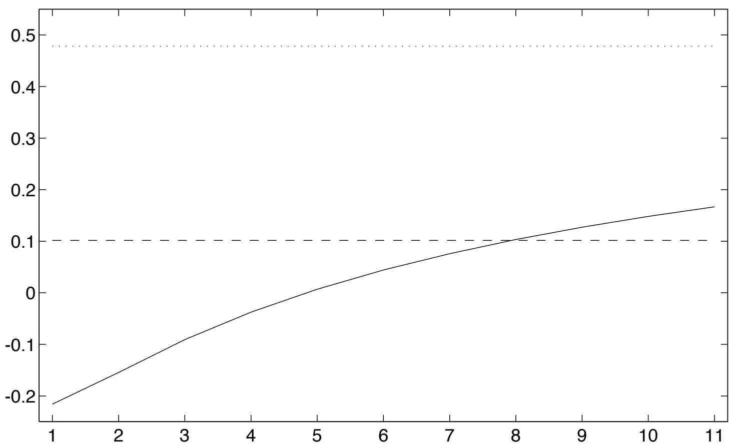


FIG. 11.2. *Regularity estimates for the subdivision mask given in table 11.1 as a function of the iteration number: the dotted line at* $r = 0.478$ *indicates the maximal regularity estimate* (10.2), *while the dashed line at* $r = 0.1015$ *indicates the lower bound* (10.1) *both computed for* $j = 4$

Four of the limit functions are plotted in figure 11.3. Notice how different they look. We have found by experiment that limit functions $\varphi_s(x)$ tend to look alike as regularity increases. More on this in [5].

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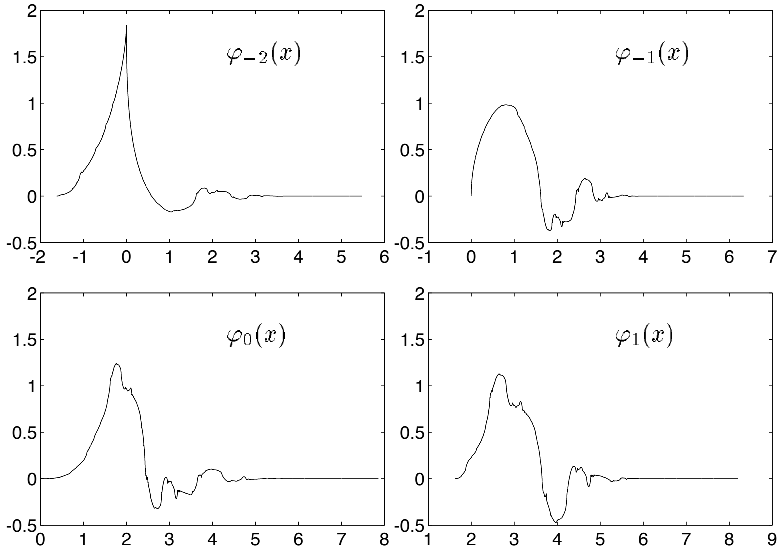

FIG. 11.3. *Limit functions $\varphi_s(x)$ for $s = -2 \ldots 1$ generated by the orthonormal filter given in table 11.1*

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