SIGNAL PROCESSING VI: Theories and Applications
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SIMPLE, OPTIMAL REGULARITY ESTIMATES FOR WAVELETS

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The new criterion of regularity is of increasing interest in applications involving wavelet decomposition schemes. In this paper, regularity is fully characterized on filter taps, resulting in easily implementable, optimal regularity estimates which can be used for any filter.

1. INTRODUCTION

Perhaps the biggest potential of wavelet theory has been claimed for signal compression schemes [1,5] in which the signal is decomposed into several resolution levels using a "discrete wavelet transform (DWT)" [3,8]. In fact, the DWT was soon recognized to be equivalent to an octave-band tree filter bank which was proposed for some time in subband coding of images [9]. In this particular context, the novelty of wavelet theory comes down to the choice of the filters present in a two-band filter bank: "Wavelet" filters are *regular*.

In order to provide an intuitive feel for what regularity represents, consider the following iterated interpolations with low-pass filter impulse response $G(z)$ which are obviously present in DWT's [8].

$$Y(z) = X(z^2)G(z). \tag{1}$$

Iterating (1) j times yields

$$Y^j(z) = X(z^{2^j})G^j(z) \tag{2}$$

where

$$G^j(z) = G(z)G(z^2)G(z^4)\dots G(z^{2^{j-1}}). \tag{3}$$

The sequence g_n^j corresponding to (3) is the equivalent impulse response at jth stage of the reconstruction. Now, for special choices of $G(z)$, the temporal shape of the g_n^j's, plotted against $n2^{-j}$ (i.e., with the same temporal extent), rapidly converges to a "regular" limit function $\varphi(t)$ as $j \to \infty$ (see Fig. 1). However, for "bad" choices of g_n, $\varphi(t)$ may be highly irregular; the iterated scheme may even diverge, even though $G(z)$ is a "good" half-band low-pass filter [8]. Note that filters are assumed FIR here, $\varphi(t)$ is compactly supported. A first definition of the regularity order of $\varphi(t)$ is the number of times it is continuously differentiable; this is clearly a smoothness requirement on the temporal waveforms of the g_n^j's.

The band-pass impulse responses present in a DWT are obtained with the same iterated interpolation procedure

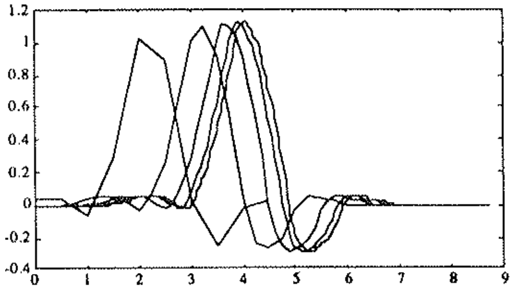

Figure 1. An example of rapidly converging iteration scheme: The g_n^j's are plotted against $n2^{-j}$ for $j = 1, 2, 3, 4$ and 5 iterations, for one of Daubechies "orthonormal" filter of length 10 [3].

as (2), which is initialized with the high-pass filter impulse response h_n [8]. The resulting limit function $\psi(t)$ is the continuous-time wavelet prototype [3,8]. Here we restrict ourselves to the convergence of the g_n^j's toward $\varphi(t)$, because $\varphi(t)$ and $\psi(t)$ share the same regularity properties [3,6].

Several intuitive arguments have been raised which hint that this property should be useful in image coding applications [1,8]. First, requiring that the signal is analyzed by smooth "basis functions" g_n^j and h_n^j ensures that no artificial discontinuity—not due to the signal itself—appears in the transform coefficients, which are inner products of the signal with these basis functions. That is, regularity would lead to a "better" representation of the signal by the transform coefficients. Second, any quantization error made in a coefficient at some resolution level results, at reconstruction, in an error signal that is proportional to the basis function corresponding to this resolution level. It is therefore natural to require that this perturbation be smooth, rather than discountinuous: A discountinuous perturbation is likely to strike the eye more than a smooth one for the same m.s.e. distortion level.

However, understanding the role of regularity in a DWT-based compression scheme requires precise evaluation of it. One difficulty is that it is a mathematical notion which is expressed on $\varphi(t)$ rather than on filters taps g_n. Therefore, the characterization of regularity on any set of coefficents g_n is a difficult problem, which was first addressed in the wavelet context by Daubechies [3]. A number of regularity order estimates, most of them based on the spectrum $|G(e^{i\omega})|^2$, have been investigated [2,3,4]. Unfortunately, these estimates turn out to be suboptimal in general and sometimes computationally expensive.

This paper presents a complete characterization of regularity on the filter taps g_n in simple terms, restricting to the one-dimensional case. This method is original in that all regularity properties of $\varphi(t)$ are translated into equivalent[1] properties of the discrete-time sequences g_n^j.

2. CONTINUITY

It can be shown [6] that as long as the resulting limit function is regular, the type of convergence of the g_n^j is "uniform," which is a strong type of convergence. Uniform convergence of the g_n^j's is in fact equivalent to the existence and *continuity* of $\varphi(t)$ [6]. Continuity (or uniform convergence) [6] is equivalent to the following intuitive conditions.

$$G(1) = 2, \tag{4}$$

$$G(-1) = 0, \tag{5}$$

$$\lim_{j\to\infty} \max_n |g_{n+1}^j - g_n^j| = 0. \tag{6}$$

Condition (4) is simply a normalization requirement on $G(z)$, while (5) is crucial for convergence and regularity, as explained in section 3.1. The basic requirement (6) is that the difference between two successive values of g_n^j tend to zero *uniformly* in n. Hence, no jumps or discontinuities should appear anywhere in the iterated sequences g_n^j as j increases, and the limit function is continuous.

However, even when $\varphi(t)$ is required to be continuous, it may not appear to be smooth at all, as shown in Fig. 2. It is therefore natural to require more, namely that $\varphi(t)$ possess $N > 0$ continuous derivatives. This is done next.

3. DERIVATIVES

The limit function $\varphi(t)$ has regularity order N if its Nth derivative, $d^N\varphi(t)/dt^N$, is continuous. To characterize this on g_n^j, consider the first-order finite difference sequence δg_n^j, defined as the sequence of the *slopes* of the

[1]except for very few pathological cases which are never encountered in practical systems [7]; we here state general results and refer the interested reader to [6] for further details.

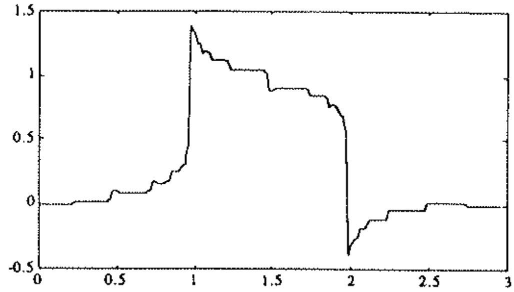

Figure 2. An example of limit function generated by an orthogonal "wavelet" filter. The optimal Sobolev regularity is negative but the Hölder regularity order is in fact 0.0146..., which implies that this limit function is continuous (see section 4).

"discrete curve" g_n^j plotted against $n2^{-j}$.

$$\delta g_n^j = \frac{g_n^j - g_{n-1}^j}{2^{-j}}. \tag{7}$$

The correponding z-transform is $\Delta G^j(z) = 2^j(1 - z^{-1})G^j(z)$. Applying N times the operator δ yields the finite difference of g_n^j of order N, $\delta^N g_n^j$, given by $\Delta^N G^j(z) = 2^{jN}(1 - z^{-1})^N G^j(z)$.

Since the role of the derivative of $\varphi(t)$ of order N is played in the discrete-time domain by $\delta^N g_n^j$, it can be shown [6] that *regularity order N is simply characterized by uniform convergence of $\delta^N g_n^j$*.

3.1. The role of zeroes at $z = -1$ in $G(z)$.

In fact, the Nth derivative of $\varphi(t)$ can obtained from the same iterated interpolation procedure as (2), where N zeroes in $G(z)$ have been removed [6]. As a side result of this and (5), $G(z)$ should have at least $N+1$ zeroes at $z = -1$ to achieve regularity order N. Note that adding one zero at $z = -1$ in $G(z)$ will increase its regularity order by one since removing one amounts to "differentiate." Therefore, zeroes at $z = -1$ have a favorable effect for regularity. This was used by Daubechies in [3] to design regular, orthonormal "wavelet" filters, by imposing as many zeroes at $z = -1$ as possible in $G(z)$ for a given filter length. Note that imposing such zeroes in $G(z)$ amounts to requiring that the frequency response $G(e^{j\omega})$ is "flat" about half the sampling frequency $\omega = \pi$.

However, the effect of zeroes at $z = -1$ may be killed by the *other* zeroes present in $G(z)$. The rest of this paper aims at quantifying the "destructive effect" of zeroes in $G(z)$ that are not located at $z = -1$ in order to quantify regularity accurately.

4. HÖLDER AND SOBOLEV REGULARITY

We first extend regularity orders to arbitrary, real-valued numbers. A popular extension [2,3] uses a spectral approach to regularity which regards it as a spectral localization. This definition is typically based on Sobolev spaces [2,6]. However, it masks the effect of regularity on the temporal waveform of $\varphi(t)$ and does not use *phase* information of $G(e^{j\omega})$. This may be inappropriate: Fig. 2 shows an example of $\varphi(t)$ for which the best Sobolev exponent r is negative, although it can shown that $\varphi(t)$ is in fact continuous.

These limitations are overcome in the following definition of Hölder regularity. The function $\varphi(t)$ is regular of order α, $0 < \alpha \leq 1$, if

$$|\varphi(t+h) - \varphi(t)| < c|h|^{\alpha}. \tag{8}$$

This controls the way infinitesimal *slopes* of $\varphi(t)$, $|\varphi(t+h) - \varphi(t)|/|h|$, grow as h becomes indefinitely small. For higher regularity orders $r = N + \alpha$, $0 < \alpha \leq 1$, the same definition is used on the Nth derivative of $\varphi(t)$. This definition is more compatible with continuity and differentiability because it can be shown that [6] if $\varphi(t)$ is a limit function of g_n^j, then $\varphi(t)$ possess N continuous derivatives *if and only if* it has some Hölder regularity order r greater than N. We have seen that this property is not shared by Sobolev regularity. In the following we therefore concentrate on Hölder regularity.

There is a slight irritation in that $\varphi(t)$ possess N continuous derivatives only when its Hölder regularity is $r = N + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small. To simplify our presentation, we drop the ε in the sequel and regard regularity orders within an arbitrarily small constant.

4.1. Hölder regularity order $0 < \alpha \leq 1$

To characterize Hölder regularity α, $0 < \alpha \leq 1$, on g_n^j, we can do an analogy with (8), replacing $\varphi(t)$ by g_n^j with $t = n2^{-j}$ and $h = 2^{-j}$. This gives

$$|g_{n+1}^j - g_n^j| < c2^{-j\alpha}. \tag{9}$$

This property, along with (4), (5), is indeed equivalent to (8) [6]. This gives an intuitive interpretation of Hölder regularity: The *slopes* of g_n^j plotted against $n2^{-j}$, $|g_{n+1}^j - g_n^j|/2^{-j}$, grow less than $2^{j(1-\alpha)}$ as $j \to \infty$. For example, bounded slopes means that regularity order is 1, i.e., $\varphi(t)$ is almost continuously differentiable. And less regularity allows slopes to increase indefinitely: This explains why $\varphi(t)$, although continuous, may sometimes be quite "nasty" as in Fig. 2.

4.2. Arbitrary regularity orders

Since derivatives of $\varphi(t)$ correspond to finite differences of g_n^j, a natural discrete-time characterization [6] of regularity order $r = N + \alpha$, $0 < \alpha \leq 1$, is (4), (5), and (9) written for $\delta^N g_n^j$, i.e.,

$$|\delta^N g_{n+1}^j - \delta^N g_n^j| < c2^{-j\alpha}. \tag{10}$$

A remarkable fact is that (10) can be extended to negative values of α [6]. That is, even if (10) "fails," i.e., gives a *negative* regularity order for $\delta^N g_n^j$, it can be used to prove that g_n^j has some (positive) regularity if $N > \alpha$. It is therefore worthwhile to consider negative regularity orders. In particular, assume that $G(z)$ has exactly N zeroes at $z = -1$. The maximum number $\alpha \leq 0$ for which (10) holds is then the exact amount of regularity lost due to the destructive effect—discussed in section 3.1—of the zeroes in $G(z)$ that are not located at $z = -1$ [7]

4.3. Regularity and rate of convergence

In practical systems involving a discrete implementation of the DWT, the number of iterations j is limited. It is therefore questionable to study the limit function as $j \to \infty$. However, the rate of convergence of g_n^j to $\varphi(t)$ is faster as regularity is high (the difference tends to 0 as $2^{-j\alpha}$ [6]). The convergence is even faster for higer regularity orders (see Fig. 1).

5. OPTIMAL REGULARITY ESTIMATES

A regularity estimate r is here said to be *optimal* if $\varphi(t)$ is at least regular of order $r - \varepsilon$ and is *not* regular of order $r + \varepsilon$, where $\varepsilon > 0$ is arbitrarily small.

A simple algorithm [6], which was independently derived using the "Littlewood-Paley theory" by Cohen and Daubechies [2], gives the optimal Sobolev regularity order. However this is not optimal for Hölder regularity in general: Hölder regularity is always greater than Sobolev regularity by at most 1/2 [6]. This gives suboptimal Sobolev lower and upper bounds for Hölder regularity. Sobolev regularity depends on the modulus of the spectrum while two filters that differ only by their phase have Hölder regularity orders that differ by at most 1/2. In the following we provide sharp lower and upper bound estimates based on characterization (10).

5.1. Lower bound

Since (10) must be satisfied for infinitely many j's and with an unknown constant c, this is impossible to check in practice. Fortunately, this task can be reduced to a finite-time computer search [6,7]:

Algorithm 1 (Lower bound on Hölder regularity). Let $N > 0$ be the exact number of zeroes at $z = -1$ in lowpass interpolation FIR filter $G(z)$, normalized such that

$G(1) = 2$. If $G(z)$ only has zeroes at $z = -1$, stop. The Hölder regularity order is N. To estimate the amount of regularity lost due to the other zeroes, compute $F(z)$, defined as

$$G(z) = 2^{-N}(1+z^{-1})^N F(z). \tag{11}$$

Let j be any positive integer. Compute the (positive) number

$$\beta_j = \frac{1}{j}\log_2 \max_{0\le n<2^j} \sum_k |f^j_{n-2^j k}|, \tag{12}$$

where f^j_n is given by

$$F^j(z) = F(z)F(z^2)\dots F(z^{2^{j-1}}). \tag{13}$$

The Hölder regularity order of $G(z)$ is at least $N - \beta_j$.

A matrix formulation can be shown [6] to be equivalent to a Hölder regularity estimate which was derived by Daubechies and Lagarias [2,4] using a very different approach. While the method they describe in [4] is only managable for very short filters $G(z)$, Algorithm 1 gives nearly optimal results (as j increases) for any filter: In fact, $N - \beta_j$ tends (at most as $1/j$) to the *optimal* Hölder regularity order as $j \to \infty$ [6]. In practice, the exact (optimal) regularity order r is generally obtained to two decimal places after $j = 20$ iterations. This algorithm can be easily implemented by recursive calls to the same small subroutine [7].

5.2. Upper bound

One possible drawback of Algorithm 1 is its exponentially increasing numerical complexity [7]. Now, assume that one retains only the values $n = 0$ and $2^j - 1$ in the computation of the maximum in (12): This results of course in a much faster algorithm. The obtained estimate clearly gives an *upper bound* of Hölder regularity as $j \to \infty$ since β_j is under-estimated. We give here the matrix formulation of this algorithm, which simplifies to the computation of a spectral radius of one matrix [6]:

Algorithm 2 (Sharp Hölder regularity upper bound). Let $G(z)$, $F(z)$, $N > 0$ be as in Algorithm 1 and let $K > 1$ be the length of $F(z)$. Form the matrix $\mathbf{F} = (\mathbf{F}_{i,j})$, $0 \le i,j \le K-2$, defined by

$$\mathbf{F}_{i,j} = f_{2i-j+1} \tag{14}$$

and compute its spectral radius ρ. The Hölder regularity order of $G(z)$ is bounded by $N - \max(|f_0|, |f_{K-1}|, \rho)$.

The resulting estimates are very close to the optimal Hölder regularity order, as seen in Fig. 3.

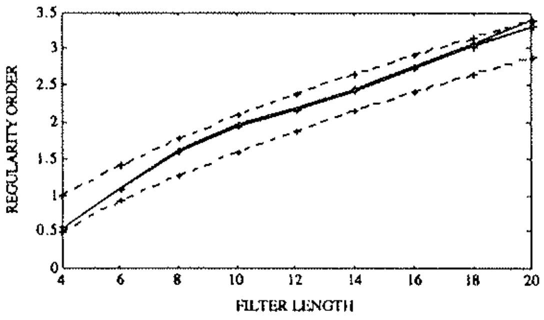

Figure 3. Comparison of regularity estimates: Sobolev lower and upper bound (dashed). Hölder upper and lower bounds (solid) for Daubechies filters given in [3].

6. Conclusion

The method presented here, which characterizes regularity on discrete-time sequences, was found to be powerful: We have provided regularity estimates that are, in contrast with earlier ones [2,3,4], easily implementable, optimal, and of general applicability. Local regularity [4] can also be studied as alternatives to global regularity using this method. [7].

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