# Weighted Automata with Discounting

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**Abstract:** We investigate weighted automata with discounting and their behaviours over semirings and finitely generated graded monoids. We characterize the discounted behaviours of weighted automata precisely as rational formal power series with a discounted form of the Cauchy product. This extends a classical result of Kleene-Schützenberger. Here we show that the very special case of Schützenberger's result for free monoids over singleton alphabets suffices to deduce our generalization.

## 1 Introduction

S.C. Kleene's famous result [Kle56] on the coincidence of regular and rational languages of words has been extended in several directions. Schützenberger [Sch61] characterized the behaviours of weighted finite automata precisely as the rational formal power series. Eilenberg [Eil74] investigated weighted automata and rational formal power series on arbitrary monoids, also cf. [Sak87]. For background on the theory of formal power series, we refer the reader to [Con71, SS78, KS86, BR88, Kui97, Sak03].

In [DK03, DK06], the authors considered formal power series with discounting which generalize classical formal power series. For this setting, the weights of the transitions of a weighted automata are taken as usual in a semiring $K$, but now along a computation sequence these weights are changed by a given endomorphism $\varphi$ of $K$. For instance, in the max-plus semiring of real numbers we might take multiplication with a number $q \geq 0$ as endomorphism; this procedure reflects that later actions might have a smaller weight in usual human evaluation. Discounting has been intensively investigated in Markov decision processes, economics, and game theory [FV97, Sha53]. Related multiplications of series have also been considered in the area of Ore series in difference and differential algebra [Ore33, Gau94, Gal85, BP96]. Formal power series with discounting have recently been further investigated in [ÉK05a, ÉK05b, Kui05, DR07].

In this paper we consider weighted automata and rational formal power series over the class of finitely generated graded monoids with a general form of discounting. A monoid is graded if it carries a length function; in particular, every free monoid is graded, but also, e.g., every trace monoid [DR95], divisibility monoid [DK01], and many others. For discounting we take a collection $(\Phi_m \mid m \in M)$ of endomorphisms $\Phi_m$ of the semiring $K$ (compare [Ulb03] for the case of free monoids). For instance, if $M$ is the free monoid $\Sigma^*$ of all words over an alphabet $\Sigma$ and if $K$ is the max-plus semiring, then our discounting could model different deflations with numbers $q_a$ for $a \in \Sigma$. We will prove that in this general situation the $\Phi$-behaviours of weighted automata coincide with the $\Phi$-rational formal power series over $K$ and $M$ (cf. Theorem 4.2). This contains the results of [Sak87] and [DK03] as special cases, and hence also Schützenberger's theorem as a very particular case. In particular, we will show how the coincidence of the Φ-behaviours of weighted automata and Φ-rational series over $K$ and $M$ can be derived from Schützenberger's theorem; somewhat surprizingly, it even suffices to use Schützenberger's theorem (which applies to arbitrary alphabets) only for singleton alphabets.

Our argument runs as follows (precise definitions are given below). Let $\mathcal{A}$ be a $K$-weighted automaton over $M$ with state set $Q$, say. We wish to show that the Φ-behaviour of $\mathcal{A}$ is a Φ-rational formal power series. In $\mathcal{A}$, the weighted transitions are described by a single $Q \times Q$-matrix with polynomials as entries. Now we proceed in two steps. First, by a diagonalization process we construct a new weighted automaton $\mathcal{A}'$ with state set $Q$ over $M$ and with weights in a polynomial semiring over $K$ and $M$ (whose multiplication reflects Φ). Second, from $\mathcal{A}'$ we construct a weighted automaton $\mathcal{A}''$, also with state set $Q$, over a singleton alphabet $\{x\}$ and with weights in the same polynomial semiring. Schützenberger's theorem, for this polynomial semiring and the alphabet $\{x\}$, shows that the behaviour of $\mathcal{A}''$ is a rational formal power series. Now we transform the rational operations back into Φ-rational operations on power series over $K$ and $M$, and obtain the Φ-rationality of the Φ-behaviour of the automaton $\mathcal{A}$ as claimed. For the converse, stating that each Φ-rational series is the Φ-behaviour of some weighted automaton, we use a similar translation and classical constructions (again for singleton alphabets only). All these transformations and constructions are effective provided all constituents are given effectively.

## 2 Monoids, semirings, and discounting

Let $(M, \circ, \mathbb{1})$ be a *monoid* with unit denoted by $\mathbb{1}$. We will need that $M$ carries a length function $|.|$, i.e., $|.|$ is a monoid morphism from $M$ into $(\mathbb{N}, +, 0)$ such that $|m| = 0$ iff $m = \mathbb{1}$. Then $(M, |.|)$ is called a *graded monoid*. A monoid $M$ is *finitely generated* if there is a finite subset $C \subseteq M$ such that each element of $M$ is a product of elements from $C$. We recall that a finitely generated graded monoid is *finitely factorizing* in the sense that, for every $m \in M$, there are only finitely many sequences $m_1, \ldots, m_k \in M \setminus \{\mathbb{1}\}$ such that $m = m_1 \circ \ldots \circ m_k$.

A *semiring* is an algebra $(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, multiplication distributes over addition, and $k \cdot 0 = 0 \cdot k = 0$ for every $k \in K$. A semiring morphism $\varphi : K \to K$ is also called an endomorphism of $K$. We denote by $\mathrm{End}(K)$ the monoid of all endomorphisms of $K$, with composition as monoid operation and the identity morphism $\mathrm{id}_K$ as unit. Now let $M$ be any monoid and $\Phi : M \to \mathrm{End}(K)$ a monoid morphism. Note that every endomorphism $\varphi \in \mathrm{End}(K)$ gives rise to a morphism $\Phi : M \to \mathrm{End}(K)$ by letting $\Phi(m) = \varphi^{|m|}$ for every $m \in M$; this case was investigated (for $M = \Sigma^*$) in [DK06]. Given any Φ, we may conceive the elements of $M$ as acting (via Φ) on $K$ by putting $k^m = \Phi(m)(k)$ for every $k \in K$ and $m \in M$. For calculations note that $k^{(m \circ m')} = (k^{m'})^m$ for any $k \in K$ and $m, m' \in M$.

**Example 2.1.** Let $K = \mathbb{R}_{\max} = (\mathbb{R}_{\geq 0} \cup \{-\infty\}, \max, +, -\infty, 0)$, the max-plus semiring, where $\mathbb{R}_{\geq 0} = [0, \infty)$ and $-\infty + x = -\infty$ for every $x \in \mathbb{R}_{\max}$. Now choose any $q \in \mathbb{R}_{\geq 0}$ and put $q \cdot (-\infty) = -\infty$. Then the mapping $\overline{q} : \mathbb{R}_{\max} \to \mathbb{R}_{\max} : x \mapsto q \cdot x$ is an endomorphism of $\mathbb{R}_{\max}$ which can be considered as a discounting of $\mathbb{R}_{\max}$. Conversely, every endomorphism of $\mathbb{R}_{\max}$ is of this form (cf. [DK06]) and $\mathrm{End}(\mathbb{R}_{\max})$ is isomorphic to $(\mathbb{R}_{\geq 0}, \cdot, 1)$. Now consider $M = \Sigma^*$, and for every $a \in \Sigma$, choose $q_a \in \mathbb{R}_{\geq 0}$. Then the mapping $\Phi : \Sigma \to \mathrm{End}(\mathbb{R}_{\max})$ with $\Phi(a) = \overline{q_a}$ extends uniquely to a monoid morphism $\Phi : \Sigma^* \to \mathrm{End}(K)$. Here the discounting Φ depends on every letter $a \in \Sigma$; more precisely, for every $x \in \mathbb{R}_{\geq 0}$ and $w \in \Sigma^*$ we have $x^w = \Phi(w)(x) = x \cdot \prod_{a \in \Sigma} q_a^{|w|_a}$ where $|w|_a$ denotes the number of $a$'s in $w$. Clearly, every morphism $\Phi : \Sigma^* \to \mathrm{End}(\mathbb{R}_{\max})$ arises in this way.

*For the rest of this paper, let $(M, \circ, \mathbb{1})$ be a finitely generated graded monoid with length function $|.| : M \to \mathbb{N}$, let $(K, +, \cdot, 0, 1)$ be a semiring, and let $\Phi : M \to \mathrm{End}(K)$ be a monoid morphism.*

## 3 Formal power series and rational operations

A *formal power series* (for short: series) over $M$ and $K$ is a mapping $S : M \to K$. We will write $\langle S, m\rangle$ for $S(m)$, for every $m \in M$. If $\langle S, m\rangle = 0$ for every $m \in M$, then $S$ is also denoted by 0, and $S$ is called *proper* if $\langle S, \mathbb{1}\rangle = 0$. The *support of* $S$ is the set $supp(S) = \{m \in M \mid \langle S, m\rangle \neq 0\}$. The series $S$ is called a *polynomial* (*monomial*) if $supp(S)$ is finite (is a singleton or empty, respectively). We will denote a monomial series $S$ with $supp(S) \subseteq \{m\}$ by $k\,m$ where $k = \langle S, m\rangle$. The *set of all series* (*polynomial series, monomial series*) *over $M$ and $K$* is denoted by $K\langle\langle M\rangle\rangle$ ($K\langle M\rangle$ and $K[M]$, respectively).

Let $S, T \in K\langle\langle M\rangle\rangle$. The *addition* $S + T \in K\langle\langle M\rangle\rangle$ is defined as usual by letting $\langle S+T, m\rangle = \langle S, m\rangle + \langle T, m\rangle$ for every $m \in M$. A family $(S_i \mid i \in I)$ of formal power series $S \in K\langle\langle M\rangle\rangle$ is called *locally finite*, if for every $m \in M$ the set $I_m = \{i \in I \mid \langle S_i, m\rangle \neq 0\}$ is finite. In this case, we define the *sum* $\sum_{i\in I} S_i$ by $\langle \sum_{i\in I} S_i, m\rangle = \sum_{i\in I_m} \langle S_i, m\rangle$ for every $m \in M$. Hence $S = \sum_{m\in M} \langle S, m\rangle\, m$ for every $S \in K\langle\langle M\rangle\rangle$.

Let again $S, T \in K\langle\langle M\rangle\rangle$. We define the $\Phi$-*Cauchy product of* $S$ *and* $T$ as the series $S \cdot_\Phi T \in K\langle\langle M\rangle\rangle$ such that for every $m \in M$ we have

$$\langle S \cdot_\Phi T, m\rangle = \sum_{m=u\circ v} \langle S, u\rangle \cdot \langle T, v\rangle^u.$$

If $\Phi$ is the trivial monoid morphism, i.e., $\Phi(m) = \mathrm{id}_K$ for every $m \in M$, then the $\Phi$-Cauchy product reduces to the usual Cauchy product of formal power series as in [Eil74, SS78]. For motivation, consider again $M = \Sigma^*$, $K = \mathbb{R}_{\max}$, and the morphism $\Phi$ as described in Example 2.1. Then in the above definition, we obtain $\langle T, v\rangle^u = \langle T, v\rangle \cdot \prod_{a\in\Sigma} q_a^{|u|_a}$ which can be considered as the value $\langle T, v\rangle$ discounted by a number depending on the word $u$, the prefix of $w$ preceeding $v$.

In fact, the $\Phi$-Cauchy product can be expressed in terms of the semidirect product of two monoids (cf. for instance [Eil76, Lal79]). For this consider the structure $K \rtimes_\Phi M = (K \times M, \odot_\Phi, (1, \mathbb{1}))$ where $(k, u) \odot_\Phi (h, v) = (k \cdot h^u, u \circ v)$ for all $k, h \in K$ and $u, v \in M$. Then $K \rtimes_\Phi M$ is a monoid, the semidirect $\Phi$-product of $(K, \cdot, 1)$ and $M$. Note that for the monomial series $k\,u, h\,v$ we have $k\,u \cdot_\Phi h\,v = (k \cdot h^u)\,uv$. Hence, if we identify $k\,u$ with the pair $(k, u) \in K \times M$, the $\Phi$-Cauchy product of monomials nicely coincides with the semidirect $\Phi$-product on $K \rtimes_\Phi M$. Also, the $\Phi$-Cauchy product $S \cdot_\Phi T$ of series $S, T \in K\langle\langle M\rangle\rangle$ is the linear extension of the $\Phi$-Cauchy product of monomials to arbitrary series in $K \rtimes_\Phi M$.

Let $S \in K\langle\langle M\rangle\rangle$ and $k \in K$. Then we define the series $k \cdot_\Phi S$ and $S \cdot_\Phi k$ in $K\langle\langle M\rangle\rangle$ by $\langle k \cdot_\Phi S, m\rangle = k \cdot \langle S, m\rangle$ and $\langle S \cdot_\Phi k, m\rangle = \langle S, m\rangle \cdot k^m$ for every $m \in M$. Observe that $k \cdot_\Phi S = k\,\mathbb{1} \cdot_\Phi S$ and $S \cdot_\Phi k = S \cdot_\Phi k\,\mathbb{1}$.

The following result was shown for free monoids $M$ in [DK06] and in [Ulb03].

**Proposition 3.1.** The algebra $K_\Phi M = (K\langle\langle M\rangle\rangle, +, \cdot_\Phi, 0, 1\,\mathbb{1})$ is a semiring (even a left and right $K$-algebra). Moreover, $K_\Phi\langle M\rangle = (K\langle M\rangle, +, \cdot_\Phi, 0, 1\,\mathbb{1})$ is a subsemiring of $K_\Phi M$.

*Proof.* By standards calculations as in Lemma 4 of [DK06]. For the associativity of the $\Phi$-Cauchy product, observe that $k^{m\circ m'} = (k^{m'})^m$ for any $k \in K$ and $m, m' \in M$. □

Let $K'$ be another semiring and $h : K \to K'$ a semiring morphism. Then $h$ is extended to a semiring morphism $h : K\langle\langle M\rangle\rangle \to K'\langle\langle M\rangle\rangle$ by defining $\langle h(S), m\rangle = h(\langle S, m\rangle)$. Moreover, if $(S_i \mid i \in I)$ is a locally finite family of series in $K\langle\langle M\rangle\rangle$, then $h(\sum_{i\in I} S_i) = \sum_{i\in I} h(S_i)$.

Let $S \in K\langle\langle M\rangle\rangle$ be proper. We put $S^{0,\Phi} = 1\,\mathbb{1}$ and $S^{n+1,\Phi} = S \cdot_\Phi S^{n,\Phi}$ for $n \geq 0$. Clearly, $\langle S^{n,\Phi}, m\rangle = 0$ whenever $m \in M$, $n \in \mathbb{N}$, and $|m| < n$. Hence, the family $(S^{n,\Phi} \mid n \geq 0)$ is locally finite and we define the $\Phi$-*star of* $S$ as the series $S^{*,\Phi} = \sum_{n\geq 0} S^{n,\Phi} \in K\langle\langle M\rangle\rangle$.

The operations $+$, $\cdot_\Phi$, and $(.)^{*,\Phi}$ are called the $\Phi$-*rational operations*. For every subset $C \subseteq K\langle\langle M\rangle\rangle$ let $Rat_\Phi(K\langle\langle M\rangle\rangle; C)$ be the smallest subset of $K\langle\langle M\rangle\rangle$ which contains $C$ and is closed under $+$, $\cdot_\Phi$, and $(.)^{*,\Phi}$ where the $\Phi$-star is applied to proper series only. The *class of* $\Phi$-*rational series over* $K$ *and* $M$ is the class $Rat_\Phi(K\langle\langle M\rangle\rangle; K[M])$, which we also denote by $Rat_\Phi(K\langle\langle M\rangle\rangle)$. Thus the $\Phi$-rational series arise by applying the $\Phi$-rational operations to the monomial series.

If $\Phi$ is the trivial monoid morphism, then we drop $\Phi$ from all these denotations and obtain the classical definitions of rational operations.

# 4 Weighted automata

A *finite* $K$-*automaton over* $M$ is a quadruple $\mathcal{A} = (Q, I, E, F)$ where $Q$ is a finite set (of *states*), $I : Q \to K$ and $F : Q \to K$ are mappings (the *initial weight function* and the *final weight function*, resp.), and $E \in (K\langle M\rangle)^{Q\times Q}$ is a matrix (the *edge weighting*, or *transition function*) such that $E_{pq}$ is a proper polynomial for every $p, q \in Q$. The matrix $E$ can be viewed as directed, labeled graph on the set $Q$ containing for any $p, q \in Q$ one edge from $p$ to $q$ labeled with the polynomial $E_{pq}$ (cf. [Eil74] for the case that $K = \mathbb{N}$; and [Sak87] where $E_{pq} \in K\langle\langle M\rangle\rangle$).

A *path* of $\mathcal{A}$ is a sequence $c = q_0 q_1 \dots q_n$ where $n \geq 0$ and $q_i \in Q$; we say that $c$ is a *path from* $q_0$ *to* $q_n$. If $n \geq 1$, then the $\Phi$-*label of* $c$ is the series $lab_\Phi(c) \in K\langle\langle M\rangle\rangle$ defined by $lab_\Phi(c) = E_{q_0 q_1} \cdot_\Phi \dots \cdot_\Phi E_{q_{n-1} q_n}$, and if $n = 0$, then $lab_\Phi(c) = 1\,\mathbb{1}$. The $\Phi$-*weight of* $c$ is the series $weight_\Phi(c) \in K\langle\langle M\rangle\rangle$ defined as

$$weight_\Phi(c) = I(q_0) \cdot_\Phi lab_\Phi(c) \cdot_\Phi F(q_n).$$

In other words, for every $m \in M$ we have $\langle weight_\Phi(c), m\rangle = I(q_0) \cdot \langle lab_\Phi(c), m\rangle \cdot F(q_n)^m$. Note that if here $|m| < n$, then $\langle lab_\Phi(c), m\rangle = 0$ because every $E_{pq}$ is proper. The $\Phi$-*behaviour of* $\mathcal{A}$ is the series $|\mathcal{A}|_\Phi \in K\langle\langle M\rangle\rangle$ defined for every $m \in M$ by

$$\langle |\mathcal{A}|_\Phi, m\rangle = \sum_{c \in P(m)} \langle weight_\Phi(c), m\rangle,$$

where $P(m) = \{c \mid \langle lab_\Phi(c), m\rangle \neq 0\}$ which is a finite set. Observe that the family $(weight_\Phi(c) \mid c$ is a path) is locally finite. Hence we can also write more succinctly

$$|\mathcal{A}|_\Phi = \sum_{c \text{ path}} weight_\Phi(c).$$

The $\Phi$-behaviour of $\mathcal{A}$ can also be expressed in terms of the iteration of $E$. Define $E^* \in (K_\Phi M)^{Q\times Q}$ by $(E^*)_{pq} = \sum_{n\geq 0}(E^n)_{pq}$ where the matrix multiplication $G \cdot H$ of two matrices $G, H \in (K_\Phi M)^{Q\times Q}$ is defined by $(G \cdot H)_{pq} = \sum_{r\in Q} G_{pr} \cdot_\Phi H_{rq}$. Using the distributivity of $K$, it follows that (cf. Proposition 6.1 of [Eil74])

$$(E^*)_{pq} = \sum\nolimits_{c \text{ path from } p \text{ to } q} lab_\Phi(c).$$

As a consequence we derive the following pleasant algebraic description of the $\Phi$-behaviour of $\mathcal{A}$.

**Corollary 4.1.** (cf. Cor. 6.2 of [Eil74]) $|\mathcal{A}|_\Phi = I \cdot_\Phi E^* \cdot_\Phi F$ where we view $I$ as row and $F$ as column vector.

The main goal of this paper is to prove the following result.

**Theorem 4.2.** *Let $M$ be a finitely generated graded monoid, $K$ be a semiring, $\Phi : M \to \mathrm{End}(K)$ be a monoid morphism, and $S \in K\langle\langle M\rangle\rangle$. Then $S$ is the $\Phi$-behaviour of a finite $K$-automaton over $M$ iff $S \in Rat_\Phi(K\langle\langle M\rangle\rangle)$.*

# 5 Diagonalization

In this section we will embed the semiring $K_\Phi M$ into the semiring $(K_\Phi M)\langle\langle M\rangle\rangle$. The latter semiring has the advantage that the Cauchy product is defined 'classically', i.e., without regard to the discounting morphism $\Phi$. Thus we push the discounting morphism $\Phi$ from the Cauchy product of the series into the coefficient semiring. Formally, we define $\omega : K\langle\langle M\rangle\rangle \to (K_\Phi M)\langle\langle M\rangle\rangle$ as follows. For every $S \in K\langle\langle M\rangle\rangle$ and $m \in M$ let

$$\langle \omega(S), m\rangle = \langle S, m\rangle\, m\,.$$

The elements of the range of $\omega$ have a particular form. We call a series $S \in (K_\Phi M)\langle\langle M\rangle\rangle$ *diagonal* if for every $m \in M$ there is a $k \in K$ such that $\langle S, m\rangle = k\,m$. The set of all *diagonal series* is denoted by $\mathbb{D}$. It is clear that $\omega(K\langle\langle M\rangle\rangle) = \mathbb{D}$.

**Lemma 5.1.** $\mathbb{D}$ is a subsemiring of $(K_\Phi M)\langle\langle M\rangle\rangle$, and $\omega : K_\Phi M \to \mathbb{D}$ is a semiring isomorphism. Moreover, $\omega(S^{*,\Phi}) = (\omega(S))^*$ for every proper $S \in K\langle\langle M\rangle\rangle$.

*Proof.* Clearly, $\omega : K_\Phi M \to \mathbb{D}$ is bijective. Now let $S, T \in K_\Phi M$. It is easy to see that $\omega(S + T) = \omega(S) + \omega(T)$. For every $m \in M$, we have

$$\begin{aligned} \langle \omega(S \cdot_\Phi T), m\rangle &= \langle S \cdot_\Phi T, m\rangle\, m = \left(\textstyle\sum_{u,v\in M, m=u\circ v} \langle S, u\rangle \cdot \langle T, v\rangle^u\right) m \\ &= \textstyle\sum_{u,v\in M, m=u\circ v} \left(\langle S, u\rangle \cdot \langle T, v\rangle^u\right)\, m = \sum_{u,v\in M, m=u\circ v} \langle S, u\rangle\, u \cdot_\Phi \langle T, v\rangle\, v \\ &= \textstyle\sum_{u,v\in M, m=u\circ v} \langle \omega(S), u\rangle \cdot_\Phi \langle \omega(T), v\rangle = \langle \omega(S) \cdot \omega(T), m\rangle. \end{aligned}$$

Hence $\omega(S \cdot_\Phi T) = \omega(S) \cdot \omega(T)$. Since $\omega$ is bijective, it follows that $\mathbb{D}$ is a semiring and $\omega$ is a semring isomorphism. Finally let $S \in K_\Phi M$ be a proper series and $m \in M$. Then we have

$$\begin{aligned} & \langle \omega(S^{*,\Phi}), m\rangle \;=\; \langle S^{*,\Phi}, m\rangle\, m \;=\; \left(\textstyle\sum_{n\geq 0}\langle S^{n,\Phi}, m\rangle\right) m \;=\; \textstyle\sum_{n\geq 0}\left(\langle S^{n,\Phi}, m\rangle\, m\right) \\ = \;& \textstyle\sum_{n\geq 0}\langle \omega(S^{n,\Phi}), m\rangle \;=\; \textstyle\sum_{n\geq 0}\langle (\omega(S))^n, m\rangle \;=\; \langle (\omega(S))^*, m\rangle. \end{aligned}$$

□

We put $\mathbb{D}_{mon} = \mathbb{D} \cap (K_\Phi M)[M]$, the *set of diagonal monomial series of* $(K_\Phi M)\langle\langle M\rangle\rangle$. Then the next lemma follows immediately from Lemma 5.1.

**Lemma 5.2.** For every $S \in K\langle\langle M\rangle\rangle$ we have $S \in Rat_\Phi(K\langle\langle M\rangle\rangle)$ iff $\omega(S) \in Rat((K_\Phi M)\langle\langle M\rangle\rangle; \mathbb{D}_{mon})$.

Let $h : K \to K'$ be a semiring morphism and $E \in K^{m\times n}$ a matrix. Then define the matrix $h(E) \in (K')^{m\times n}$ such that $(h(E))_{ij} = h(E_{ij})$. Now we show that $\omega$ preserves the behaviour of weighted automata.

**Lemma 5.3.** Let $\mathcal{A} = (Q, I, E, F)$ be a finite $K$-automaton over $M$ and $\mathcal{A}' = (Q, I', E', F')$ be a finite $K_\Phi M$-automaton over $M$ such that $E' = \omega(E)$, and $I'(q) = I(q)\,\mathbb{1}$ and $F'(q) = F(q)\,\mathbb{1}$ for every $q \in Q$. Then we obtain $\omega(|\mathcal{A}|_\Phi) = |\mathcal{A}'|$.

*Proof.* Let $c = q_0 q_1 \dots q_n$ be a path. Then $weight_{\mathcal{A},\Phi}(c) = I(q_0) \cdot_\Phi lab_\Phi(c) \cdot_\Phi F(q_n) = (I(q_0)\,\mathbb{1}) \cdot_\Phi lab_\Phi(c) \cdot_\Phi (F(q_n)\,\mathbb{1})$. Since $\omega : K_\Phi M \to \mathbb{D}$ is a semiring isomorphism, we can calculate as follows.

$$\begin{aligned} & \omega(weight_{\mathcal{A},\Phi}(c)) \;=\; \omega(I(q_0)\,\mathbb{1}) \cdot \omega(lab_\Phi(c)) \cdot \omega(F(q_n)\,\mathbb{1}) \\ = \;& (I'(q_0)\,\mathbb{1}) \cdot \omega(E_{q_0 q_1} \cdot_\Phi \dots \cdot_\Phi E_{q_{n-1} q_n}) \cdot (F'(q_n)\,\mathbb{1}) \;=\; I'(q_0) \cdot E'_{q_0 q_1} \cdot \ldots \cdot E'_{q_{n-1} q_n} \cdot F'(q_n) \\ = \;& weight_{\mathcal{A}'}(c). \end{aligned}$$

Then $\omega(|\mathcal{A}|_\Phi) = \omega(\sum_{c \text{ path}} weight_{\mathcal{A},\Phi}(c)) = \sum_{c \text{ path}} \omega(weight_{\mathcal{A},\Phi}(c)) = \sum_{c \text{ path}} weight_{\mathcal{A}'}(c) = |\mathcal{A}'|$. □

# 6 Uniformity

Here we replace in the semiring $\mathbb{D}$ of diagonal series the underlying monoid $M$ by the particular free monoid $x^* = \{x^n \mid n \geq 0\}$ as follows. We define the mapping $\psi : M \to x^*$ by $\psi(m) = x^{|m|}$ for every $m \in M$. Clearly, $\psi$ is a monoid morphism. Next we extend $\psi$ to a mapping $\psi : (K_\Phi M)\langle\langle M\rangle\rangle \to (K_\Phi M)\langle\langle x^*\rangle\rangle$ by defining

$$\langle \psi(S), x^n\rangle = \sum_{m\in M, |m|=n} \langle S, m\rangle.$$

Note that the sum is taken over a finite set.

As for $\omega$, the elements of the range of $\psi|_{\mathbb{D}}$ have a particular form. Let us call a series $T \in (K_\Phi M)\langle\langle x^*\rangle\rangle$ *uniform* if $\mathrm{supp}(\langle T, x^n\rangle) \subseteq \{m \in M \mid |m| = n\}$ for every $n \geq 0$. The *set of all uniform series in* $(K_\Phi M)\langle\langle x^*\rangle\rangle$ is denoted by $\mathbb{U}$. Then clearly, $\psi(\mathbb{D}) = \mathbb{U}$. We note that [HK91] used this notion of uniformity to prove that the equivalence of multitape finite automata is decidable (also compare [Sak03]).

**Lemma 6.1.** $\mathbb{U}$ is a subsemiring of $(K_\Phi M)\langle\langle x^*\rangle\rangle$, and the mapping $\psi|_{\mathbb{D}} : \mathbb{D} \to \mathbb{U}$ is a semiring isomorphism. Moreover, $\psi(S^*) = (\psi(S))^*$ for every proper $S \in \mathbb{D}$.

*Proof.* First, let us prove that $\psi|_{\mathbb{D}}$ is injective. Let $S \in \mathbb{D}$. Then for every $m \in M$ there is a $k_m \in K$ such that $\langle S, m\rangle = k_m\, m$. Hence $\langle \psi(S), x^n\rangle = \sum_{m\in M, |m|=n} k_m\, m$ for every $n \geq 0$. Now for every $m \in M$ and $|m| = n$ we have $\langle\langle S, m\rangle, m\rangle = k_m = \langle\langle \psi(S), x^n\rangle, m\rangle$. Now if $S, T \in \mathbb{D}$ with $\psi(S) = \psi(T)$, then the above equality implies that $\langle\langle S, m\rangle, m\rangle = \langle\langle T, m\rangle, m\rangle$ for every $m \in M$. Since $S, T \in \mathbb{D}$, we get $S = T$. Hence $\psi|_{\mathbb{D}} : \mathbb{D} \to \mathbb{U}$ is bijective. The fact that $\psi$ is a semiring morphism follows from general results, compare [SS78], page 13. Hence, $\mathbb{U}$ is a semiring and $\psi|_{\mathbb{D}}$ is a semiring isomorphism. It is easy to calculate that $\psi(S^*) = (\psi(S))^*$ for every proper $S \in \mathbb{D}$. □

We put $\mathbb{U}_{mon} = \mathbb{U} \cap (K_\Phi M)[x^*]$, the *set of all uniform monomial series of* $(K_\Phi M)\langle\langle x^*\rangle\rangle$. Then the following lemma follows immediately from Lemma 6.1.

**Lemma 6.2.** For every $S \in \mathbb{D}$ we have $S \in Rat((K_\Phi M)\langle\langle M\rangle\rangle; \mathbb{D}_{mon})$ iff $\psi(S) \in Rat((K_\Phi M)\langle\langle x^*\rangle\rangle; \mathbb{U}_{mon})$.

Observe that if $h : K \to K'$ is a semiring morphism and $E_1, E_2 \in K^{m\times n}$ are matrices, then $h(E_1+E_2) = h(E_1) + h(E_2)$, and if $E_1 \in K^{l\times m}, E_2 \in K^{m\times n}$, then $h(E_1 \cdot E_2) = h(E_1) \cdot h(E_2)$. Now we show that $\psi$ preserves the behaviour of weighted automata.

**Lemma 6.3.** Let $\mathcal{A} = (Q, I, E, F)$ be a finite $K_\Phi M$-automaton over $M$ and $\mathcal{A}' = (Q, I, E', F)$ a finite $K_\Phi M$-automaton over $x^*$ such that $E' = \psi(E)$. Then $\psi(|\mathcal{A}|) = |\mathcal{A}'|$.

*Proof.* By Corollary 4.1 we have $|\mathcal{A}| = I \cdot E^* \cdot F = (I\,\mathbb{1}) \cdot E^* \cdot (F\,\mathbb{1})$ where $I\,\mathbb{1} = (I(p)\,\mathbb{1})_{p\in Q}$ is a row vector and $F\,\mathbb{1} = (F(q)\,\mathbb{1})_{q\in Q}$ is a column vector. Thus we obtain $\psi(|\mathcal{A}|) = \psi(I\,\mathbb{1})\cdot\psi(E^*)\cdot\psi(F\,\mathbb{1})$. Since $(E^*)_{pq} = \sum_{n\geq 0}(E^n)_{pq}$ we have $\psi((E^*)_{pq}) = \sum_{n\geq 0}\psi((E^n)_{pq}) = \sum_{n\geq 0}(\psi(E^n))_{pq} = \sum_{n\geq 0}((\psi(E))^n)_{pq}$ $= (E'^*)_{pq}$. Finally, $\psi(|\mathcal{A}|) = (I\,\varepsilon) \cdot E'^* \cdot (F\,\varepsilon) = \bar{I} \cdot E'^* \cdot F = |\mathcal{A}'|$. □

# 7 Proof of the main result

Here we prove our main result. First let us recall the classical Schützenberger-result for formal power series. It belongs to folklore and can be shown by standard constructions of automata.

**Proposition 7.1.** (cf. [Sch61]) Let $\Sigma$ be an alphabet, $K$ any semiring, and $S \in K\langle\langle\Sigma^*\rangle\rangle$. Then $S$ is the behaviour of a finite $K$-automaton over $\Sigma^*$ iff $S \in \mathit{Rat}(K\langle\langle\Sigma^*\rangle\rangle)$. Moreover, the following holds.

(1) Let $\mathcal{A} = (Q, I, E, F)$ be a finite $K$-automaton over $\Sigma^*$. Then $|\mathcal{A}| \in \mathit{Rat}(K\langle\langle\Sigma^*\rangle\rangle; C)$ where $C = \{\langle E_{pq}, w\rangle\, w \mid p, q \in Q, w \in \Sigma^*\} \cup \{I(q)\,\varepsilon, F(q)\,\varepsilon \mid q \in Q\}$.

(2) Let $S \in \mathit{Rat}(K\langle\langle\Sigma^*\rangle\rangle; C)$ with a subsemiring $C \subseteq K\langle\Sigma^*\rangle$. Then there exists a finite $K$-automaton $\mathcal{A} = (Q, I, E, F)$ over $\Sigma^*$ with proper $E_{pq} \in C$ for every $p, q \in Q$ such that $|\mathcal{A}| = S$.

Now we can prove the main result of our paper.

*Proof of Theorem 4.2* First, let $\mathcal{A} = (Q, I, E, F)$ be a finite $K$-automaton over $M$ such that $S = |\mathcal{A}|_\Phi$. Then the finite $K_\Phi M$-automaton $\mathcal{A}' = (Q, I', E', F')$ over $M$ described in Lemma 5.3 satisfies $\omega(|\mathcal{A}|_\Phi) = |\mathcal{A}'|$. Moreover, $(E')_{pq} \in \mathbb{D}$ for every $p, q \in Q$. Then by Lemma 6.3 the finite $K_\Phi M$-automaton $\mathcal{A}'' = (Q, I', E'', F')$ over $x^*$ where $E'' = \psi(E')$, has the property $\psi(|\mathcal{A}'|) = |\mathcal{A}''|$. Since $(E')_{pq} \in \mathbb{D}$ we obtain that $(E'')_{pq} \in \mathbb{U}$. Then by Proposition 7.1, part 1 for $\Sigma = \{x\}$, we obtain that $|\mathcal{A}''| \in \mathit{Rat}((K_\Phi M)\langle\langle x^*\rangle\rangle; \mathbb{U}_{mon})$. Then by Lemma 6.2 we obtain that $|\mathcal{A}'| \in \mathit{Rat}((K_\Phi M)\langle\langle M\rangle\rangle; \mathbb{D}_{mon})$, and by Lemma 5.2 that $|\mathcal{A}|_\Phi \in \mathit{Rat}_\Phi(K\langle\langle M\rangle\rangle)$.

Conversely, let $S \in \mathit{Rat}_\Phi(K\langle\langle M\rangle\rangle)$. Then by Lemma 5.2 we obtain $S' = \omega(S) \in \mathit{Rat}((K_\Phi M)\langle\langle M\rangle\rangle; \mathbb{D}_{mon})$. Then by Lemma 6.2, $S'' = \psi(S') \in \mathit{Rat}((K_\Phi M)\langle\langle x^*\rangle\rangle; \mathbb{U}_{mon})$. Let $C = \mathbb{U} \cap (K_\Phi M)\langle x^*\rangle$, the set of all uniform polynomial series of $(K_\Phi M)\langle\langle x^*\rangle\rangle$; this is a subsemiring of $(K_\Phi M)\langle x^*\rangle$. Now by Proposition 7.1, part 2 with $\Sigma = \{x\}$, there exists a finite $(K_\Phi M)$-automaton $\mathcal{A}'' = (Q, I', E'', F')$ over $x^*$ with proper $(E'')_{pq} \in C$ for every $p, q \in Q$ and $|\mathcal{A}''| = S''$. Now define the finite $(K_\Phi M)$-automaton $\mathcal{A}' = (Q, I', E', F')$ over $M$ with $E' = (\psi|_{\mathbb{D}})^{-1}(E'')$; this is possible since all entries $(E'')_{pq}$ $(p, q \in Q)$ are proper uniform polynomials, and hence the entries $(E')_{pq}$ are proper diagonal polynomials. By Lemma 6.3 we have $\psi(|\mathcal{A}'|) = |\mathcal{A}''| = S'' = \psi(S')$. Since $|\mathcal{A}'|$ and $S'$ are in $\mathbb{D}$ and $\psi$ is injective on $\mathbb{D}$, we obtain $|\mathcal{A}'| = S'$. Now define the finite $K$-automaton $\mathcal{A} = (Q, I, E, F)$ over $M$ by letting $E = \omega^{-1}(E')$ and $I(q) = \langle I'(q), \mathbb{1}\rangle$ and $F(q) = \langle F'(q), \mathbb{1}\rangle$ for every $q \in Q$. By Lemma 5.3 we obtain $\omega(|\mathcal{A}|_\Phi) = |\mathcal{A}'| = S' = \omega(S)$. Since $\omega$ is injective, $S = |\mathcal{A}|_\Phi$ as needed.

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