Sequential ?

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Abstract

This paper is a survey where we try to organise the known answers to the question whether a given finite automaton with multiplicity in a semiring $\mathbb{K}$ is equivalent to a sequential, or *input deterministic*, one. We shall see that depending on $\mathbb{K}$, the question goes from obvious to open, that the answer goes from yes to undecidable. We review results on sequentiality in the cases of series of finite image, of series with multiplicity in fields, and of series with multiplicity in idempotent semirings.

Key words: Finite automata, sequential machine, twinning property

Dedicated to Christian Choffrut
on the occasion of his 60th birthday

Being sequential — or input deterministic — is a fundamental property of machines, understood here as automata with output, the output being taken in an arbitrary semiring of multiplicities (or weights) $\mathbb{K}$. To some extent, it amounts to say that the model can readily translate into a physical device. Whether a function realised by a given finite automaton can be realised by an equivalent sequential one is thus a natural as well as a dramatic question: a negative answer somehow implies that there is no finite realisation of the machine or that performing the computations described by the model may be very costly.

That question of sequentiality has both a theoretical and a practical aspect. Every finite automaton may be seen as an automaton with output, as it was the case at the dawn of the theory with the Moore or Mealy machines and the first characterisation of sequentiality, due to Raney, goes back to the late fifties. Somehow, it is still the point of view adopted in the unorthodox and

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profound work of Pippenger [31]. The Elgot–Mezei characterisation of functional transducers gives the sequential ones a central position [13,5,34]. On the other hand, automata with multiplicity — in particular with multiplicity in some idempotent semirings — have proved to be powerful modelisation tools for many applications, in the field of task scheduling for instance, and, above all, in natural language processing, especially for speech recognition (*cf.* references given in [1,8,26]). A lively publishing activity ensued.

In this paper, we try to survey and organise the known answers to the question of sequentiality. We shall see that depending on $\mathbb{K}$, the question goes from obvious to open, that the answer goes from yes to undecidable. One may note that two distinct complexity functions then arise, according to whether one wants just to know the answer or to compute the equivalent sequential automaton (in case it exists).

In the first section, we fix the notation and recall the necessary definitions. In particular, we emphasise on the matrix representation of automata which will be the basis of the main constructions. On few points, our terminology sligthly diverges from the most common usage and we explain the reason for our choice.

In the second section, we try to set up a general framework in which the known results of the area take place naturally. As usual in automata theory, we distinguish two levels: the objects themselves, here the formal power series, that correspond to an *abstract* level where properties such as sequentiality may be defined, and the representations of these objects, here the automata with multiplicity, which form the *effective* level where constructions are performed and properties can be checked. The problem does not thus consists only in designing the adequate constructions and defining the right properties to be checked but also in understanding to what extend the properties checked at the effective level are a faithful characterisation of the properties that are to be verified at the abstract level.

Our presentation brings to light the differences and similarities between results stated in diverse structures, allows to express them in a unified language, to relate one to the others and to give each of them their role. This section is probably the most original part of this survey although a number of statements in it will be considered as folklore by many. In the sequel we review results on sequentiality in the cases of locally finite multiplicity semirings, of series of finite image, of series with multiplicity in fields, and of series with multiplicity in idempotent semirings respectively.

We first mention the case of locally finite semirings in order to state that fundamentally there is nothing to state. The case of series with finite image is more interesting. We explain how in three cases: the field case, the tropical case, and the transducer case, series of finite image are realised by finite state machines (and thus deterministic ones) in the sense that no computation in the multiplicity semiring is involved in the determination of the coefficient attached to a word. If the answer is simple, the way to get it is not elementary. It relies on the reduction theory in the case of field, and on the ‘limitedness’ theorem in the case of tropical semirings; the case of transducers is much simpler but is known to be undecidable.

The problem of sequentiality of automata with multiplicity in a field does not seem to have ever attracted much attention. We solely describe how, by way of the reduction theory, the two levels, ‘abstract’ and ‘effective’, coincide in this case. The subject is open to further investigations.

The situation is completely opposite in the case of automata with multiplicity in idempotent semirings. This is the case where most of the work on sequentiality has been done, again both on theoretical and practical side, the case where the demand for efficient algorithms is the strongest. It is no surprise that it is the case which takes the best part of our survey. To tell the truth, we are not interested in stating results for *all* idempotent semirings. We focus on two classes of automata which have been extensively studied from the point of view of sequentiality: the automata with multiplicity in a tropical semiring and the functional transducers.

The framework we have set up in Section 2 is particularly well-fitted to organize the rich publication activity devoted to these automata. Not only it makes clear the comparison between results in different semirings but it allows to point out with precision where erroneous assumptions could have been made or misleading terminology may have been adopted, leading to false statements or hopeless algorithms.

At the heart of the study of these automata, the *twinning property* defined by Choffrut and which is both decidable and characteristic of sequentiality for functional transducers. With respect to sequentiality, the functional transducers happen then to be simpler than tropical automata as it is still an *open problem* to know whether sequentiality is decidable for the latter. We end the paper with the study of some particular cases (unary, finitely ambiguous, heap model) for which sequentiality is decidable.

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It is a pleasure to dedicate this survey to our colleague and friend Christian Choffrut — a forty year friendship this year for the second author. We hope that, among other things, it will show how the idea of *the twinning property* developed by Christian in the late seventies has proved to be so fruitful and central to the design of many algorithms.

1 From automata to representations: fixing the terminology

Automata with multiplicity — or weighted automata [1] — are the subject of this paper. We recall here most of the definitions and the notations that will be used in the sequel. For missing definitions and related results, we refer to [13,6,34].

1.1 *Multiplicity semirings*

We shall be concerned mostly with the following four classes of multiplicity semirings: first, the *Boolean semiring* $\mathbb{B}$, which basically means no multiplicity; second, the classical *semirings of numbers*: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}_+$, $\mathbb{Q}$, $\mathbb{R}_+$, $\mathbb{R}$, that is, the non negative integers, the integers, the non negative rationals, the rationals, the non negative reals, and the reals; third, the so-called *tropical semirings*: $\mathcal{M} = \langle\, \mathbb{N}, \mathsf{min}, + \,\rangle$, $\langle\, \mathbb{N}, \mathsf{max}, + \,\rangle$, $\langle\, \mathbb{Z}, \mathsf{max}, + \,\rangle$, $\langle\, \mathbb{Q}_+, \mathsf{max}, + \,\rangle$, *etc.* with the convention that the adequate infinity element is added when necessary to have a neutral element for the addition of the semiring, that is, min or max; and finally the *semirings of power series*: $\langle\, \mathfrak{P}(A^*), \cup, \cdot \,\rangle$, the semiring of subsets of the free monoid, its subsemiring of rational languages $\operatorname{Rat} A^*$, $\mathbb{K}\langle\langle A^* \rangle\rangle$, the semiring of series over A^* with multiplicity in $\mathbb{K}$, *etc.*

1.2 *Automata with multiplicity*

A classical, or *Boolean*, automaton $\mathcal{A}$ is a 5-tuple $\mathcal{A} = \langle\, Q, A, E, I, T \,\rangle$ where Q is the set of *states*, A the (input) alphabet, I and T the sets of *initial* and *final* states, and $E \subseteq Q \times A \times Q$ is the set of *transitions* of $\mathcal{A}$.

An *automaton with multiplicity* in a semiring $\mathbb{K}$ — or $\mathbb{K}$-automaton — is a generalisation of the former: $\mathcal{A} = \langle\, Q, A, \mathbb{K}, E, I, T \,\rangle$ where $E \subseteq Q{\times}(A{\times}\mathbb{K}){\times}Q$ is the set of *transitions with multiplicity*, and I and T instead of being *subsets* of Q are *maps* from Q into $\mathbb{K}$. Let

$$c := q_0 \xrightarrow{(a_1,k_1)} q_1 \xrightarrow{(a_2,k_2)} q_2 \cdots q_{n-1} \xrightarrow{(a_n,k_n)} q_n$$

be a computation in $\mathcal{A}$. The *label* $|c|$ of c is the *monomial* $|c| = k\, w$ where $w = a_1 a_2 \cdots a_n$ is the *word label* of c and $k = I(q_0)\, k_1 k_2 \cdots k_n\, T(q_n)$ its *multiplicity*. The *behaviour* $|\mathcal{A}|$ of $\mathcal{A}$ is the sum of the labels of all computations in $\mathcal{A}$; this sum is well defined if every word of A^* is the word label of

[1] Multiplicity or weight: these are perfectly synonymous. We use here the former, for it seems that the latter is more currently attached to 'numerical' multiplicity.

only a *finite* number of computations in $\mathcal{A}$, which holds in particular when Q is finite or when $\mathcal{A}$ is *deterministic*.

Thus $|\mathcal{A}|$ is a (formal) power series on A^* with coefficients in $\mathbb{K}$, which means also *a map* from A^* into $\mathbb{K}$. The *coefficient* of a word w in $|\mathcal{A}|$, which is the same as *the image* of w by $|\mathcal{A}|$, or *the value* of $|\mathcal{A}|$ at w, is denoted by $<|\mathcal{A}|, w>$ (the series notation); it is the sum of the multiplicities of all computations in $\mathcal{A}$ whose word label is w.

Figure 1 shows a labeled graph, that is, an automaton, where the multiplicity of transitions are integers. By convention, incoming (resp. outgoing) arrows without label indicate that the initial (resp. final) map gives the corresponding state the value $1_{\mathbb{K}}$, that is, the *multiplicative* identity element of the multiplicity semiring. If the integers are seen as being elements of the semiring $\mathbb{N} = \langle \mathbb{N}, +, \times \rangle$, then we denote this $\mathbb{N}$-automaton by $\mathcal{A}_1$ and for every w in $\{a, b\}^*$, $<|\mathcal{A}_1|, w>$ is the number written w in binary (if a is interpreted as 0 and b as 1). If the same coefficients are considered as elements of $\mathcal{M} = \langle \mathbb{N}, \mathsf{min}, + \rangle$, then we denote this $\mathcal{M}$-automaton by $\mathcal{B}_1$ and $<|\mathcal{B}_1|, w>$ is $+\infty = 0_{\mathcal{M}}$ if w does not contain a b and the *length* of w plus the number of a's that follow the rightmost b otherwise. If finally we interpret the graph as a $\langle \mathbb{N}, \mathsf{max}, + \rangle$-automaton $\mathcal{C}_1$ then $<|\mathcal{C}_1|, w>$ is $-\infty$ if w does not contain a b and twice the *length* of w minus 1 minus the number of a's that precede the leftmost b otherwise.

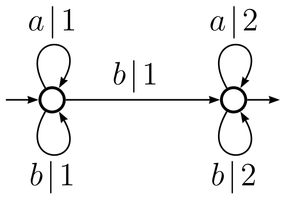


Fig. 1. The automata with multiplicity $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{C}_1$

A Boolean automaton is exactly a $\mathbb{B}$-automaton and will be denoted as such to avoid ambiguity. Every $\mathbb{K}$-automaton $\mathcal{A}$ can be transformed into a $\mathbb{B}$-automaton, called the *support* of $\mathcal{A}$, denoted by $\mathsf{supp}\,\mathcal{A}$, and obtained by replacing every non-zero (non $0_{\mathbb{K}}$) multiplicity on transitions by $1 = 1_{\mathbb{B}}$.

1.3 Matrix representation of automata

The definition we gave above for $\mathbb{K}$-automata corresponds to a description of automata as *labeled graphs*. In two steps (and one theorem), this description can be transformed into the *matrix representation* of automata, that proves to be very effective for the definition and *for the study* of sequential automata.

A $\mathbb{K}$-automaton $\mathcal{A} = \langle Q, A, \mathbb{K}, E, I, T \rangle$ may be denoted as well by:

$$\mathcal{A} = \langle I, E, T \rangle \ ,$$

where I, a map from Q into $\mathbb{K}$, is seen as *a row vector* of dimension Q with entries in $\mathbb{K}$, where similarly T is a *column vector* of $\mathbb{K}^Q$, and where E is a $Q \times Q$*-matrix* whose entries are linear combinations of letters of A with coefficients in $\mathbb{K}$. For example $\mathcal{C}_1$ is written as:

$$\left\langle \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} a+b & b \\ 0 & 2a+2b \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$$

(when seen as an $\mathbb{N}$-automaton). With this notation it then comes:

$$|\mathcal{A}| = \sum_{k \in \mathbb{N}} I\, E^k\, T = I\, E^*\, T \ .$$

The set Q is also called the *dimension* of $\mathcal{A}$. It might be disturbing to use a *set* as a *dimension* for vectors and matrices. But this easiness in writing — which puts the emphasis on the fact that a matrix is rather about *indexing* entries than comparing their numbers — proves to be very convenient.

Definition 1 *Let* $\mathcal{A} = \langle I, E, T \rangle$ *be a* $\mathbb{K}$*-automaton of dimension* Q*. The* (matrix) representation *of* $\mathcal{A}$ *is the triple* (I, μ, T) *where* $\mu \colon A^* \longrightarrow \mathbb{K}^{Q \times Q}$ *is* the morphism *defined by*

$$E = \sum_{a \in A} \mu(a)\, a \ .$$

In the sequel we shall equally write $\mathcal{A} = \langle Q, A, \mathbb{K}, E, I, T \rangle$, or $\mathcal{A} = \langle I, E, T \rangle$, for a $\mathbb{K}$-automaton and $\mathcal{A} = (I, \mu, T)$ for a $\mathbb{K}$-representation. For instance, $\mathcal{C}_1 = (I_1, \mu_1, T_1)$ with:

$$I_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \ , \quad \mu_1(a) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \ , \quad \mu_1(b) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ .$$

As A^* is a free monoid, we have:

Proposition 2 *If* $\mathcal{A} = \langle I, E, T \rangle$ *is a* $\mathbb{K}$*-automaton on* A^* *and* (I, μ, T) *the associated* $\mathbb{K}$*-representation, then, for every* w *of* A^**, it holds:*

$$<\!|\mathcal{A}|, w\!> = I\, \mu(w)\, T \ .$$

A series s of $\mathbb{K}\langle\langle A^*\rangle\rangle$ is $\mathbb{K}$-*recognisable* if it is realised by a $\mathbb{K}$-representation, that is, if it exists (I, μ, T) such that for every w in A^* it holds: $<s, w> = I\,\mu(w)\,T$. The set of $\mathbb{K}$-*rational* series is the smallest subalgebra of $\mathbb{K}\langle\langle A^*\rangle\rangle$ that contains the polynomials and that is closed for the star operation. And the celebrated Kleene–Schützenberger theorem states that a series of $\mathbb{K}\langle\langle A^*\rangle\rangle$ is $\mathbb{K}$-rational if, and only if, it is $\mathbb{K}$-recognisable [6]. This reminder of classics would have been useless if we were not going to consider also relations on words for which the distinction between rational and recognisable becomes meaningful.

1.4 A word on transducers

One of the purposes of this paper is to deal with relations, or functions, between words that are realised by finite automata at the same time as the behaviour of other families of automata with multiplicity in order to bring to light the similarities and the differences between the two kinds of machines.

A transducer is an automaton whose transitions are labeled by elements in a *product* of two free monoids $M = A^*{\times}B^*$. The *behaviour* of such a automaton is a subset of $A^*{\times}B^*$ and thus *the graph of a relation* from A^* into B^*. Now, a relation is said to be *rational* if its graph is a rational subset of $A^*{\times}B^*$ and a relation is rational if, and only if, it is the behaviour of a (finite) transducer.

To a relation α from A^* into B^*, whose graph is denoted $\widehat{\alpha}$ corresponds a map $\tau_\alpha\colon A^* \to \mathfrak{P}(B^*)$ by

$$\forall u \in A^* \qquad \tau_\alpha(u) = \{v \in B^* \mid (u, v) \in \widehat{\alpha}\} \ ,$$

and conversely, with every map $\tau\colon A^* \to \mathfrak{P}(B^*)$ is associated a relation α_τ from A^* into B^* by

$$\widehat{\alpha_\tau} = \{(u, v) \in A^* \times B^* \mid v \in \tau(u)\} \ .$$

Rationality is preserved under this correspondance in the sense that the relation α is rational if, and only if, τ_α is a rational map $\tau_\alpha\colon A^* \to \operatorname{Rat} B^*$, that is, if, and only if, τ_α is a $\operatorname{Rat} B^*$-*recognizable series* over A^*.

We shall be mostly concerned in *rational functions* α. In this case, the $\operatorname{Rat} B^*$-recognizable series τ_α is realised by a representation (I, μ, T) whose non zero entries are just *words* in B^* instead of *rational subsets* of B^*. By abuse — and because it is common usage — we shall also call *transducers* the automata on A^* with multiplicity in $\operatorname{Rat} B^*$ (that correspond to representations (I, μ, T)). We even call them *functional transducers* when the relation associated to their behaviour is a rational function. They can indeed hardly be distinguished then from automata over $A^* \times B^*$.

A matrix representation (I, μ, T) is *row*-monomial, if for every a in A, every row of the matrix $\mu(a)$ has at most one non zero entry and if I has (at most) one non zero entry (and no condition on T). The same is then true of $\mu(f)$ and of $I\mu(f)$ for every f in A^*.

Definition **3** *An automaton* $\mathcal{A}$ *is* deterministic*, or* sequential*, if its matrix representation* (I, μ, T) *is* row-*monomial,*

A function, or a series, $\tau\colon A^* \to \mathbb{K}$ *is* sequential *if it can be realised by a sequential, or deterministic finite* $\mathbb{K}$*-automaton over* A^**.*

It is to be acknowledged that *deterministic automata* — and the derived terms such as 'determinisability' — is a very common terminology in the field. As there are *deterministic transducers* that are a different thing than the sequential ones, and since we want a uniform treatment between transducers and other automata with multiplicity, we have prefered to use the word 'sequential' throughout in this paper, but for a Boolean automaton. [2]

If $\mathcal{A}$ is sequential, then $\mathsf{supp}\,\mathcal{A}$ is a deterministic Boolean automaton and we borrow to the latter the notation for transitions: for every state p and every letter a in A, the unique state q such that $\mu(a)_{p,q} \neq 0_{\mathbb{K}}$ in the representation of $\mathcal{A}$ is denoted as $q = p \cdot a$.

Remark **1** *To tell the truth, we have changed the terminology for transducers as well and we call here* sequential *that which is commonly called* sub-sequential*, as proposed in the seminal paper by Schützenberger* [35]. [3] *We do so because we think that the* fundamental object *is indeed that which is called here a* sequential function*, and that it thus merits the basic term. We thus meet Bruyre and Reutenauer who wrote in* [7]*: 'the word* sub-sequential *is unfortunate since these functions should be called simply* sequential*; [...] we hope that someone will find some day a definite terminology.' It is also the terminology adopted in [26].*

[2] There are other places indeed where 'sequential' seems more adequate or natural than 'deterministic': deterministic map, for instance, sounds a bit awkward, and we shall also distinguish between the 'universal determinisation process' and the 'general sequentialisation procedure'. On the other hand, it is impossible to say that a classical automaton, that is an automaton with multiplicity in the Boolean semiring, is 'sequential' instead of 'deterministic'.

[3] And we would call *pure sequential* what is usually called *sequential* (*cf.* [34]).

As for any other structure, it is possible to define morphisms for $\mathbb{K}$-automata (which we call $\mathbb{K}$-quotient in [25,4]). But here we shall be concerned only with the simpler notion of quotient of the *support automata*, that is, of Boolean automata. Let $\mathcal{C} = \langle\, Q, A, E, I, T \,\rangle$ and $\mathcal{D} = \langle\, R, A, F, J, U \,\rangle$ be two $\mathbb{B}$-automata. A (surjective) map $\varphi\colon Q \longrightarrow R$ induces (or *is*) a morphism from $\mathcal{C}$ onto $\mathcal{D}$ if $(p, a, q) \in E$ implies $(\varphi(p), a, \varphi(q)) \in F$ and this morphism is a *($\mathbb{B}$)-quotient* if moreover $(r, a, s) \in F$ and $p \in \varphi^{-1}(r)$ implies that there exists q in $\varphi^{-1}(s)$ such that $(p, a, q) \in E$. A quotient of a deterministic (resp. complete) automaton is deterministic (resp. complete). Every automaton has a unique minimal quotient.

If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathbb{K}$-automata, we say that $\mathcal{B}$ is a *pre-quotient* of $\mathcal{A}$ if $\mathsf{supp}\,\mathcal{B}$ is a quotient of $\mathsf{supp}\,\mathcal{A}$. If $\mathcal{B}$ is a pre-quotient of $\mathcal{A}$, $\mathcal{B}$ is not necessarily equivalent to $\mathcal{A}$. Figure 2 shows an example of the pre-quotient of an $\mathbb{N}$-automaton.

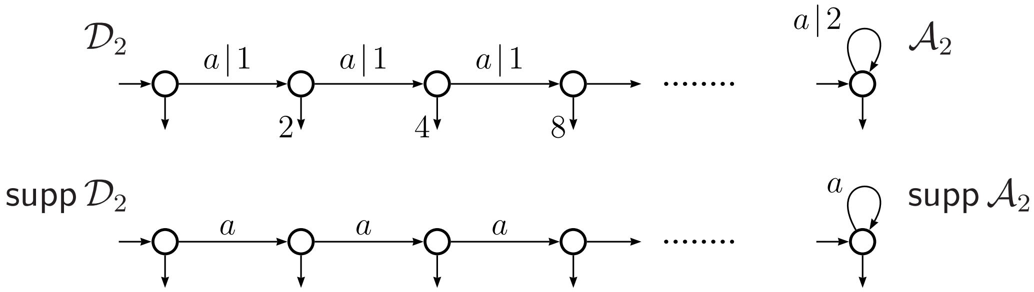


Fig. 2. Two $\mathbb{N}$-automata and their supports: $\mathcal{A}_2$ is a pre-quotient of $\mathcal{D}_2$

2 The general setting

We try to give a characterisation of sequential series in a way that is as independent as possible of the multiplicity semiring. We first do it at the level of the series itself, and then at the level of a finite $\mathbb{K}$-automaton that realises the series.

2.1 The 'abstract' level

2.1.1 The automaton of residuals

Let s be a series in $\mathbb{K}\langle\!\langle A^* \rangle\!\rangle$; the *residual* of s by u in A^* is the series denoted by $u^{-1}s$ and defined by $\langle u^{-1}s, v \rangle = \langle s, uv \rangle$ for every v in A^*. Hence, if s is realised by a representation $(\, I, \mu, T\,)$, then $u^{-1}s$ is realised by $(\, I\mu(u), \mu, T\,)$.

The set of residuals of s, $\mathbf{R}_s = \{w^{-1}\, s \mid w \in A^*\}$, is naturally the set of states of a deterministic automaton denoted by $\mathcal{A}_s$, where $w^{-1}s \cdot a = a^{-1}w^{-1}s = (w\,a)^{-1}s$, the initial state is s, and the final function is defined by $T(w^{-1}s) = \langle w^{-1}s, 1_{A^*}\rangle = \langle s, w\rangle$. This automaton $\mathcal{A}_s$, a priori infinite, realises s. If s is a $\mathbb{B}$-series, that is, if s is a language L, then $\mathcal{A}_L$ is the *minimal automaton* of L.

A subset $\mathbf{U}$ of $\mathbb{K}\langle\!\langle A^*\rangle\!\rangle$ is said to be *stable* if $w^{-1}t$ is an element of $\mathbf{U}$ for every w in A^* and every t in $\mathbf{U}$.

PROPOSITION **4 (Jacob *cf.* [6])** *A series s is recognisable if, and only if, s, and thus $\mathbf{R}_s$, is contained in a finitely generated stable submodule of $\mathbb{K}\langle\!\langle A^*\rangle\!\rangle$.*

We have an analogous statement for the characterisation of sequential series, based on the notion of lines. Let us call a *line* in $\mathbb{K}\langle\!\langle A^*\rangle\!\rangle$ a set $\ell = \mathbb{K}\, r$, for r in $\mathbb{K}\langle\!\langle A^*\rangle\!\rangle$. Every t such that $\mathbb{K}\, t = \ell$ is a *generator* of ℓ.

PROPOSITION **5** *A series s is sequential if, and only if, s, and thus $\mathbf{R}_s$, is contained in a stable finite set $\mathbf{L}$ of lines of $\mathbb{K}\langle\!\langle A^*\rangle\!\rangle$.*

Proof. If s is sequential, realised by a row-monomial representation $(\, I, \mu, T\,)$ of dimension Q, let us denote by I_q the row vector whose entries are all $0_{\mathbb{K}}$ but the q-th one which is $1_{\mathbb{K}}$ and by t_q the series realised by $(\, I_q, \mu, T\,)$ for every q in Q. For every u in A^*, we have $u^{-1}s = k\, t_q$ where q is the coordinate of *the* non zero entry of $I\, \mu(u)$ and k its value. Likewise, for every u in A^* and every p in Q we have $u^{-1}t_p = k\, t_q$, where q is the coordinate of the non zero entry of $I_p\, \mu(u)$ and k its value.

Conversely, let G be a set of generators of the set $\mathbf{L}$ of lines. The hypothesis on G translates into the following *two* equations:

$$\forall w \in A^*,\ \exists r \in G,\ \exists k \in \mathbb{K} \qquad w^{-1}s = k\, r \tag{1}$$
$$\forall w \in A^*,\ \forall r \in G,\ \exists r' \in G,\ \exists h' \in \mathbb{K} \qquad w^{-1}r = h'\, r' \tag{2}$$

From (1) and (2) the sequential $\mathbb{K}$-representation $\mathcal{D}_s = (\, J, \rho, U\,)$ of dimension G is naturally built:[4]

$$\begin{aligned} \forall a \in A,\ \forall r, r' \in G \qquad \rho(a)_{r,r'} &= h && \text{if} \quad a^{-1}r = h\, r' \\ J_r &= k && \text{if} \quad s = k\, r \\ U_r &= \langle r, 1_{A^*}\rangle && \end{aligned}$$

[4] This notation is slightly improper since $\mathcal{D}_s$ is not canonically associated with s but also depends on $\mathbf{L}$ and G.

and it comes $<s,w> = <w^{-1}s, 1_{A^*}> = J\,\rho(w)\,U$ by induction on the length of w: s is realised by $\mathcal{D}_s$. ■

Since the states of $\mathcal{A}_s$ correspond to residuals of s and so do the states of $\mathcal{D}_s$, it is immediate to verify the following.

Proposition **6** *Let s be a sequential series. Then $\mathcal{D}_s$ is a pre-quotient of $\mathcal{A}_s$.*

2.1.2 The automaton of translations

One cannot go further without any assumption on $\mathbb{K}$. Let us suppose now that $\mathbb{K}$ admits a *greatest common divisor* (gcd) operation.

The existence of a gcd on the whole semiring $\mathbb{K}$ is indeed too strong an hypothesis; it will suffice that the gcd exists on a subset of $\mathbb{K}$ containing all the entries of the matrix representation of the automaton. This is the case in particular for functional transducers: the multiplicity semiring is $\mathfrak{P}(B^*)$ on which there is no gcd, but the entries are all in B^* and the *longest common prefix* is a gcd.

If s is a series in $\mathbb{K}\langle\langle A^*\rangle\rangle$, let $\mathring{s}$ be the gcd of all non zero coefficients of s and $s^\sharp$ be the series such that $\mathring{s}\,s^\sharp = s$.[5] In some cases (*e.g.* if $\mathbb{K}$ is a field), it may happen that $\mathring{s}$ is not uniquely determined; in such cases, one can take as a convention that the first non zero coefficient of $s^\sharp$ is equal to $1_\mathbb{K}$.

Definition **7** *Let s be in $\mathbb{K}\langle\langle A^*\rangle\rangle$. The* translation *of s by w in A^* is the series $[w^{-1}s]^\sharp$.*

Theorem **8 (Raney [32])** *A series s in $\mathbb{K}\langle\langle A^*\rangle\rangle$ is sequential if, and only if, the set of translations of s is finite.*[6]

Proof. If s is sequential, realised by a row-monomial representation (I, μ, T), and with the notation of Proposition 5, it holds:

$$\forall w \in A^* \qquad [w^{-1}s]^\sharp = [t_q]^\sharp \ ,$$

where q is the coordinate of *the* non zero entry of $I\,\mu(w)$, hence the conclusion.

[5] Beware that the notation may be a little confusing: s and $s^\sharp$ are *series* whereas $\mathring{s}$ is an *element* of $\mathbb{K}$.

[6] Raney's original proposition applies to the case of letter-to-letter functions but the substance of this generalised statement is identical.

Conversely, let :

$$\mathbf{G}_s = \{[w^{-1}s]^\sharp \mid w \in A^*\}$$

be the finite set of translations of s. This set of series generates a stable set of lines containing s. By Proposition 5, s is sequential. ∎

Thus, if s is sequential and as in the proof of Proposition 5, the automaton $\mathcal{D}_s = (\,J, \rho, U\,)$ of dimension $\mathbf{G}_s$ is defined by:

$$\forall a \in A\,,\ \forall r, r' \in \mathbf{G}_s \qquad \rho(a)_{r,r'} = h \qquad \text{with} \quad h = [\overset{\circ}{a^{-1}r}] \quad \text{and} \quad r' = [a^{-1}r]^\sharp$$
$$J_r = \overset{\circ}{s} \qquad \text{if} \quad r = s^\sharp$$
$$U_r = \langle r, 1_{A^*}\rangle$$

From the proof of Theorem 8, $\mathcal{D}_s$ is a pre-quotient of any sequential automaton that realises s.

To summarise the content of the subsection, the problem we address can be reformulated as whether it is possible, given a representation of s, to effectively compute a set G that meets (1) and (2). Let us thus consider what can be computed from a representation, in full generality.

2.2 The 'effective' level

The classical 'subset construction' may be generalised to automata with multiplicity. Afterwards, and modulo the *gcd* hypothesis on the multiplicities of the automaton, a sequential automaton (possibly infinite) may be built.

2.2.1 The universal determinisation process

Let $\mathcal{A} = (\,I, \mu, T\,)$ be a $\mathbb{K}$-automaton over A^* of dimension Q. The subset $\mathbf{R}_\mathcal{A}$ of $\mathbb{K}^Q$ that consists of all (row) vectors that can be accessed by $\mathcal{A}$:

$$\mathbf{R}_\mathcal{A} = \{I\,\mu(w) \mid w \in A^*\}$$

can be turned into a deterministic automaton, in the same way as $\mathbf{R}_s$ was into $\mathcal{A}_s$. This automaton is denoted by $\widehat{\mathcal{A}}$, its transitions are defined by $[I\,\mu(w)]{\cdot}a = I\,\mu(w)\,\mu(a) = I\,\mu(w\,a)$, its unique initial state is $I = I\,\mu(1_{A^*})$, and its final function $\widehat{T}$ maps each state $I\,\mu(w)$ to $I\,\mu(w)\,T = \langle|\mathcal{A}|, w\rangle$. This automaton $\widehat{\mathcal{A}}$, a priori infinite, realises the same series as $\mathcal{A}$.

As $\mathcal{A}_s$, the automaton $\widehat{\mathcal{A}}$ outputs nothing while reading a word w and performs the output at the end of the computation, when w has been read completely.

But the state reached with w is the vector of values computed by $\mathcal{A}$ on all paths labeled by w. For instance, if $\mathcal{A}_2$ is as in Figure 2, then $\widehat{\mathcal{A}_2} = \mathcal{D}_2$.

The well-known relation between a determinised automaton and the minimal automaton of the recognised language generalises to series.

PROPOSITION **9** *Let s be a $\mathbb{K}$-recognisable series and $\mathcal{A}$ any finite $\mathbb{K}$-automaton that realises s. Then $\mathcal{A}_s$ is the* minimal quotient *of $\widehat{\mathcal{A}}$.*

Strictly writing, $\mathcal{A}_s$ and $\widehat{\mathcal{A}}$ are $\mathbb{K}$-automata and we should have written 'minimal $\mathbb{K}$-quotient'. But as the coefficients in all transitions of $\mathcal{A}_s$, as well as in those of $\widehat{\mathcal{A}}$ are all equal to $1_{\mathbb{K}}$ (and as they are deterministic), these two automata behave exactly as if they were Boolean automata, but for the final function. Two states of $\widehat{\mathcal{A}}$ can be merged into a single state of $\mathcal{A}_s$ if they are given the same value by the final function and afterwards the classical Moore procedure for minimisation applies.

2.2.2 The general sequentialisation procedure

The states of $\mathcal{A}_s$ were defined in an abstract way whereas the states of $\widehat{\mathcal{A}}$ are described concretely, which is a first step towards an effective computation. But there are, in general, an infinite number of states in $\widehat{\mathcal{A}}$. The problem is thus to transform it into an equivalent finite one while keeping the sequentiality.

Once again, we suppose that $\mathbb{K}$ admits a *gcd* operation. And we define for vectors of $\mathbb{K}^n$ notations as for series. If ζ is in $\mathbb{K}^n$, let $\mathring{\zeta}$ be the gcd of all non zero entries of ζ and let $\zeta^\sharp$ be the vector of $\mathbb{K}^n$ such that $\mathring{\zeta}\,\zeta^\sharp = \zeta$.

We can now define a new sequential automaton $\check{\mathcal{A}}$ associated with a finite $\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$. Let us start from $\mathbf{R}_\mathcal{A}$, the set of states of $\widehat{\mathcal{A}}$, and define the set

$$\mathbf{G}_\mathcal{A} = [\mathbf{R}_\mathcal{A}]^\sharp = \{\zeta^\sharp \mid \zeta \in \mathbf{R}_\mathcal{A}\} \ .$$

For every ζ in $\mathbf{G}_\mathcal{A}$ and every a in A, let $\eta = \zeta\,\mu(a)$; there is a transition from ζ to $\eta^\sharp$ labeled by a and with multiplicity $\mathring{\eta}$. The initial state is $I^\sharp$, with multiplicity $\mathring{I}$ and the final function is defined by $\check{T}(\zeta) = \zeta\, T$. The properties of a *gcd* imply that $\eta^\sharp$ belongs to $\mathbf{G}_\mathcal{A}$ and that $\check{\mathcal{A}}$ is equivalent to $\mathcal{A}$.

EXAMPLE **1** *Let $\mathcal{B}_3 = \langle\, J_3, \kappa_3, U_3 \,\rangle$ be the $\mathcal{M}$-automaton shown at Figure 3 (a). It realises the function that maps every word onto its length, an obviously sequential one. In $\mathcal{M}$, $\mathsf{min}(m,n)$ is a gcd. It comes $J_3 = (0, +\infty)$, and for every positive integer n, $J_3\,\kappa_3(a^n) = (n, n)$. Hence, for every positive integer n, $[J_3\,\kappa_3(a^n)]^\sharp = (0, 0)$ and $\check{\mathcal{B}_3}$ shown at Figure 3 (b) is finite.*

(a) The automata $\mathcal{A}_3$, $\mathcal{B}_3$, and $\mathcal{C}_3$ (b) The automaton $\breve{\mathcal{B}}_3$

Fig. 3. The sequentialisation procedure

Proposition **10** *Let s be a series realised by $\mathcal{A}$. Then $\breve{\mathcal{A}}$ is a pre-quotient of $\widehat{\mathcal{A}}$ that realises s. Moreover, if s is* sequential*, then $\mathcal{D}_s$ is a pre-quotient of $\breve{\mathcal{A}}$.*

The overall relationships between all the automata we have defined in this section are shown at Figure 4. Although we have defined it from $\widehat{\mathcal{A}}$ — which is infinite in general — $\breve{\mathcal{A}}$ can be effectively computed from $\mathcal{A}$, if it is finite. In particular, if $\mathcal{A}$ is already sequential then $\breve{\mathcal{A}} = \mathcal{A}$. If $\breve{\mathcal{A}}$ is finite, finding $\mathcal{D}_s$ is the problem of *minimisation*, that we do not address here (a complete account on this problem for transducers has been given by Choffrut in [11]).

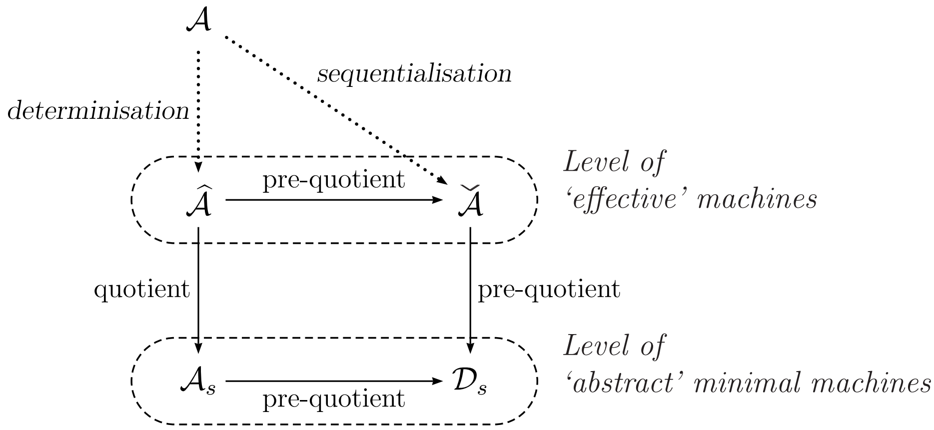


Fig. 4. The sequentialisation and minimisation scheme for a series s realised by an automaton $\mathcal{A}$

One of the difficulty of the problem is that it may happen, even in 'gcd-semiring', that s is sequential *without* $\breve{\mathcal{A}}$ being finite.

For instance, let $\mathcal{A}_3 = (\, I_3, \mu_3, T_3 \,)$ be the $\mathbb{N}$-automaton shown at Figure 3 (a). It realises the function that maps every word of length n onto 2^n: this function is sequential as well (it is realised by $\mathcal{A}_2$). But for every positive integer n, $I_3\,\mu_3(a^n) = (1, 2^n - 1)$ and $\breve{\mathcal{A}}_3$ is not finite.

Remark that even when deciding whether $\breve{\mathcal{A}}$ is finite is of low complexity, it may happen that the construction of $\breve{\mathcal{A}}$ from $\mathcal{A}$ is of exponential complexity (*e.g.* $\mathbb{K} = \mathbb{B}$).

Let us note finally that this 'sequentialisation procedure' is a natural construct on automata whose relevance goes beyond the sole question of sequentiality: for instance, we have made use of it in the study of the conjugacy of equivalent functional transducers [3].

3 The trivial case

If $\mathbb{K}$ is finite, $\mathbb{K}^Q$ is finite and so is $\widehat{\mathcal{A}}$. In particular, if $\mathbb{K} = \mathbb{B}$, the construction of $\widehat{\mathcal{A}}$ is *exactly* the celebrated *subset construction* which implies that every finite ($\mathbb{B}$)-automaton is equivalent to a finite deterministic one.

This property can be generalised as follow. Say that a semiring is *locally finite* if any finitely generated subsemiring is finite. Examples of infinite but locally finite semirings are the "fuzzy" semirings $\langle\, \mathbb{Z}, \mathsf{max}, \mathsf{min} \,\rangle$ or $\langle\, [0,1], \mathsf{max}, \mathsf{min} \,\rangle$. If $\mathcal{A} = (\, I, \mu, T \,)$ is a finite automaton with multiplicity in a locally finite semiring $\mathbb{K}$, the set $\{I\, \mu(w) \mid w \in A^*\}$ is finite albeit $\mathbb{K}^Q$ may be infinite and we thus have the following

Proposition 11 *Every rational map from A^* into a locally finite semiring $\mathbb{K}$ is sequential and, if $\tau \colon A^* \to \mathbb{K}$ is such a map, $\tau^{-1}(k)$ is a rational language of A^* for every k in $\mathbb{K}$.*

Another way to state Proposition 11 is to say that finite automata with multiplicity in a (locally) finite semiring are *finite state machines*, which they are not anymore if the multiplicity semiring is is arbitrary.

4 The finite image case

This case is induced by the trivial case. As before let $\tau \colon A^* \to \mathbb{K}$ be a rational map realised by a finite $\mathbb{K}$-automaton $\mathcal{A}$ and suppose that $\mathsf{Im}\, \tau$ is finite ($\mathbb{K}$ not finite). In which case (relative to $\mathbb{K}$) can we infer that τ is a sequential function? that the automaton $\widehat{\mathcal{A}}$ is finite? or that $\widehat{\mathcal{A}}$ has a finite quotient?

Altogether, these questions have the flavour of a *Fatou property* — a term that is probably to explain. Saying that a series is rational amounts to give a mean — a $\mathbb{K}$-representation — to compute the coefficients of the series. The Fatou Lemma states (*cf.* [6]) that *a $\mathbb{Q}$-rational series that has all its coefficients in $\mathbb{Z}$ is a $\mathbb{Z}$-rational series.* In contrast, for instance, it is known ([13,6]) that a $\mathbb{Z}$-rational series that has all its coefficients in $\mathbb{N}$ is not necessarily an $\mathbb{N}$-rational series. The hypothesis of Fatou Lemma is a condition *on the result* of the computation: the coefficients are integers, and the conclusion is a property of *the way this result can be computed*: the entries of the representation can be chosen to be integers. Here, the condition on the result is: the coefficients form a finite set, and we want to know whether they can be obtained in the same way as in the trivial case, that is, with a finite state machine which delivers the coefficient of every w according to the state reached at the end of the reading of w but without any computation in the semiring $\mathbb{K}$.

A simple example will show that the answer cannot be positive independently of $\mathbb{K}$. Let us first define the *polycyclic* monoid P_X generated by the alphabet $X = \{x, y\}$. Write $\widetilde{X} = \{x, \overline{x}, y, \overline{y}\}$ and P_X is isomorpic to the quotient of $\widetilde{X}^*$ by the relations $x\,\overline{x} = y\,\overline{y} = 1_{\widetilde{X}^*}$, $x\,\overline{y} = y\,\overline{x} = 0$. Let $\mathbb{P} = \mathfrak{P}(P_X)$ be the subset semiring of P_X and consider the behaviour of the sequential $\mathbb{P}$-automaton[7] $\mathcal{P}_1$ of Figure 5. It is easy to check that $\mathsf{Im}\,|\mathcal{P}_1| = \{1_\mathbb{P}, 0_\mathbb{P}\}$ and that $\widehat{\mathcal{P}_1}$ is infinite and minimal (*i.e.* has no proper quotient).

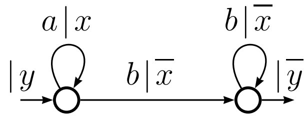


Fig. 5. The $\mathbb{P}$-automaton $\mathcal{P}_1$

4.1 The field case

The first case where a meaningful answer to the above questions can be given is when $\mathbb{K}$ is a field. It is known ([21]) that it is decidable whether a rational series with coefficients in an "effective" field[8] $\mathbb{F}$, that is whether a rational map $\tau \colon A^* \to \mathbb{F}$, has a finite image.

Moreover, if $\mathcal{A} = (I, \mu, T)$ is a *reduced* representation — *i.e.* a representation of minimal dimension n — of the function $|\mathcal{A}|$, then by a theorem of Schützenberger (*cf.* [6]), there exist some polynomials $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ in $\mathbb{F}\langle A^* \rangle$ such that for every i, j in $[1, n]$ it holds:

$$\forall w \in A^* \qquad \mu(w)_{i,j} = \langle |\mathcal{A}|, P_i\, w\, Q_j \rangle \ .$$

Hence, if $|\mathcal{A}|$ is of finite image there is a finite number of distinct matrices $\mu(w)$ and $\widehat{\mathcal{A}}$ is finite. All this can be summarised in the following

[7] Nivat [29,30] has shown that any pushdown automaton is strongly equivalent (bijection between the computations) to a finite $\mathbb{P}$-automaton and conversely.

[8] We say that a field is effective if there is a finite representation for every element that makes the field operations computable. Typically, $\mathbb{Q}$ is effective whereas $\mathbb{R}$ is not.

Proposition 12 *Let $\mathbb{F}$ be an effective field. It is decidable whether a rational map $\tau\colon A^* \to \mathbb{F}$ has finite image. Such a rational map τ with finite image is sequential, and if $\mathcal{A}$ is a reduced automaton that realises τ, then $\widehat{\mathcal{A}}$ is finite and $\tau^{-1}(k)$ is a rational language of A^* for every k in $\mathbb{F}$.*

4.2 The tropical case

In [19], Hashiguchi proved that it is decidable whether a rational map $s\colon A^* \to \mathcal{M}$ has finite image (in this context, the function, or the automaton, was said to be *limited*). This result, of paramount importance for the solution of the star height problem, has been reworked by a number of people, including Courcelle, Leung, and Simon.

This decidability result has been extended to other tropical semirings by Gaubert ([17]). Gaubert also showed ([15]) that in this case, $\mathcal{A}_s$ is finite. It then holds:

Proposition 13 *Let $\mathbb{T}$ be an effective tropical semiring. It is decidable whether a rational map $\tau\colon A^* \to \mathbb{T}$ has finite image. Such a rational map τ with finite image is sequential. From an automaton $\mathcal{A}$ that realises τ, it is possible to effectively compute an equivalent automaton $\mathcal{B}$ such that $\widehat{\mathcal{B}}$ is finite and $\tau^{-1}(k)$ is a rational language of A^* for every k in $\mathbb{T}$.*

4.3 The transducer case

We mentioned earlier that to every relation α from A^* into B^* corresponds a map $\tau_\alpha\colon A^* \to \mathfrak{P}(B^*)$ and that conversely, every map $\tau\colon A^* \to \mathfrak{P}(B^*)$ is associated with a relation α_τ from A^* into B^*. Moreover, *rationality* is preserved under this correspondance. But the notions of *image*, *inverse*, and *recognisability* take different meanings on the two terms of the correspondance and this has to be precisely described before stating a result which involves these notions.

The image of a relation α is understood to be *a subset of* B^*:

$$\mathsf{Im}\,\alpha = \{v \in B^* \mid \exists u \in A^* \quad (u,v) \in \widehat{\alpha}\} \ ,$$

whereas the image of a map τ is *a subset of* $\mathfrak{P}(B^*)$:

$$\mathsf{Im}\,\tau = \{\tau(w) \mid w \in A^*\} \ .$$

When we say that a map τ is of finite image we mean that $\mathsf{Im}\,\tau$ is a finite subset of $\mathfrak{P}(B^*)$ and it may well be the case that the image of the associated relation α_τ is infinite.

The inverse of α is the relation α^{-1} from B^* into A^* defined by

$$\forall v \in B^* \qquad \alpha^{-1}(v) = \{u \in A^* \mid (u,v) \in \widehat{\alpha}\} \ ,$$

whereas the inverse of τ is defined by

$$\forall X \subseteq B^* \qquad \tau^{-1}(X) = \{u \in A^* \mid \tau(u) = X\} \ .$$

For instance, $(\tau_\alpha)^{-1}$ and $\tau_{\alpha^{-1}}$ are different applications into $\mathfrak{P}(A^*)$: the former is a partial one from $\mathfrak{P}(B^*)$, the latter a total one from B^*.

We also recalled that rational relations are not all recognisable, that is, their graph is not necessarily a recognisable subset of $A^* \times B^*$ [eventhough every rational relation corresponds to a rational[9] map from A^* into $\mathrm{Rat}\, B^*$, that is, to a $\mathrm{Rat}\, B^*$-*recognizable series* over A^*]. Recognisable relations α are characterised by the so-called *Mezei's Theorem* (*cf.* [13,5]): $\widehat{\alpha}$ is a recognisable subset of $A^* \times B^*$ if, and only if, there exists a finite set I of pairs (X_i, Y_i) of subsets, X_i in $\mathrm{Rat}\, A^*$ and Y_i in $\mathrm{Rat}\, B^*$ such that

$$\widehat{\alpha} = \bigcup_{i \in I} X_i \times Y_i \ . \tag{3}$$

We are now ready to state and prove the following.

Proposition **14** *A rational map* $\tau \colon A^* \to \mathrm{Rat}\, B^*$ *is of finite image if and only if the corresponding relation* $\alpha_\tau \colon A^* \to B^*$ *is recognisable.*

Proof. The condition is sufficient for if α is recognisable, τ_α has at most $2^{|I|}$ distinct images, where I is the finite set in (3).

The formula:

$$\forall X \in \mathsf{Im}\,\tau \qquad \tau^{-1}(X) = \bigcap_{U \subseteq X} \alpha_\tau^{-1}(U) \setminus \bigcup_{V \not\subseteq X} \alpha_\tau^{-1}(V) \ , \tag{4}$$

where U and V range over the *atoms* of the (finite) Boolean algebra generated by $\mathsf{Im}\,\tau$, shows that $X\tau^{-1}$ is in $\mathrm{Rat}\, A^*$ and thus α_τ is recognisable since:.

$$\widehat{\alpha_\tau} = \bigcup_{X \in \mathsf{Im}\,\tau} \tau^{-1}(X) \times X \ . \qquad \blacksquare$$

[9] As noted in [12], it is an open problem to know whether a rational map $\tau \colon A^* \to \mathfrak{P}(B^*)$ with values in $\mathrm{Rat}\, B^*$ always defines a rational relation, another Fatou problem.

Equation (4) amounts to say that if $\mathcal{A}$ realises τ of finite image, $\mathcal{A}_\tau$ is finite and since it is undecidable whether a rational relation is recognisable ([14]), we can state:

COROLLARY **15** (i) *It is undecidable whether a rational relation corresponds to a rational map of finite image.*

(ii) *If a* Rat B^**-automaton* $\mathcal{A}$ *realises a recognisable relation then a quotient of* $\widehat{\mathcal{A}}$ *is finite.*

Figure 6 shows a transducer $\mathcal{T}_1$ which realises a recognisable relation ω although $\widehat{\mathcal{T}_1}$ is not finite, and the finite minimal quotient of $\widehat{\mathcal{T}_1}$.

Fig. 6. Two transducers for the universal relation ω on a^*

5 The field case

Let us suppose now that $\mathbb{K}$ is an infinite skew field (division ring), which we denote $\mathbb{F}$. The characterisation of recognisability by the residuals (Proposition 4) takes then the following form.

PROPOSITION **16 (Schützenberger *cf.* [6])** *An* $\mathbb{F}$*-series* s *is recognisable if, and only if,* $\mathbf{R}_s$ *generates a subspace of* $\mathbb{F}\langle\langle A^*\rangle\rangle$ *of finite dimension.*

The subspace generated by $\mathbf{R}_s$ is stable — since $\mathbf{R}_s$ is stable — and canonically associated with s, contrary to the finitely generated stable submodule whose existence is garanteed by Proposition 4 in the general case. The *dimension* of the subspace generated by $\mathbf{R}_s$ is known as the *rank* of s; it is a lower bound for the dimension of any $\mathbb{F}$-automaton which realises s.

From now on, we suppose that $\mathbb{F}$ is an effective field. The preceding proposition can be given then an effective version.

THEOREM **17 (Schützenberger *cf.* [6])** *Let* $\mathcal{A}$ *be a finite* $\mathbb{F}$*-automaton and* s *the series it realises. An equivalent* $\mathbb{F}$*-automaton of minimal dimension rank of* s *is effectively computable from* $\mathcal{A}$*.*

An $\mathbb{F}$-automaton $\mathcal{A} = (I, \mu, T)$ of dimension n is said to be *reduced* if n is the rank of $|\mathcal{A}|$, that is, if $\mathcal{A}$ is of minimal dimension. It is known (*cf.* [6,34])

that $\mathcal{A}$ is reduced if, and only if, $I\,\mu(A^*) = \{I\,\mu(w) \mid w \in A^*\}$ and $\mu(A^*)\,T$ both generate $\mathbb{F}^n$.

Proposition **18** *Let $\mathcal{A}$ be a finite $\mathbb{F}$-automaton and s the series it realises. If $\mathcal{A}$ is reduced, then $\widehat{\mathcal{A}}$ is isomorphic to $\mathcal{A}_s$.*

Proof. Let $\mathcal{A} = (\,I, \mu, T\,)$ and $s = |\mathcal{A}|$. Let us consider then the application $\Phi_{\mathcal{A}}$ defined by

$$\begin{aligned} \Phi_{\mathcal{A}} \colon \mathbb{F}^n &\longrightarrow \mathbb{F}\mathrm{Rat}\,A^* \\ \xi &\longmapsto \sum_{w \in A^*} (\xi\,\mu(w)\,T)\,w \end{aligned}$$

For every ξ, $\Phi_{\mathcal{A}}(\xi) = 0_{\mathbb{F}}$ if, and only if, $\xi\,\mu(w)\,T = 0_{\mathbb{F}}$ for every w, which implies that the image of ξ by any linear form is null since $\mu(A^*)\,T$ generates $\mathbb{F}^n$. Hence, $\Phi_{\mathcal{A}}(\xi) = 0_{\mathbb{F}}$ if and only if $\xi = 0_{\mathbb{F}}$ and $\Phi_{\mathcal{A}}$ is injective. The application $\Phi_{\mathcal{A}}$ is therefore a one-to-one mapping from $\mathbf{R}_{\mathcal{A}}$ onto $\mathbf{R}_s$. ■

Propositions 16 and 18 have their counterparts for the sequentialisation. First, Proposition 5 becomes:

Proposition **19 (Reutenauer [33])** *An $\mathbb{F}$-series s is sequential if, and only if, $\mathbf{R}_s$ is projectively finite.*

The bijection between $\mathbf{R}_{\mathcal{A}}$ and $\mathbf{R}_s$, gives then:

Proposition **20** *Let $\mathcal{A}$ be a finite $\mathbb{F}$-automaton and s the series it realises. If $\mathcal{A}$ is reduced, then $\check{\mathcal{A}}$ is isomorphic to $\mathcal{D}_s$.*

In other words, if $\mathcal{A}$ is a reduced $\mathbb{F}$-automaton, it realises a sequential series if and only if the set $\mathbf{G}_{\mathcal{A}}$ is finite. If we refer to the general sheme of Figure 4, Propositions 18 and 20 state that in the field case, and by way of reduction, it is possible to bring the level of 'enumerable' machines at the level of 'abstract' minimal machines (which are thus not abstract anymore).

The problem of deciding whether a series is sequential amounts thus to decide whether $\mathbf{G}_{\mathcal{A}}$ is finite. The decidability depends both on algebraic properties of the field and on the effectivity of the operations in the field.

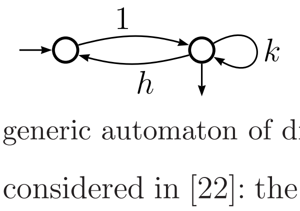


Fig. 7. A generic automaton of dimension 2

The simplest case has been considered in [22]: the alphabet has only one letter and $\mathbf{R}_s$ has dimension 2; a reduced automaton for s is thus given at Figure 7.

If h^2/k is equal to -1, -2, or -3 then s is realised by a sequential automaton with 3, 4, and 6 states respectively. It can be shown that these are the only possibilities if s is $\mathbb{Z}$ or $\mathbb{Q}$-recognisable. If s is $\mathbb{R}$-recognisable, there are pairs of values for h and k such that s is realised by a sequential automaton with n states, for any n — but one goes then beyond effectivity.

We thus end up with the following problem; we conjecture a positive answer, at least in the case of one letter alphabet.

PROBLEM **1** *Is it decidable whether a $\mathbb{Z}$ or $\mathbb{Q}$-recognizable series is sequential?*

6 The idempotent case

As we said in the introduction, we focus here on two classes of automata which have been extensively studied from the point of view of sequentiality:

(a) the tropical automata,

(b) the functional transducers.

In both cases one can say that the multiplicities admit a *gcd*: min or max is a gcd for the tropical semirings (according to their domain), and for functional transducers the longest common prefix plays the role of a gcd, eventhough there is no gcd on the multiplicity semiring $\mathfrak{P}(B^*)$.

The functional transducers happen then to be *simpler* than the tropical automata for they correspond to a special class of the latter, called 1-valued. The decision of sequentiality for such automata goes through a characteristic topological property and then an effective procedure to decide whether the property holds. If the hypothesis of 1-valuedness is removed, such a decision procedure is not known:

PROBLEM **2** *Is it decidable whether a recognisable tropical series is sequential?*

After presenting the decision procedure for 1-valued automata, we consider a few particular cases where a positive answer has been given to Problem 2.

6.1 A topological criterion

In the sequel, a $\mathbb{K}$-automaton (over A^*) is either a tropical automaton or a functional transducer. And $d(y, z)$, for y, z in A^* or in $\mathbb{K}$, denotes the Euclidean distance on tropical semirings, or the prefix distance on words.

Definition **21** *Let s be in $\mathbb{K}\langle\!\langle A^* \rangle\!\rangle$.*

(i) *The series s is* uniformely bounded *if*

$$\forall M > 0\,,\ \exists K > 0\,,\ \forall u, v \in \mathsf{supp}\, s \qquad d(u,v) \leqslant M \Rightarrow d(<s,u>, <s,v>) \leqslant K.$$

(ii) *Let k be an integer. The series s is k-*Lipschitzian *if*

$$\forall M > 0\,,\ \forall u, v \in \mathsf{supp}\, s \qquad d(u,v) \leqslant M \Rightarrow d(<s,u>, <s,v>) \leqslant k\,M.$$

The series s is Lipschitzian *if s is k-Lipschitzian for a certain k.*

Clearly, if a series s is Lipschitzian then s is uniformely bounded. And the definition of sequential series directly implies:

Proposition **22** *A sequential series s is Lipschitzian.*

The converse of Proposition 22 does not hold in general. The $\langle\,\mathbb{N}, \mathsf{max}, +\,\rangle$-automaton $\mathcal{C}_4$ of Figure 8 realises the series:

$$\forall w \in A^* \qquad <|\mathcal{C}_4|, w> = \mathsf{max}\{|w|_a, |w|_b\}$$

which is *not sequential*. Indeed, as

$$\forall n, m \in \mathbb{N} \qquad <[(a^n)^{-1}|\mathcal{C}_4|]^\sharp, b^m> = \mathsf{max}(0, m-n) \qquad\qquad ,$$

the number of translations of $|\mathcal{C}_4|$ is infinite .

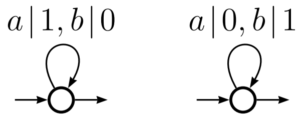


Fig. 8. An automaton $\mathcal{C}_4$ for a Lipschitzian but non sequential series

The converse of Proposition 22 holds for functional transducers and for a subfamily of tropical automata: the 1-valued ones. A tropical automaton is *1-valued if all computations spelling the same word give that word the same multiplicity.* (Functional transducers are clearly also the 1-valued automata among the $\mathfrak{P}(B^*)$-automata whose entries are restricted to B^*.)

Theorem **23 (Small Sequentiality Theorem [9,35])** *Let s be a 1-valued recognisable tropical series or a recognisable word function. The series s is uniformely bounded if, and only if, s is sequential.*

We call this theorem *small* to contrast with another one which we call *large* and which only holds for word functions. It states that the same equivalence holds true for those word functions with the only property that their inverse preserves recognisable sets: a generalisation of Ginsburg–Rose Theorem also due to Choffrut ([10], *cf.* also [7]).

Theorem 23 can be proved directly for transducers and for automata with multiplicity in a *discrete* topological semiring. The idea of the proof ([35]) is that if $\mathcal{A}$ is trim and $\lfloor\mathcal{A}\rfloor$ uniformely bounded, then the entries of the vectors that are the states of $\breve{\mathcal{A}}$ are bounded and $\breve{\mathcal{A}}$ is therefore finite under the current hypothesis.

If the tropical semiring is not discrete the above argument of boundedness does not implies finiteness. In any case, we need another tool which gives a procedure to test whether an automaton realises an uniformely bounded series: it is the role of the twinning property.

6.2 *The twinning property*

The twinning property has been introduced by Choffrut in [9] for transducers. This notion has been adapted to tropical automata by Mohri [28]. We present it now under a form that applies both to transducers and tropical semirings, and makes it directly decidable.

For transducers, the conjugacy of words is involved in the twinning property. Since tropical semirings are commutative the twinning property is implied by a weaker property, that we call the *weak* twinning property.

Since the twinning property amounts to compare paths with the same input label, it seems us appropriate to describe and study it on the square of a $\mathbb{K}$-automaton (as it is done for instance in [2]).

6.2.1 *Squaring automata*

Let $\mathcal{A} = \langle I, \mu, T\rangle$ be a $\mathbb{K}$-automaton of dimension Q. We define *the square of* $\mathcal{A}$, $\mathcal{A}\times\mathcal{A} = \langle J, \nu, U\rangle$ to be *the accessible part* of the $\mathbb{K}\times\mathbb{K}$-automaton of dimension $Q\times Q$ defined by:

$$J_{p,q} = \begin{cases} (I_p, I_q) & \text{if } I_p \neq 0_{\mathbb{K}} \text{ and } I_q \neq 0_{\mathbb{K}}, \\ 0_{\mathbb{K}\times\mathbb{K}} & \text{otherwise;} \end{cases}$$

$$\nu(a)_{(p,q),(p',q')} = \begin{cases} (\mu(a)_{p,p'}, \mu(a)_{q,q'}) & \text{if } \mu(a)_{p,p'} \neq 0_{\mathbb{K}} \text{ and } \mu(a)_{q,q'} \neq 0_{\mathbb{K}}, \\ 0_{\mathbb{K}\times\mathbb{K}} & \text{otherwise;} \end{cases}$$

$$U_{p,q} = \begin{cases} (T_p, T_q) & \text{if } T_p \neq 0_{\mathbb{K}} \text{ and } T_q \neq 0_{\mathbb{K}}, \\ 0_{\mathbb{K}\times\mathbb{K}} & \text{otherwise.} \end{cases}$$

DEFINITION **24 (Weak Twinning Property)** *Let $\mathcal{A}$ be a $\mathbb{K}$-automaton.*

(i) *Two states p and q of $\mathcal{A}$ are* weakly twin *if every circuit of $\mathcal{A}\times\mathcal{A}$ that contains the state (p,q) has multiplicity (h,k), where h and k are* conjugate *in $\mathbb{K}$.*

(ii) *The automaton $\mathcal{A}$ itself is said to have the* weak twinning property *if for any (p,q) in $\mathcal{A}\times\mathcal{A}$ p and q are weakly twin.*

For functional transducers, this property is not sufficient to guarantee the sequentiality of the series.

6.2.2 The advance/delay action

In order to define a stronger property, we introduce *the advance/delay action* on the square of a $\mathbb{K}$-automaton ([2,34]). Let us first define the set $\Delta_{\mathbb{K}} = 1_{\mathbb{K}}\times\mathbb{K} \cup \mathbb{K}\times 1_{\mathbb{K}} \cup \mathbf{0}$ and the canonical mapping δ from $\mathbb{K}\times\mathbb{K}$ onto $\Delta_{\mathbb{K}}$:

$$\delta : (u,v) \longmapsto \begin{cases} (u,v)^\sharp & \text{if } (u,v)^\sharp \in \Delta_{\mathbb{K}} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The semiring $\mathbb{K}\times\mathbb{K}$ naturally acts on $\Delta_{\mathbb{K}}$ by an action ω that we prefer ton denote by a '$\cdot$':

$$\forall u,v \in \mathbb{K} \quad \mathbf{0}\cdot(u,v) = \mathbf{0}\,, \quad \forall (x,y)\in\Delta_{\mathbb{K}} \quad (x,y)\cdot(u,v) = \delta(x\,u, y\,v)$$

We denote by $\mathcal{A}_\Delta$ the *accessible part* of a kind of product of $\mathcal{A}\times\mathcal{A}$ by the action ω; it is the $\mathbb{B}$-automaton with states in $Q\times Q\times\Delta_{\mathbb{K}}$ such that:

(i) the initial states of $\mathcal{A}_\Delta$ are the $(p,q,\delta(I_{p,q}))$ for (p,q) in $\mathcal{A}\times\mathcal{A}$ and $\delta(I_{p,q}) \neq \mathbf{0}$;

(ii) if (p,q,x) is a state of $\mathcal{A}_\Delta$ and $\mu(a)_{(p,q),(p',q')} = z \neq 0_{\mathbb{K}\times\mathbb{K}}$ then $(p',q',x\cdot z)$ is a state of $\mathcal{A}_\Delta$ and there is a transition from (p,q,x) to $(p',q',x\cdot z)$ labeled by a.

The original definition of the twinning property takes then the equivalent following form.

DEFINITION **25 (Twin Property [9])** *Let $\mathcal{A}$ be a $\mathbb{K}$-automaton. Two states p and q of $\mathcal{A}$ are* twin *if whenever (h,k) is the multiplicity of a circuit around (p,q) in $\mathcal{A}\times\mathcal{A}$,* different from $(1_{\mathbb{K}}, 1_{\mathbb{K}})$*, the two assertions hold true:.*

(i) *h and k are* conjugate *in $\mathbb{K}$ (i.e. p and q are* weakly twin*):*

(ii) *if (p,q,x) is a state of $\mathcal{A}_\Delta$ then $x \neq \mathbf{0}$ and $x \cdot (h,k) = x$.*

The automaton $\mathcal{A}$ has the twinning property *if any two states p and q such that (p,q) is in $\mathcal{A}\times\mathcal{A}$ are twin.*

It is not difficult to verify that $\mathcal{A}$ has the twinning property if, and only if, $\mathcal{A}_\Delta$ is finite and for every state $(p,q,\mathbf{0})$ in $\mathcal{A}_\Delta$ the only possible multiplicity of a circuit around (p,q) in $\mathcal{A}\times\mathcal{A}$ is $(1_\mathbb{K}, 1_\mathbb{K})$. It is immediate that a tropical automaton $\mathcal{A}$ has the twinning property if and only if it has the weak twinning property.

Proposition **26** **([9])** *The twinning property is decidable.*

Because of the importance of the sequentiality, the design of algorithms for deciding the twinning property has attracted much attention. It has been shown to be of polynomial complexity for transducers ([38,2,1]) and for tropical automata ([8,1]).

6.2.3 Decision procedure

Proposition **27** **([9,28])** *If a $\mathbb{K}$-automaton $\mathcal{A}$ has the twinning property, then $\check{\mathcal{A}}$ is finite.*

Hence, the following chain of implications holds:

$$\begin{array}{llll} \mathcal{A} \text{ has twin. p.} & \Rightarrow \ \check{\mathcal{A}} \text{ is finite} & \Rightarrow \ |\mathcal{A}| \text{ is sequential} & \Rightarrow \\ & \Rightarrow \ |\mathcal{A}| \text{ is Lipschitzian} & \Rightarrow \ |\mathcal{A}| \text{ is uniformely bounded} & \end{array} \tag{5}$$

For 1-valued tropical automata and functional transducers, the twinning property is a characterisation of sequentiality by way of the following:

Proposition **28** *Let $\mathcal{A}$ be a 1-valued $\mathbb{K}$-automaton. If the series realised by $\mathcal{A}$ is uniformely bounded, then $\mathcal{A}$ has the twinning property.*

And Proposition 26 implies then:

Corollary **29** *The sequentiality is decidable for 1-valued tropical automata and functional transducers.*

If the hypothesis of 1-valuedness is removed, then the first three implications in (5) are strict. The $\mathcal{M}$-automaton $\mathcal{B}_3$ of Figure 3 (a) has not the twinning property and still $\check{\mathcal{B}}_3$ is finite. The same graph, interpreted as the $\langle\,\mathbb{N}, \mathsf{max}, +\,\rangle$-automaton $\mathcal{C}_3$ realises the function that maps every word of length n onto $2n-1$ if n is positive (and 1_{A^*} onto 0); this function is sequential, but for every positive n, $I\,\mu(a^n) = (n, 2n-1)$ and $\check{\mathcal{C}}_3$ is not finite. As already mentioned, the automaton $\mathcal{C}_4$ shown at Figure 8 realises a Lipschitzian series which is not sequential.

The two handsides of the last implication in (5) do not refer to rational series. For general series s, s uniformely bounded does not imply s Lipschitzian (*e.g.* $s\colon a^* \to a^*$ with $<\!s, a^{n^2}\!> = a^{n^3}$ and s undefined otherwise).

But, at least under the hypothesis that the multiplicity semiring $\mathbb{K}$ is positive, a $\mathbb{K}$-rational series is Lipschitzian as soon as it is uniformely bounded. It follows from a 'no arbitrary large hole' property in rational language that is stated in the next lemma.

Lemma **30** *Let L be a rational language of A^*. There exists an integer M such that for every u and v in L there exists a sequence of $k+1$ words in L, $w_0 = u$, w_1, w_2, $\ldots$, w_{k-1}, and $w_k = v$ — where $k = d(u,v)$ — such that $d(w_{j-1}, w_j) < M$ for every j in $[1,k]$.*

Proof. Let $\mathcal{D}$ be the trim minimal deterministic automaton that accepts L, n its number of states, i its initial state and take $M = 2n$. For every state p, let t_p be a shortest word that labels a path from p to a final state. It holds $|t_p| < \frac{M}{2}$.

Let u and v be two words in L and z their longest common prefix: $u = z\,u'$ and $v = z\,v'$; it holds $d(u,v) = |u'|+|v'|$. Let $q = i \cdot z$. Let $k' = |u'|$ and, for every j in $[0,k']$, write u'_j for the prefix of length j of u' and let $p_j = q \cdot u'_j$, and $x_j = z\,u'_j\,t_{p_j}$.

Clearly, x_j is in L, $d(x_{j-1}, x_j) < M$ for every j in $[0,k']$ and $x_{k'} = u$.

Doing the same thing on the v side, we find a sequence of $|v'|+1$ words y_ℓ in L such that $x_0 = y_0$, $y_{|v'|} = v$ and $d(y_{\ell-1}, y_\ell) < M$ for every ℓ in $[0,|v'|]$. Up to a renaming, we have found the desired sequence. ■

Proposition **31** *Let $\mathbb{K}$ be a positive semiring and s a $\mathbb{K}$-rational series. If s is uniformely bounded, then s is Lipschitzian.*

Proof. Let $\mathcal{A}$ be a $\mathbb{K}$-automaton that realises s. As $\mathbb{K}$ is positive, $\mathsf{supp}\,s$ is a rational language and let M be the integer associated with it by the preceding lemma. By hypothesis, there exists K such that for every x and y in the support of s $d(x,y) \leqslant M$ implies $d(<\!s,x\!>, <\!s,y\!>) \leqslant K$. Now, for every u and

v in the support of s, we have:

$$d(\langle s,u\rangle,\langle s,v\rangle)\leqslant\sum_{k=1}^{d(u,v)}d(\langle s,w_{k-1}\rangle,\langle s,w_k\rangle)\leqslant K d(u,v)\ ,$$

where the w_k are the words in the support of s the existence of which is assured by Lemma 30. Hence, s is Lipschitzian. ■

The tropical semirings and $\operatorname{Rat} B^*$ are obviously positive semirings: if $\mathbb{K}$ is idempotent, then $x\oplus y=0_{\mathbb{K}}$ implies $x=x\oplus x\oplus y=x\oplus y=0_{\mathbb{K}}$.

6.3 Particular cases

From now on, $\mathbb{K}$ is a tropical semiring. In some particular cases, sequentiality is decidable even for $\mathbb{K}$-automata which are not 1-valued.

6.3.1 Unary tropical automata

We call *unary* an automaton, or a series, over a *one letter* alphabet.

A (unary tropical) series s is said to be *arithmetic*, with *period* p and *ratio* r, if there exist two integers n and m such that $\mathsf{supp}\, s$ is the arithmetic sequence $\{p\,i+n\mid i\in\mathbb{N}\}$ and the image of s is another arithmetic sequence: $\langle s,a^{p\,i+n}\rangle=r\,i+m$ for every i in $\mathbb{N}$. A series is *ultimately arithmetic* if it is the union of a series of finite support and of an arithmetic series.

It is quite obvious that a sequential (unary tropical) series is a finite union of ultimately arithmetic series with same period and ratio.

Proposition **32 (Gaubert [16])** *Any rational unary tropical series is effectively decomposable into a finite disjoint union of ultimately arithmetic series.*

What is proved in [16] is that the set of series that are finite disjoint union of ultimately arithmetic series is rationally closed. This gives a procedure to decide the sequentiality, since it is sufficient to check that all infinite ultimately arithmetic sequences have the same ratio.

Corollary **33** *Sequentiality is decidable for recognisable unary tropical series.*

Let us note that the sequentiality can indeed be decided by an algorithm that works directly on an automaton $\mathcal{A}$ that realises the series (*cf.* [24]). First, critical circuits — *i.e.* circuits with maximal average multiplicity if $\mathbb{K}$ is a $\langle\, \mathsf{max}, + \,\rangle$ semiring — have to be detected. Then, the decision depends only on $\mathsf{supp}\, \mathcal{A}$; let L_1 be the set of words that label a path containing at least one state of a critical circuit and let L_2 be the complement of L_1 in the language accepted by $\mathcal{A}$. The series realised by $\mathcal{A}$ is sequential if and only if L_2 is finite.

6.3.2 Finitely ambiguous tropical automata

An automaton is *finitely ambiguous* if the number of paths labeled by any word is uniformely bounded.

PROPOSITION **34 ([23])** *Sequentiality is decidable on finitely ambiguous automata.*

Actually, it is proved in [23] that it is decidable whether the series realised by a finitely ambiguous automaton can be realised by a 1-valued automaton. The decidability of sequentiality follows from Corollary 29.

Let us note that proving a decidability result inside the family of finitely ambiguous tropical automata makes sense as it is decidable whether a tropical automaton is finitely ambiguous [39]. Moreover, the equivalence is decidable in this family [20]. Along the same line, let us also recall the following stronger statement (an automaton is *finitely valued* if the number of values of paths labeled by any word is uniformely bounded).

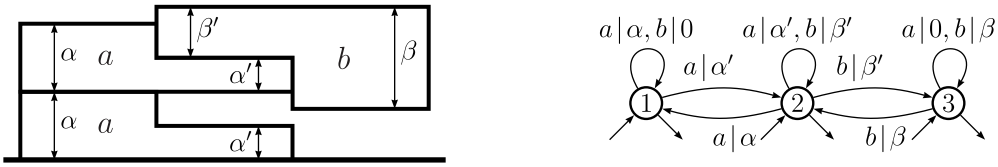
PROPOSITION **35 ([37])** *A finitely valued tropical automaton can be turned into a finite union of unambiguous tropical automata.*

6.3.3 Heap automata

A certain subfamily of tropical automata can represent a computation model, called *heap model*, that is used to modelise resource allocation problems and that can be described as follow.

A heap model — heap stands for ‘heap of pieces’ — consists of a finite set of slots and a finite set A of pieces (*cf.* [36]). Every piece is assigned a subset of slots: its support, and has a fixed shape. Pieces fall one on the other, thus forming a heap. The configuration of a heap is given by its *profile*, that is, the vector whose entries are the height at every slot (*cf.* Figure 9 (a)). [10]

[10] The interpretation for the resource allocation problem goes as follow: every slot represents a resource and every piece modelises a task. The support of the piece is the set of resources needed to complete the task: the thickness of a piece on a slot measures the acces time of the task on that resource and the shape of the piece



(a) A heap model... (b) ...and its heap automaton

Fig. 9. A heap model

A heap of pieces with n slots can be represented by a $\langle \mathsf{max}, + \rangle$-automaton $\mathcal{A} = (I, \mu, T)$ of dimension n on the alphabet A of pieces. For every i in $[1, n]$, $I_i = T_i = 0$. A piece a is characterised by a matrix $\mu(a)$, where $\mu(a)_{i,j}$ is the difference between the bottom of the piece a on the slot i and the top of the piece on the slot j. For convenience, when the column i is not in the support of a, $\mu(a)_{i,i} = 0$ and, for every $j \neq i$, $\mu(a)_{i,j} = \mu(a)_{j,i} = -\infty$. (Recall that $-\infty$ is the identity for 'addition' and 0 the identity for 'multiplication' in such multiplicity semiring.)

It is easy to verify that if the vector ξ is the profile of the heap, $\xi\,\mu(a)$ is the profile after the fall of the piece a. Thus, the automaton $\mathcal{A}$ realises the series s in which the coefficient of the word $w = a_1 a_2 \ldots a_n$ is the maximal height of the heap obtained by stacking a_1, a_2 up to a_n; it is the minimal time taken by the sequence of tasks a_1, a_2 to a_n to be completed. Such an automaton is called a heap automaton.

Example **2** *The heap automaton corresponding to the heap model of Figure 9 (a) is shown at Figure 9 (b). The pieces a and b are represented by the matrices:*

$$\mu(a) = \begin{bmatrix} \alpha & \alpha' & -\infty \\ \alpha & \alpha' & -\infty \\ -\infty & -\infty & 0 \end{bmatrix} \qquad \mu(b) = \begin{bmatrix} 0 & -\infty & -\infty \\ -\infty & \beta' & \beta' \\ -\infty & \beta & \beta \end{bmatrix} .$$

A heap automaton is generally not finitely ambiguous, but the decision of sequentiality can be improved on this particular class. The vector $I\,\mu(w)$ is the profile of the heap after the sequence w. Instead of computing the states of $\breve{\mathcal{A}}$ (*i.e* the vectors $(I\,\mu(w))^\sharp$), the normalisation of vectors is improved. For every vector ξ, the *completed* vector $\overline{\xi}$ is the maximal vector such that

$$\forall a \in A\,,\ \forall i \in \mathsf{supp}\, a \qquad \left(\overline{\xi}\,\mu(a)\right)_i = \left(\xi\,\mu(a)\right)_i \ .$$

how these resources are combined for the task. The pieces falling one on the other correspond to the fact that one task begins as soon as the previous one ends. The profile of the heap of pieces describes the earliest time at which every resource is free after the completion of the corresponding sequence of tasks.

For every ξ, $\overline{\xi}$ is computable from ξ and the shape of pieces of A. Figure 10 shows a heap of height ξ and the corresponding vector $\overline{\xi}$.

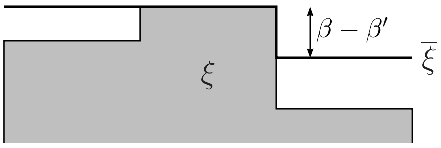


Fig. 10. The normalisation of the height vector

THEOREM **36 (Gaubert and Mairesse [18])** *Let $\mathcal{A} = \langle I, \mu, T \rangle$ be a heap automaton. If the set $\left[\overline{I\,\mu(w)}\right]^\sharp$ is finite, then the series realised by $\mathcal{A}$ is sequential. In this case, this finite set is the set of states of a sequential automaton equivalent to $\mathcal{A}$.*

It is an open question to know whether this property is necessary for the sequentiality. For the *two piece case*, Mairesse and Vuillon have answered the question. They have shown in [27] that any heap automaton over a two letter alphabet belongs to one of the three families:

- The support of the pieces are disjoint: the automaton is not sequentialisable except if the height of a piece is equal to zero.
- The support of the pieces are equal: the automaton is sequentialisable.
- Otherwise, the automaton is equivalent to the case solved in Theorem 37.

THEOREM **37 ([27])** *Let $\mathcal{A}$ be a heap automaton over a two letter alphabet of pieces with a single common slot. The automaton $\mathcal{A}$ is sequentialisable if and only if the height of each piece on this slot is positive or if the ratio of the maximal height of each of the both pieces is a rational number.*

Acknowledgements A first version of this survey was written while the second author was visiting the LaCIM at UQAM. The friendly hospitality of Christophe Reutenauer, as well as the illuminating discussions with him on the subject, are gratefully acknowledged. The comments of two careful referees were of great help in the writing of the second version.

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