The rational skimming theorem

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Abstract

We define the notion of $\mathbb{K}$-covering of automata with multiplicity (in a semiring $\mathbb{K}$) that extend the one of covering of automata. We make use of this notion, together with the Schützenberger construct that we have explained in a previous work and that we briefly recall here, in order to give a direct and constructive proof of a fundamental theorem on $\mathbb{N}$-rational power series.

In a previous work (*cf.* [4]), we have shown how a construction, proposed by Schützenberger (in [8] and [9]) in order to prove that rational functions are unambiguous, can be given a central position in the theory of relations and functions realized by finite automata. The other basic results such as the "Rational Cross-section Theorem", the "Rational Uniformisation Theorem" (that is dual to the preceeding one), and the "Decomposition Theorem " (of rational functions into sequential functions) appear then as direct and formal consequences of it.

We have explained that this construction is indeed *a construction on finite automata* and we have described it in the framework of *covering of automata* — which is derived from the notion of covering of graphs that was proposed by Stallings ([10]) — and which makes (in our opinion) the whole subject much clearer.

The purpose of the present communication is to extend the concept of covering to the one of $\mathbb{K}$-covering that apply to automata *with multiplicity* in a semiring $\mathbb{K}$. And to make use of this notion together with the Schützenberger construct quoted above, in order to establish another result, due to Schützenberger as well, and that we call the *Rational Skimming Theorem*.

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Theorem 1 [7] *If s is a $\mathbb{N}$-rational power series on A^*, then the series s' obtained from s by substracting 1 to every non-zero coefficient of s, i.e. the series*

$$s' = s - \underline{\mathsf{supp}\, s}$$

is a $\mathbb{N}$-rational power series as well.

This result is not new, by far. In [2, Theorem VI.11.1], it is obtained as the consequence of the Rational Cross-section Theorem quoted above (and of some other results such as the division theorem). In [6, Theorem II.8.6] and in [1, Theorem V.2.1], more direct proofs are given (the attribution to Schützenberger is made in the latter reference).

The proof presented here is hopefully simpler than the preceeding ones and corresponds to an explicit construction on automata. A complete exposition of all that matter, $\mathbb{K}$-coverings and their use in the theory of $\mathbb{K}$-rational series will be found in [5].

1 The Schützenberger covering

We basically follow the definitions and notation of [2] which we use without further notice. Those that follow in this section and that are more original have been described in detail in [4].

A (finite) automaton over a finite alphabet A, $\mathcal{A} = \langle\, Q, A, E, I, T\,\rangle$, is a directed labelled graph where Q, I and T are respectively the (finite) sets of states, initial states and terminal states, and E is the set of labelled edges. The *language accepted* by $\mathcal{A}$, that is the set of the labels of the successful computations in $\mathcal{A}$, also called *the behaviour* of $\mathcal{A}$, is denoted by $|\mathcal{A}|$.

A *morphism* φ *from an automaton* $\mathcal{B} = \langle\, R, A, F, J, U\,\rangle$ *into an automaton* $\mathcal{A} = \langle\, Q, A, E, I, T\,\rangle$ is indeed a pair of mappings (both denoted by φ): one between the set of states $\varphi\colon R \to Q$, and one between the set of edges $\varphi\colon F \to E$, which are consistent with the structure of the automata, that is, for every f in F:

i) the origin of $f\varphi$ is the image (by φ) of the origin of f;

ii) the label of $f\varphi$ is equal to the label of f;

iii) and $\quad J\varphi \subseteq I \quad$ and $\quad U\varphi \subseteq T$.

These conditions imply that the image of a successful computation in $\mathcal{B}$ is a successful computation in $\mathcal{A}$, that their labels are equal, and thus that $|\mathcal{B}| \subseteq |\mathcal{A}|$ holds.

For every state q of an automaton $\mathcal{A} = \langle Q, A, E, I, T \rangle$, let us denote by $\mathsf{Out}_{\mathcal{A}}(q)$ the set[1] of edges of $\mathcal{A}$ *the origin of which* is q, that is edges that are "going out" of q. One defines dually $\mathsf{In}_{\mathcal{A}}(q)$ as the set of edges of $\mathcal{A}$ *the end of which* is q, that is edges that are "going in" q.

If φ is a morphism from $\mathcal{B} = \langle R, A, F, J, U \rangle$ into $\mathcal{A} = \langle Q, A, E, I, T \rangle$ then for every r in R, φ maps $\mathsf{Out}_{\mathcal{B}}(r)$ into $\mathsf{Out}_{\mathcal{A}}(r\varphi)$, and $\mathsf{In}_{\mathcal{B}}(r)$ into $\mathsf{In}_{\mathcal{A}}(r\varphi)$. We say that φ is *Out-surjective* (resp. *Out-bijective*, *Out-injective*) if for every r in R the restriction of φ to $\mathsf{Out}_{\mathcal{B}}(r)$ is surjective onto $\mathsf{Out}_{\mathcal{A}}(r\varphi)$ (resp. bijective between $\mathsf{Out}_{\mathcal{B}}(r)$ and $\mathsf{Out}_{\mathcal{A}}(r\varphi)$, injective). Accordingly, we say that φ is *In-surjective* (resp. *In-bijective*, *In-injective*) if for every r in R the restriction of φ to $\mathsf{In}_{\mathcal{B}}(r)$ is surjective onto $\mathsf{In}_{\mathcal{A}}(r\varphi)$ (resp. bijective between $\mathsf{In}_{\mathcal{B}}(r)$ and $\mathsf{In}_{\mathcal{A}}(r\varphi)$, injective).

Definition 1 *Let $\mathcal{B} = \langle R, A, F, J, U \rangle$ and $\mathcal{A} = \langle Q, A, E, I, T \rangle$; a morphism $\varphi\colon \mathcal{B} \to \mathcal{A}$ is a* covering *(resp. a* co-covering*) if the following conditions hold:*

i) *φ is* Out-bijective *(resp.* In-bijective*);*

ii) *for every i in I, there exists a* unique *j in J such that $j\varphi = i$ (resp. for every t in T, there exists a* unique *s in S such that $s\varphi = t$);*

iii) *$T\varphi^{-1} = U$ (resp. $I\varphi^{-1} = J$).*

Proposition 1 *Any covering (resp. any co-covering) $\varphi\colon \mathcal{B} \to \mathcal{A}$ induces a bijection between the successful computations in $\mathcal{B}$ and those in $\mathcal{A}$.* ■

Theorem & Definition 2 *Let $\mathcal{A}$ be an automaton and $\mathcal{A}_{\mathsf{det}}$ the determinized automaton of $\mathcal{A}$. We call* Schützenberger covering of $\mathcal{A}$ *the accessible part $\mathcal{S}$ of $\mathcal{A}_{\mathsf{det}} \times \mathcal{A}$. Then:*

i) *$\pi_{\mathcal{A}}$ is a covering from $\mathcal{S}$ onto $\mathcal{A}$.*

ii) *$\pi_{\mathcal{A}_{\mathsf{det}}}$ is an In-surjective morphism from $\mathcal{S}$ onto $\mathcal{A}_{\mathsf{det}}$.* ■

We call *immersion of* $\mathcal{A}$ a sub-automaton of a covering of $\mathcal{A}$. From all these definitions and result, one derives easily the result which is the basis of the present work.

Corollary 2 *Let $\mathcal{A}$ be an automaton on A^*. Then there exists an* unambiguous *automaton that is equivalent to $\mathcal{A}$ and that is a sub-automaton of a covering of $\mathcal{A}$.*

[1] Stallings denotes it "$\mathrm{Star}_{\mathcal{A}}(q)$". As the star is the common denomination for the generated submonoid, we cannot keep it, though it nicely conveys the idea of "a set of edges going out" of q.

Proof. Let $\mathcal{S}$ be the Schützenberger covering of $\mathcal{A}$. As $\pi_{\mathcal{A}_{\mathsf{det}}}$ is *In-surjective* from $\mathcal{S}$ onto $\mathcal{A}_{\mathsf{det}}$, one can delete edges in $\mathcal{S}$ (and possibly suppress the quality of being terminal to some of its states) in such a way that the sub-automaton $\mathcal{T}$ that is obtained is a *co-covering of* $\mathcal{A}_{\mathsf{det}}$. The automaton $\mathcal{T}$ is then *unambiguous* — as there is a one-to-one correspondence between its successful computations and those of $\mathcal{A}_{\mathsf{det}}$ — and equivalent to $\mathcal{A}_{\mathsf{det}}$, hence to $\mathcal{A}$. ■

The essence of this statement lies of course in the fact that the quoted unambiguous automaton is at the same time *equivalent to* and an *immersion of* $\mathcal{A}$. For otherwise, the deterministic automaton $\mathcal{A}_{\mathsf{det}}$ associated to $\mathcal{A}$ by the subset construction is obviously unambiguous and equivalent to $\mathcal{A}$; but it can not be immersed in $\mathcal{A}$: there is no relationships between the pathes in $\mathcal{A}$ and those in $\mathcal{A}_{\mathsf{det}}$.

Example 1 : The Figure 1 represents an automaton $\mathcal{A}_1$ that accepts all words of $\{a, b\}^*$ which contain at least one b (vertically, on the left), its determinized automaton $\mathcal{A}_{1\mathsf{det}}$, the Schützenberger covering of $\mathcal{A}_1$, and the two possible immersions that can be derived from it. □

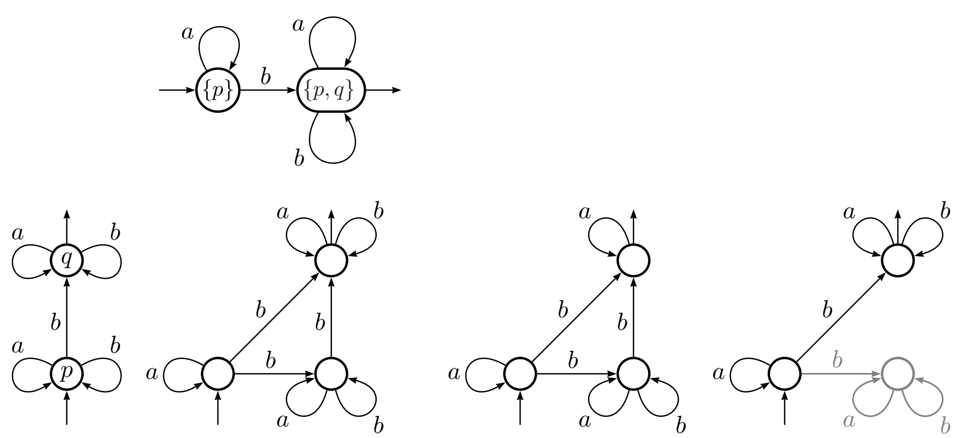

Figure 1: An automaton, its Schützenberger covering, and two immersions.

2 $\mathbb{K}$-automata

As far as *polynomials* and *power series* are concerned, we follow the definitions and notation of [1]. The set of polynomials over A^* with multiplicity in a semiring $\mathbb{K}$ is denoted by $\mathbb{K}\langle A^*\rangle$. A (finite) automaton $\mathcal{A}$ over A^* with multiplicity in a semiring $\mathbb{K}$, or $\mathbb{K}$-*automaton* for short, is a straightforward generalization of a classical automaton. It is adequatly described as a triple $\mathcal{A} = \langle\, I, E, T\,\rangle$ where

- E is a square matrix of dimension Q whose entries are polynomial over A^* with coefficients in $\mathbb{K}$, *i.e.* elements of $\mathbb{K}\langle A^*\rangle$.
- I and T are vectors of dimension Q (respectively a row vector and a column vector) with entries in $\mathbb{K}\langle A^*\rangle$.

The dimension Q is called *the set of states of* $\mathcal{A}$, every entry $E_{p,q}$ of E is the *label of the transition* that goes from p to q in $\mathcal{A}$.[2] The *behaviour* of $\mathcal{A}$, denoted by $|\mathcal{A}|$, is defined if and only if the star of the matrix E, E^*, is defined and it holds:

$$|\mathcal{A}| = I \cdot E^* \cdot T$$

A power series is $\mathbb{K}$-*rational* if and only if it is the behaviour of a finite $\mathbb{K}$-automaton[3].

A polynomial is *proper* if its constant term (*i.e.* the coefficient of 1_{A^*}) is zero, a $\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, E, T\,\rangle$ is *proper* if every entry of E is proper and every entry of I and T are in $\mathbb{K}$. It is known that the behaviour of $\mathcal{A}$ is defined if and only if it is equivalent to a proper $\mathbb{K}$-automaton.

Example 2 : Let us consider the $\mathbb{N}$-automaton over $\{a,b\}^*$, $\mathcal{C}_1$, defined by:

$$\mathcal{C}_1 = \langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2a+2b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle$$

and represented (in two ways) at the Figure 2. If every word f of $\{a,b\}^*$ is viewed as the writing of an integer in the binary system, where a is interpreted as 0 and b as 1, then $\mathcal{C}_1$ "computes" the integer written by f, which we denote by $\overline{f}$, in the sense that

$$|\mathcal{C}_1| = \sum_{f \in A^*} \overline{f}\, f \qquad \text{and the first terms of } |\mathcal{C}_1| \text{ reads then}$$

$$|\mathcal{C}_1| = b + ab + 2\,ba + 3\,bb + aab + 2\,aba + 3\,abb + 4\,baa + 5\,bab + \cdots \qquad \square$$

[2] This definition coincides then with the classical one when $\mathbb{K} = \mathbb{B}$, the Boolean semiring.

[3] In the context of this paper, we can take this statement as a definition for the $\mathbb{K}$-rational series.

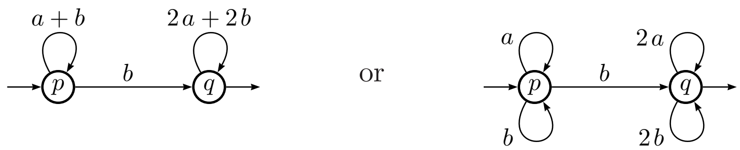

Figure 2: The $\mathbb{N}$-automaton $\mathcal{C}_1$.

The *support* of a power series, or of a polynomial, over A^* is the set of words of A^* whose coefficient is not zero in the series or in the polynomial. The *support* of a $\mathbb{K}$-automaton $\mathcal{A} = \langle I, E, T \rangle$ is the (classical) automaton obtained by taking the support of every entry of I, E and T. Conversely, to any (classical) automaton $\mathcal{A} = \langle Q, A, E, I, T \rangle$ is associated its *characteristic automaton* that is defined as the $\mathbb{K}$-automaton whose support is $\mathcal{A}$ and whose non-zero coefficients are all equal to $1_{\mathbb{K}}$ (generally, $\mathbb{K} = \mathbb{N}$).

Property 1 *The support of the behaviour of a $\mathbb{K}$-automaton $\mathcal{A}$ is contained in the behaviour of the support of $\mathcal{A}$. If $\mathbb{K}$ is a* positive *semiring, these two languages are equal.*

Property 2 *An automaton over A^* is unambiguous if and only if the behaviour of its characteristic $\mathbb{N}$-automaton is a characteristic power series.*

3 $\mathbb{K}$-coverings

The notion of covering seems to fit perfectly with the one of automaton with multiplicity. If $\mathcal{A}$ and $\mathcal{B}$ are two (classical) automata, the existence of a covering $\varphi \colon \mathcal{B} \to \mathcal{A}$ implies not only that $\mathcal{B}$ is equivalent to $\mathcal{A}$, *i.e.* that they both recognize the same language, but also that there exists a 1-to-1 correspondence between their successful computations, that is they are equivalent even if multiplicity is taken into account, *i.e.* they are equivalent as $\mathbb{N}$-automata — with the natural hypothesis that the label of every transition has multiplicity 1_N.

But it may be the case that we have two equivalent $\mathbb{K}$-automata $\mathcal{A}$ and $\mathcal{B}$ such that there exists an (automaton) morphism φ from the support of $\mathcal{B}$ into the support of $\mathcal{A}$ *which is not a covering*. As we said, an automaton morphism is a covering if its restriction to the corresponding "outgoing bouquets" is bijective. This condition is not adequate anymore for automata with multiplicity.

Example 3 : Let us consider the $\mathbb{N}$-automata $\mathcal{C}_2$ and $\mathcal{V}_2$ of the Figure 3. There exists an obvious morphism from the support of $\mathcal{C}_2$ onto the support of $\mathcal{V}_2$ that is not a covering: there is no bijection between $\mathsf{Out}_{\mathcal{C}_2}(j)$ and $\mathsf{Out}_{\mathcal{V}_2}(i)$, neither a co-covering: there is no bijection between $\mathsf{In}_{\mathcal{C}_2}(u)$ and $\mathsf{In}_{\mathcal{V}_2}(t)$.

These two $\mathbb{N}$-automata are equivalent nevertheless. The reason is that the *sum* of the labels of the transitions that go from j into the *set of states* whose image by φ is q is equal to the label of the transition that go from $i = j\varphi$ to q. □

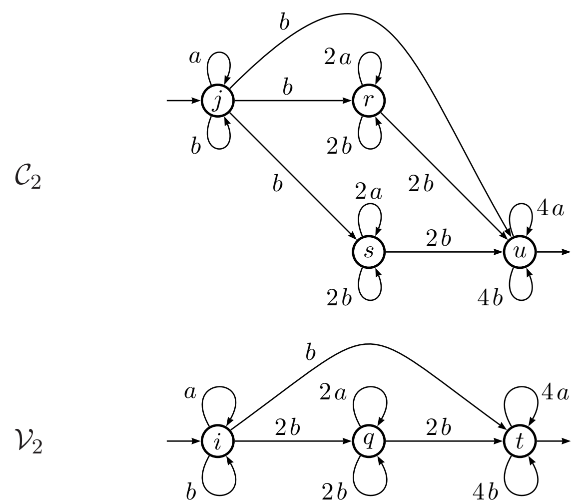

Figure 3: The *vertical* is a morphism, not a covering.

We now formalize the observation made in the example. Let $\mathcal{A} = \langle I, E, T \rangle$ and $\mathcal{B} = \langle J, F, U \rangle$ be two $\mathbb{K}$-automata of dimension Q and R respectively. Let $\varphi\colon R \to Q$ be a surjective mapping. Let F' be the matrix obtained from F by adding together the *columns* whose index have the same image by φ. Then φ is a $\mathbb{K}$-covering if any *row* of index r of F' is equal to the row of index $r\varphi$ of E.

Example 3 (continued)**:** If we write the above automata $\mathcal{C}_2$ and $\mathcal{V}_2$ as

$\mathcal{C}_2 = \langle\, J_2, F_2, U_2 \,\rangle$ and $\mathcal{V}_2 = \langle\, I_2, E_2, T_2 \,\rangle$, then:

$$F_2 = \begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix} .$$

We add the two mid columns and we get the matrix

$$F_2' = \begin{pmatrix} a+b & 2b & b \\ 0 & 2a+2b & 2b \\ 0 & 2a+2b & 2b \\ 0 & 0 & 4a+4b \end{pmatrix}$$

whose rows of index r and s are equal to the mid row of

$$E_2 = \begin{pmatrix} a+b & 2b & b \\ 0 & 2a+2b & 2b \\ 0 & 0 & 4a+4b \end{pmatrix} . \qquad \Box$$

Once it is understood that an image of a $\mathbb{K}$-automaton under a $\mathbb{K}$-covering is obtained by adding together some of the "columns of the $\mathbb{K}$-automaton", the definition of a $\mathbb{K}$-covering is best written under a matrix expression. To any surjective mapping $\varphi\colon R \to Q$ we associate the *row monomial* $R{\times}Q$-matrix H_φ defined by:

$$(H_\varphi)_{r,q} = \begin{cases} 1 & \text{if} \quad r\varphi = q \\ 0 & \text{otherwise} \end{cases}$$

Since φ is surjective, every column of H_φ contains at least one 1. From H_φ, a matrix K_φ is built by transposing H_φ and by making some entries equal to 0 in such a way that K_φ is *row monomial* (with exactly one 1 in every row). The matrix K_φ is not uniquely defined by φ (as is H_φ) but also by the arbitrary choice of a representative in every class modulo the mapping equivalence of φ.

Example 3 (continued)**:** If φ_2 is the mapping from $\{j, r, s, u\}$ onto $\{i, q, t\}$ such that $j\varphi_2 = i$, $u\varphi_2 = t$ and $r\varphi_2 = s\varphi_2 = q$, then:

$$H_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad K_{\varphi_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Box$

The multiplication of an $X \times R$-matrix Z by H_φ *on the right* yields an $X \times Q$-matrix whose column q is the sum of the columns of Z of index s such that $s\varphi = q$. The multiplication of a $R \times X$-matrix Z by K_φ *on the left* yields a $Q \times X$-matrix whose row p is chosen among the rows of Z of index r such that $r\varphi = p$. We can then state:

Definition 2 *A mapping $\varphi \colon R \to Q$ is a* $\mathbb{K}$-covering *from $\mathcal{B} = \langle J, F, U \rangle$ onto $\mathcal{A} = \langle I, E, T \rangle$ if $\mathcal{A}$ is defined by:*

$$E = K_\varphi \cdot F \cdot H_\varphi\,, \qquad I = J \cdot H_\varphi \qquad \textit{and} \qquad T = K_\varphi \cdot U \quad . \tag{1}$$

and if the following equations hold:

$$H_\varphi \cdot K_\varphi \cdot F \cdot H_\varphi = F \cdot H_\varphi \tag{2}$$

$$\textit{and} \qquad H_\varphi \cdot K_\varphi \cdot U = U \tag{3}$$

The definition of $\mathbb{K}$-covering *via* matrix expressions makes the proof of the following basic result particularly easy.

Proposition 3 *Any $\mathbb{K}$-automaton is equivalent to any of its $\mathbb{K}$-coverings.*

Proof. If $\varphi \colon \mathcal{B} \to \mathcal{A}$ is a $\mathbb{K}$-covering, it holds, for every n in $\mathbb{N}$:

$$\begin{aligned}
I \cdot E^n \cdot T &= J \cdot H_\varphi \cdot (K_\varphi \cdot F \cdot H_\varphi)^n \cdot K_\varphi \cdot U && \text{by (1)}\\
&= J \cdot (H_\varphi \cdot K_\varphi \cdot F)^n \cdot H_\varphi \cdot K_\varphi \cdot U && \\
&= J \cdot F^n \cdot H_\varphi \cdot K_\varphi \cdot U && \text{by (2)}\\
&= J \cdot F^n \cdot U && \text{by (3)}
\end{aligned}$$

and this implies the equality $|\mathcal{B}| = J \cdot F^* \cdot U = I \cdot E^* \cdot T = |\mathcal{A}|$. ■

To the $\mathbb{K}$-covering corresponds the *dual notion* of $\mathbb{K}$-co-covering. Roughly speaking, some rows will be added together, instead of some columns. More precisely we have:

Definition 3 *A mapping $\varphi \colon R \to Q$ is a* $\mathbb{K}$-co-covering *from $\mathcal{B} = \langle J, F, U \rangle$ onto $\mathcal{A} = \langle I, E, T \rangle$ if $\mathcal{A}$ is defined by:*

$$E = H_\varphi^{\mathsf{t}} \cdot F \cdot K_\varphi^{\mathsf{t}}\,, \qquad I = J \cdot K_\varphi^{\mathsf{t}} \qquad \textit{and} \qquad T = H_\varphi^{\mathsf{t}} \cdot U \quad .$$

and if the following equations hold:

$$H_\varphi^{\mathsf{t}} \cdot F \cdot K_\varphi^{\mathsf{t}} \cdot H_\varphi^{\mathsf{t}} = H_\varphi^{\mathsf{t}} \cdot F \qquad \textit{and} \qquad J \cdot K_\varphi^{\mathsf{t}} \cdot H_\varphi^{\mathsf{t}} = J$$

Proposition 4

Any $\mathbb{K}$-automaton is equivalent to any of its $\mathbb{K}$-co-coverings. ■

Example 3 (continued)**:** Let us consider $\mathcal{C}_2 = \langle\, J_2, F_2, U_2 \,\rangle$ again: if we add the two mid rows of

$$F_2 = \begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}$$

we get the matrix

$$\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 2a+2b & 4b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}$$

whose columns r and s are equal to the mid column of the matrix

$$E_2' = \begin{pmatrix} a+b & b & b \\ 0 & 2a+2b & 4b \\ 0 & 0 & 4a+4b \end{pmatrix}$$

which defines *another* $\mathbb{N}$-automaton $\mathcal{V}_2' = \langle\, I_2, E_2', T_2 \,\rangle$ equivalent to $\mathcal{C}_2$ (*cf.* Figure 4). □

Coming back to our first intuition, we then have:

Property 3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two (classical) automata and let $\varphi\colon \mathcal{B} \to \mathcal{A}$ be a covering (resp. a co-covering). Then, for any $\mathbb{K}$, φ is a $\mathbb{K}$-covering (resp. a $\mathbb{K}$-co-covering) from the characteristic automaton of $\mathcal{B}$ onto the characteristic automaton of $\mathcal{A}$.* ■

The following two properties are also easily verified.

Proposition 5 *Let $\mathcal{A}$ be a $\mathbb{K}$-automaton. Among all the $\mathbb{K}$-automata of which $\mathcal{A}$ is a $\mathbb{K}$-covering (resp. a $\mathbb{K}$-co-covering) there exists a unique one, effectively computable, that has a minimal number of states and of which all these $\mathbb{K}$-automata are $\mathbb{K}$-coverings (resp. $\mathbb{K}$-co-coverings).* ■

Proposition 6 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $\mathbb{K}$-automata such that $\mathcal{A}$ is a $\mathbb{K}$-covering of $\mathcal{C}$ and $\mathcal{B}$ is a $\mathbb{K}$-co-covering of $\mathcal{C}$. Then there exists a $\mathbb{K}$-automaton $\mathcal{D}$ which is a $\mathbb{K}$-co-covering of $\mathcal{A}$ and a $\mathbb{K}$-covering de $\mathcal{B}$.* ■

Remark 1 The terminology may be slightly misleading inasmuch as if a $\mathbb{K}$-automaton is exactly a classical automaton when $\mathbb{K} = \mathbb{B}$, a $\mathbb{B}$-covering *is not* a covering of classical automata, but only an Out-surjective morphism.

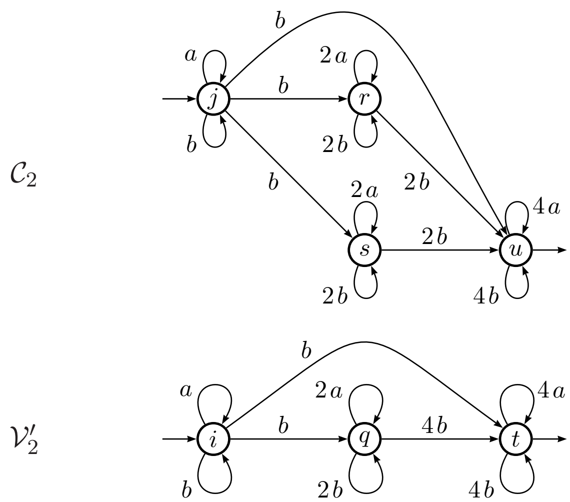

Figure 4: $\mathcal{C}_2$ is an $\mathbb{N}$-co-covering of $\mathcal{V}'_2$.

4 The skimming theorem

The Schützenberger construct, applied to a $\mathbb{N}$-automaton $\mathcal{A}$, yields an unambiguous $\mathbb{N}$-automaton $\mathcal{T}$ whose behaviour is the characteristic series of the support of the behaviour of $\mathcal{A}$. (*i.e.* $|\mathcal{T}| = \underline{\mathsf{supp}\,|\mathcal{A}|}$), and this is not surprising indeed. What is remarkable is that *the same construction*, together with the notion of $\mathbb{N}$-covering, yields a $\mathbb{N}$-automaton $\mathcal{P}$ which is the complement of $\mathcal{T}$ with respect to $\mathcal{A}$ *i.e.* $|\mathcal{A}| = |\mathcal{T}| + |\mathcal{P}|$, and this is the theorem we are aiming at:

Theorem 1 *If s is a $\mathbb{N}$-rational power series on A^*, the series* $s' = s - \underline{\mathsf{supp}\, s}$ *is a $\mathbb{N}$-rational power series as well.*

In other words, the series obtained by substracting 1 to every non zero coefficient of a $\mathbb{N}$-rational power series on A^* is still a $\mathbb{N}$-rational power series on A^* and this can be represented as on Figure 5. The series s is represented as the sequence of the values of the coefficients, adequatly oriented downwards; the upper layer is taken off[4]; what is left is the representation of *another* $\mathbb{N}$*-rational power series*.

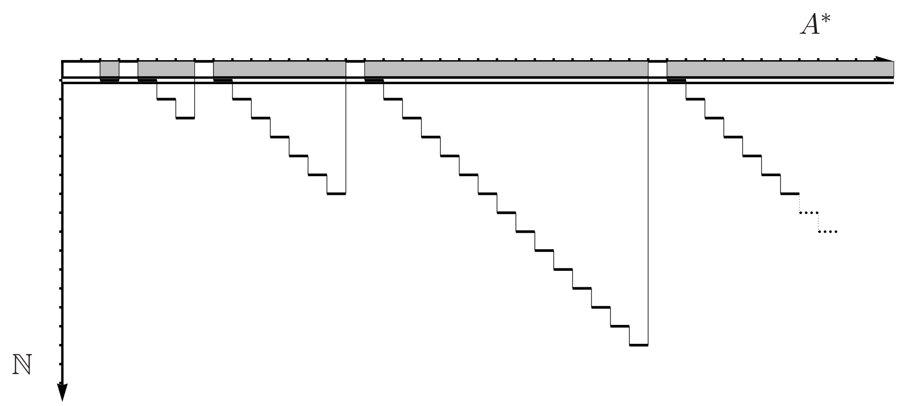

Figure 5: Skimming the $\mathbb{N}$-series $|\mathcal{C}_1|$.

Proof of Theorem 1. Let $\mathcal{A} = \langle I, E, T \rangle$ be a (proper) $\mathbb{N}$-automaton on A^* whose behaviour is equal to s, $\mathcal{S} = \langle J, F, U \rangle$ its Schützenberger covering (which is a $\mathbb{N}$-automaton of dimension R), and $\mathcal{T} = \langle J, G, V \rangle$ a S-immersion in $\mathcal{A}$, of dimension R as well. By definition, $\mathcal{T}$ is a sub-automaton of $\mathcal{S}$ and there exist a matrix H with coefficients in $\mathbb{N}\langle\langle A^* \rangle\rangle$ and a vector W with coefficients in $\mathbb{N}$ such that $F = G + H$ and $U = V + W$.

It is then observed that the automaton $\mathcal{S}'$ below, of dimension $R\times\{1,2,3\}$ is equivalent to $\mathcal{S}$, hence to $\mathcal{A}$:

$$\mathcal{S}' = \langle \begin{pmatrix} J & J & 0 \end{pmatrix}, \begin{pmatrix} G & 0 & H \\ 0 & G & 0 \\ 0 & 0 & F \end{pmatrix}, \begin{pmatrix} V \\ W \\ U \end{pmatrix} \rangle = \langle \begin{pmatrix} J & J & 0 \end{pmatrix}, F', \begin{pmatrix} V \\ W \\ U \end{pmatrix} \rangle \ .$$

Indeed, if we add *the rows* of $\mathcal{S}'$ of index $(r,1)$ and $(r,2)$ for every r in R we then get the matrices

$$\begin{pmatrix} J & J & 0 \end{pmatrix} \ , \quad \begin{pmatrix} G & G & H \\ 0 & 0 & F \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} U \\ U \end{pmatrix} \ ,$$

whose columns of index $(r,1)$ et $(r,2)$ (for every r in R) are equal: $\mathcal{S}'$ is a $\mathbb{N}$-*co-covering* of

$$\mathcal{S}'' = \langle \begin{pmatrix} J & 0 \end{pmatrix}, \begin{pmatrix} G & H \\ 0 & F \end{pmatrix}, \begin{pmatrix} U \\ U \end{pmatrix} \rangle \ .$$

[4]As one skims the cream from a milk jar.

The automaton $\mathcal{S}''$, of dimension $R{\times}\{1,2\}$, is itself an $\mathbb{N}$-*covering* of $\mathcal{S}$ since if we add *the columns* of index $(r,1)$ and $(r,2)$ (for every r in R) we get the matrices

$$J \ , \quad \begin{pmatrix} F \\ F \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} U \\ U \end{pmatrix} \ ,$$

whose rows of index $(r,1)$ et $(r,2)$ (for every r in R) are equal to the row of index r in $\mathcal{S}$. Hence, it holds:

$$|\mathcal{A}| = |\mathcal{S}| = |\mathcal{S}'| = |\langle \begin{pmatrix} J & J & 0 \end{pmatrix}, F', \begin{pmatrix} V \\ 0 \\ 0 \end{pmatrix} \rangle| + |\langle \begin{pmatrix} J & J & 0 \end{pmatrix}, F', \begin{pmatrix} 0 \\ W \\ U \end{pmatrix} \rangle| \ .$$

and

$$|\mathcal{T}| = |\langle J, G, V \rangle| = |\langle \begin{pmatrix} J & J & 0 \end{pmatrix}, F', \begin{pmatrix} V \\ 0 \\ 0 \end{pmatrix} \rangle| = \underline{\mathsf{supp}\,|\mathcal{A}|}$$

The behaviour of the automaton

$$\mathcal{P} = \langle \begin{pmatrix} J & J & 0 \end{pmatrix}, F', \begin{pmatrix} 0 \\ W \\ U \end{pmatrix} \rangle$$

is then equal to $s - \underline{\mathsf{supp}\, s}$. ■

Example 2 (continued)**:** The above construction is applied to the automaton

$$\mathcal{C}_1 = \langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2a+2b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle \ .$$

This case is made simple by the fact that (with notation of the proof) $V = U$ and thus it holds directly that $\mathcal{S}' = \mathcal{S}''$ is a $\mathbb{N}$-covering of the Schützenberger covering of $\mathcal{C}_1$. The corresponding automaton $\mathcal{P}_1$ is drawn at the Figure 6. □

Theorem 1 yields directly a series of well-known corollaries that give useful insights on the structure of $\mathbb{N}$-rational series and that are worth recalling.

An $\mathbb{N}$-series is said to be *bounded (by k)* if the set of its coefficients is bounded (by k). Let s and t be two $\mathbb{N}$-series. We write $s \leqslant t$ if $\langle s, f\rangle \leqslant \langle t, f\rangle$ for every f in A^*, *i.e.* if there exists an $\mathbb{N}$-series u such that $s + u = t$; in this case we write $u = t - s$. More generally, the operation $t \dot{-} s$ is defined by

$$\langle t \dot{-} s, f\rangle = \mathsf{sup}\{0, (\langle t, f\rangle - \langle s, f\rangle)\}$$

for every f in A^*.

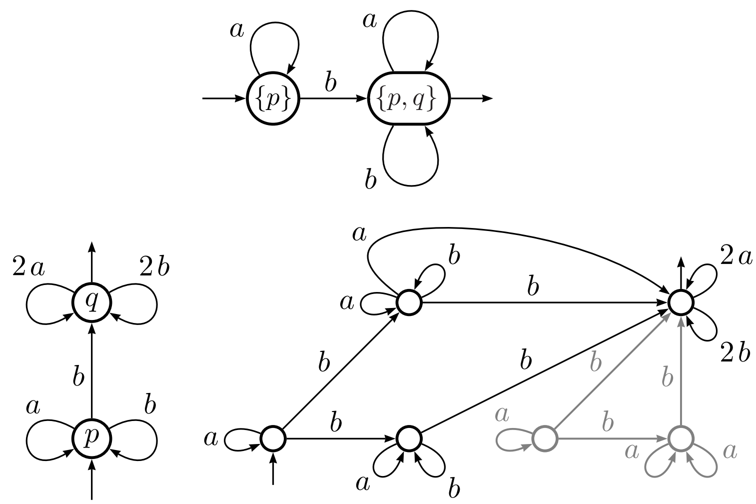

Figure 6: An automaton $\mathcal{P}_1$ whose behaviour is equal to $|\mathcal{C}_1| - \underline{\mathsf{supp}\,|\mathcal{C}_1|}$.

Corollary 7 *An $\mathbb{N}$-rational series bounded by k is the sum of at most k $\mathbb{N}$-rational characteristic series* ■

Corollary 8 *Let s and t be two $\mathbb{N}$-rational series such that s is bounded. Then $t \dot{-} s$ is an $\mathbb{N}$-rational series.* ■

Corollary 9 *Let s be an $\mathbb{N}$-rational series on A^*. For every integer k the languages*

$$\{f \in A^* \mid <\!s,f\!> \geqslant k\}, \quad \{f \in A^* \mid <\!s,f\!> = k\} \quad \textit{and} \quad \{f \in A^* \mid <\!s,f\!> \leqslant k\}$$

are rational. ■

Remark 2 The proof of Corollary 7 is indeed immediate: if $|\mathcal{A}|$ is bounded by k, we write $|\mathcal{A}| = |\mathcal{T}| + |\mathcal{P}|$ as above, $|\mathcal{P}|$ is bounded by $k-1$ and we iterate the procedure. But it conceals a problem: at every step of that procedure, we have to perform a determinization (for the construction of the Schützenberger covering) which means an exponentiation. And this easy proof yields then a tower of k exponentiation. However, the work of Weber ([11]) on the decomposion of k valued transducers leads to think that a double exponentiation is sufficient in any case but this is still a conjecture.

Remark 3 The definition of $\mathbb{K}$-covering we have given as well as the construction of the automaton $\mathcal{P}$ in the proof of Theorem 1 may ring some bells to the reader who is familiar with symbolic dynamical system theory and who is reminiscent of the technic of *state splitting* and *state amalgamation* (*cf.* [3, §2.4] for instance).

If $\mathcal{B}$ is obtained from $\mathcal{A}$ by an In-splitting, then $\mathcal{B}$ is an $\mathbb{N}$-covering of $\mathcal{A}$ and, dually, $\mathcal{B}$ is an $\mathbb{N}$-co-covering of $\mathcal{A}$ if it is obtained by an Out-splitting. But the converse is not true. Roughly speaking, $\mathcal{B} = \langle J, F, U \rangle$ is an $\mathbb{N}$-covering of $\mathcal{A}$ if the rows of "equivalent" index of the matrix F' are equal, where F' is obtained from F by adding the columns of equivalent index. Whereas $\mathcal{B}$ is obtained from $\mathcal{A}$ by an In-splitting if the rows of "equivalent" index of F are equal (and thus they are equal in F').

Proposition 5 can then be seen as the equivalent of Williams' theorem in this setting.

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